Analysis of Algorithms

Piyush Kumar

(Lecture 9: NP Completeness)

Welcome to COP 4531

Based on Kevin Wayne's slides

Announcements

- Programming Assignment due: April 25^{th}
- Submission: email your project.tar.gz to Vinod Akula: akula at cs dot fsu dot edu
- Last Homework due: April 19^{th} ?
- Project presentations : April 27^{th}. (1/2 hour)
- Final Exams : April 26th

NP

- Certification algorithm intuition.
  - Certifier views things from "managerial" viewpoint.
  - Certifier doesn't determine whether $s \in X$ on its own; rather, it checks a proposed proof t that $s \in X$.
- Def. Algorithm C(s, t) is a certifier for problem X if for every string s, $s \in X$ iff there exists a string t such that C(s, t) = yes.

  "certificate" or "witness"
- NP. Decision problems for which there exists a poly-time certifier.

  C(s, t) is a poly-time algorithm and $|t| \le p(|s|)$ for some polynomial p(·).
- Remark. NP stands for nondeterministic polynomial-time.

Certifiers and Certificates: Composite

- COMPOSITES. Given an integer s, is s composite?
- Certificate. A nontrivial factor t of s. Note that such a certificate exists iff s is composite. Moreover $|t| \le |s|$.
- Certifier.

```
boolean C(s, t) {
   if (t ≤ 1 or t ≥ s)
      return false
   else if (s is a multiple of t)
      return true
   else
      return false
}
```

- Instance. s = 437,669.
- Certificate. t = 541 or 809. 437,669 = 541 × 809
- Conclusion. COMPOSITES is in NP.

Certifiers and Certificates: 3-Satisfiability

- SAT. Given a CNF formula Φ, is there a satisfying assignment?
- Certificate. An assignment of truth values to the n boolean variables.
- Certifier. Check that each clause in Φ has at least one true literal.

$$(\overline{x_1} \vee x_2 \vee x_3) \wedge (x_1 \vee \overline{x_2} \vee x_3) \wedge (x_1 \vee x_2 \vee x_4) \wedge (\overline{x_1} \vee \overline{x_3} \vee \overline{x_4})$$

instance s

- Ex.

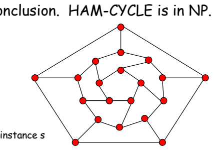
$$x_1 = 1,\ x_2 = 1,\ x_3 = 0,\ x_4 = 1$$

certificate t

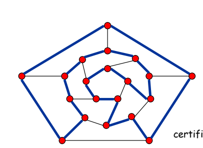
- Conclusion. SAT is in NP.

Certifiers and Certificates: Hamiltonian Cycle

- HAM-CYCLE. Given an undirected graph G = (V, E), does there exist a simple cycle C that visits every node?
- Certificate. A permutation of the n nodes.
- Certifier. Check that the permutation contains each node in V exactly once, and that there is an edge between each pair of adjacent nodes in the permutation.
- Conclusion. HAM-CYCLE is in NP.



instance s



certificate t

P, NP, EXP

- P. Decision problems for which there is a poly-time algorithm.
- EXP. Decision problems for which there is an exponential-time algorithm.
- NP. Decision problems for which there is a poly-time certifier.

- Claim. $P \subseteq NP$.
- Pf. Consider any problem X in P.
  - By definition, there exists a poly-time algorithm A(s) that solves X.
  - Certificate: $t = \varepsilon$, certifier $C(s, t) = A(s)$. ▪

- Claim. $NP \subseteq EXP$.
- Pf. Consider any problem X in NP.
  - By definition, there exists a poly-time certifier C(s, t) for X.
  - To solve input s, run C(s, t) on all strings t with $|t| \le p(|s|)$.
  - Return yes, if C(s, t) returns yes for any of these. ▪

The Main Question: P Versus NP

- Does P = NP? [Cook 1971, Edmonds, Levin, Yablonski, Gödel]
  - Is the decision problem as easy as the certification problem?
  - Clay $1 million prize.

would break RSA cryptography
(and potentially collapse economy)

- If yes: Efficient algorithms for 3-COLOR, TSP, FACTOR, SAT, …
- If no: No efficient algorithms possible for 3-COLOR, TSP, SAT, …

Consensus opinion on P = NP? Probably no.

Quantum machines can FACTOR in poly time!! But…

NP Completeness

Polynomial Transformation

- Def. Problem X polynomial reduces (Cook) to problem Y if arbitrary instances of problem X can be solved using:
  - Polynomial number of standard computational steps, plus
  - Polynomial number of calls to oracle that solves problem Y.

- Def. Problem X polynomial transforms (Karp) to problem Y if given any input x to X, we can construct an input y such that x is a yes instance of X iff y is a yes instance of Y.

  ↑ we require |y| to be of size polynomial in |x|

- Note. Polynomial transformation is polynomial reduction with just one call to oracle for Y, exactly at the end of the algorithm for X. Almost all previous reductions were of this form.

- Open question. Are these two concepts the same?

  ↑ we abuse notation $\le_p$ and blur distinction

NP-Complete

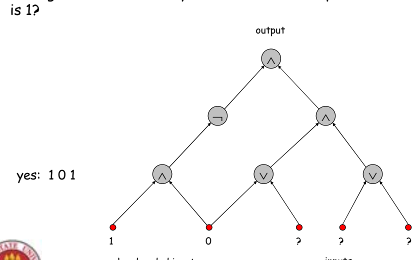
- NP-complete. A problem Y in NP with the property that for every problem X in NP, $X \le_p Y$. (Hardest problems in NP)

- Theorem. Suppose Y is an NP-complete problem. Then Y is solvable in poly-time iff P = NP.
- Pf. ⇐ If P = NP then Y can be solved in poly-time since Y is in NP.
- Pf. ⇒ Suppose Y can be solved in poly-time.
  - Let X be any problem in NP. Since $X \le_p Y$, we can solve X in poly-time. This implies $NP \subseteq P$.
  - We already know $P \subseteq NP$. Thus P = NP. ▪

- Fundamental question. Do there exist "natural" NP-complete problems?

Circuit Satisfiability

- CIRCUIT-SAT. Given a combinational circuit built out of AND, OR, and NOT gates, is there a way to set the circuit inputs so that the output is 1?



The "First" NP-Complete Problem

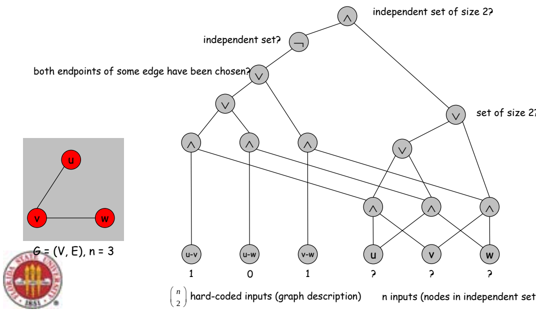
- Theorem. CIRCUIT-SAT is NP-complete. [Cook 1971, Levin 1973]
- Pf. (sketch)
  - Any algorithm that takes a fixed number of bits n as input and produces a yes/no answer can be represented by such a circuit. Moreover, if algorithm takes poly-time, then circuit is of poly-size.

    sketchy part of proof; fixing the number of bits is important, and reflects basic distinction between algorithms and circuits

  - Consider some problem X in NP. It has a poly-time certifier C(s, t). To determine whether s is in X, need to know if there exists a certificate t of length p(|s|) such that C(s, t) = yes.
  - View C(s, t) as an algorithm on |s| + p(|s|) bits (input s, certificate t) and convert it into a poly-size circuit K.
    - first |s| bits are hard-coded with s
    - remaining p(|s|) bits represent bits of t
  - Circuit K is satisfiable iff C(s, t) = yes.

Example

- Ex. Construction below creates a circuit K whose inputs can be set so that K outputs true iff graph G has an independent set of size 2.



Establishing NP-Completeness

- Remark. Once we establish first "natural" NP-complete problem, others fall like dominoes.

- Recipe to establish NP-completeness of problem Y.
  - Step 1. Show that Y is in NP.
  - Step 2. Choose an NP-complete problem X.
  - Step 3. Prove that $X \le_p Y$.

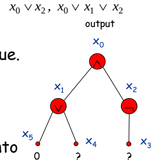
- Justification. If X is an NP-complete problem, and Y is a problem in NP with the property that $X \le_P Y$ then Y is NP-complete.

- Pf. Let W be any problem in NP. Then $W \le_P X \le_P Y$.
  - By transitivity, $W \le_P Y$.
  - Hence Y is NP-complete. ▪

  (by definition of NP-complete; by assumption)

3-SAT is NP-Complete

- Theorem. 3-SAT is NP-complete.
- Pf. Suffices to show that CIRCUIT-SAT $\le_P$ 3-SAT since 3-SAT is in NP.
  - Let K be any circuit.
  - Create a 3-SAT variable x_i for each circuit element i.
  - Make circuit compute correct values at each node:
    - $x_2 = \neg x_3 \Rightarrow$ add 2 clauses: $x_2 \vee x_3$, $\overline{x_2} \vee \overline{x_3}$
    - $x_1 = x_4 \vee x_5 \Rightarrow$ add 3 clauses: $x_1 \vee \overline{x_4}$, $x_1 \vee \overline{x_5}$, $\overline{x_1} \vee x_4 \vee x_5$
    - $x_0 = x_1 \wedge x_2 \Rightarrow$ add 3 clauses: $\overline{x_0} \vee x_1$, $\overline{x_0} \vee x_2$, $x_0 \vee \overline{x_1} \vee \overline{x_2}$
  - Hard-coded input values and output value.
    - $x_5 = 0 \Rightarrow$ add 1 clause: $\overline{x_5}$
    - $x_0 = 1 \Rightarrow$ add 1 clause: x_0
  - Final step: turn clauses of length < 3 into clauses of length exactly 3. ▪



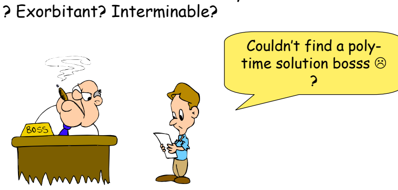
Final Step?

- We force z1 = z2 = 0
- For single terms t : $t \vee z_1 \vee z_2$
- For two term clauses : $t \vee w \vee z_1$

- How can we force z1 = z2 = 0 in a 3-sat?
- Hence we now have
  - 3-SAT $\le_p$ Independent Set $\le_p$ Vertex Cover $\le_p$ Set Cover
  - In our NP-Complete Bank

X problems?

- X = Hard ? Tough? Herculean? Formidable? Arduous? NPC?
- X = Impractical? Bad? Heavy? Tricky? Intricate? Prodigious ? Difficult? Intractable? Costly ? Obdurate? Obstinate ? Exorbitant? Interminable?



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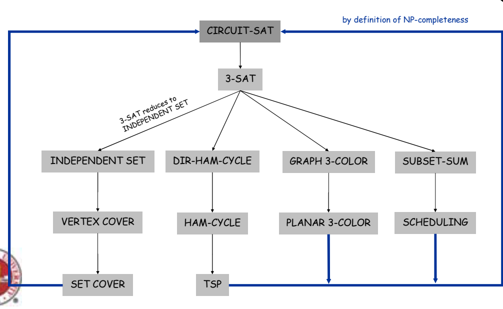
NP-Completeness

- Observation. All problems below are NP-complete and polynomial reduce to one another!



Some NP-Complete Problems

- Six basic genres of NP-complete problems and paradigmatic examples.
  - Packing problems: SET-PACKING, INDEPENDENT SET.
  - Covering problems: SET-COVER, VERTEX-COVER.
  - Constraint satisfaction problems: SAT, 3-SAT.
  - Sequencing problems: HAMILTONIAN-CYCLE, TSP.
  - Partitioning problems: 3D-MATCHING 3-COLOR.
  - Numerical problems: SUBSET-SUM, KNAPSACK.
- Practice. Most NP problems are either known to be in P or NP-complete.
- Notable exceptions. Factoring, graph isomorphism, Nash equilibrium.

Extent and Impact of NP-Completeness

- Extent of NP-completeness. [Papadimitriou 1995]
  - Prime intellectual export of CS to other disciplines.
  - 6,000 citations per year (title, abstract, keywords).
    - more than "compiler", "operating system", "database"
  - Broad applicability and classification power.
  - "Captures vast domains of computational, scientific, mathematical endeavors, and seems to roughly delimit what mathematicians and scientists had been aspiring to compute feasibly."
- NP-completeness can guide scientific inquiry.
  - 1926: Ising introduces simple model for phase transitions.
  - 1944: Onsager solves 2D case in tour de force.
  - 19xx: Feynman and other top minds seek 3D solution.
  - 2000: Istrail proves 3D problem NP-complete.

More Hard Computational Problems

- Aerospace engineering: optimal mesh partitioning for finite elements.
- Biology: protein folding.
- Chemical engineering: heat exchanger network synthesis.
- Civil engineering: equilibrium of urban traffic flow.
- Economics: computation of arbitrage in financial markets with friction.
- Electrical engineering: VLSI layout.
- Environmental engineering: optimal placement of contaminant sensors.
- Financial engineering: find minimum risk portfolio of given return.
- Game theory: find Nash equilibrium that maximizes social welfare.
- Genomics: phylogeny reconstruction.
- Mechanical engineering: structure of turbulence in sheared flows.
- Medicine: reconstructing 3-D shape from biplane angiocardiogram.
- Operations research: optimal resource allocation.
- Physics: partition function of 3-D Ising model in statistical mechanics.
- Politics: Shapley-Shubik voting power.
- Pop culture: Minesweeper consistency.
- Statistics: optimal experimental design.

8.9 co-NP and the Asymmetry of NP

Asymmetry of NP

- Asymmetry of NP. We only need to have short proofs of yes instances.
- Ex 1. SAT vs. TAUTOLOGY.
  - Can prove a CNF formula is satisfiable by giving such an assignment.
  - How could we prove that a formula is not satisfiable?
- Ex 2. HAM-CYCLE vs. NO-HAM-CYCLE.
  - Can prove a graph is Hamiltonian by giving such a Hamiltonian cycle.
  - How could we prove that a graph is not Hamiltonian?
- Remark. SAT is NP-complete and SAT $\equiv_P$ TAUTOLOGY, but how do we classify TAUTOLOGY?
  ↑ not even known to be in NP

NP and co-NP

- NP. Decision problems for which there is a poly-time certifier.
- Ex. SAT, HAM-CYCLE, COMPOSITES.
- Def. Given a decision problem X, its complement $\overline{X}$ is the same problem with the yes and no answers reverse.
- Ex. X = { 0, 1, 4, 6, 8, 9, 10, 12, 14, 15, … }
  $\overline{X}$ = { 2, 3, 5, 7, 11, 13, 17, 23, 29, … }
- co-NP. Complements of decision problems in NP.
- Ex. TAUTOLOGY, NO-HAM-CYCLE, PRIMES.

Reverse the yes/no answers for the decision problem.

NP and co-NP

- NP : Problems that have succinct certificates
  - (Ex: Hamiltonian Cycle)
- co-NP : Problems that have succinct disqualifiers.
  - (Ex: No-Hamiltonian Cycle)

No Hamiltonian Cycle → yes / no (short proof)

$X \in NP \Leftrightarrow \overline{X} \in co-NP$

NP = co-NP ?

- Fundamental question. Does NP = co-NP?
  - Do yes instances have succinct certificates iff no instances do?
  - Consensus opinion: no.
- Theorem. If NP ≠ co-NP, then P ≠ NP.
- Pf idea.
  - P is closed under complementation.
  - If P = NP, then NP is closed under complementation.
  - In other words, NP = co-NP.
  - This is the contrapositive of the theorem.

Good Characterizations

- Good characterization. [Edmonds 1965] NP I co-NP.
  - If problem X is in both NP and co-NP, then:
    - for yes instance, there is a succinct certificate
    - for no instance, there is a succinct disqualifier
  - Provides conceptual leverage for reasoning about a problem.
- Ex. Given a bipartite graph, is there a perfect matching.
  - If yes, can exhibit a perfect matching.
  - If no, can exhibit a set of nodes S such that |N(S)| < |S|.

Good Characterizations

- Observation. P ⊆ NP I co-NP.
  - Proof of max-flow min-cut theorem led to stronger result that max-flow and min-cut are in P.
  - Sometimes finding a good characterization seems easier than finding an efficient algorithm.

- Fundamental open question. Does P = NP I co-NP?
  - Mixed opinions.
  - Many examples where problem found to have a non-trivial good characterization, but only years later discovered to be in P.
    - linear programming [Khachiyan, 1979]
      - Still open if its strongly poly !
    - primality testing [Agrawal-Kayal-Saxena, 2002]

- Fact. Factoring is in NP I co-NP, but not known to be in P.

if poly-time algorithm for factoring, can break RSA cryptosystem

PRIMES is in NP ∩ co-NP

- Theorem. PRIMES is in NP ∩ co-NP.
- Pf. We already know that PRIMES is in co-NP, so it suffices to prove that PRIMES is in NP.

- Pratt's Theorem. An odd integer s is prime iff there exists an integer 1 < t < s s.t.

$$t^{s-1} \equiv 1 \pmod{s}$$
$$t^{(s-1)/p} \not\equiv 1 \pmod{s}$$

for all prime divisors p of s-1

Input. s = 437,677
Certificate. t = 17, $2^2 \times 3 \times 36{,}473$

prime factorization of s-1
also need a recursive certificate to assert that 3 and 36,473 are prime

Certifier.
- Check s-1 = 2 × 2 × 3 × 36,473.
- Check 17^{s-1} = 1 (mod s).
- Check $17^{(s-1)/2} \equiv 437{,}676$ (mod s).
- Check $17^{(s-1)/3} \equiv 329{,}415$ (mod s).
- Check $17^{(s-1)/36{,}473} \equiv 305{,}452$ (mod s).

use repeated squaring

FACTOR is in NP ∩ co-NP

- FACTORIZE. Given an integer x, find its prime factorization.
- FACTOR. Given two integers x and y, does x have a nontrivial factor less than y?

- Theorem. FACTOR $\equiv_P$ FACTORIZE.

- Theorem. FACTOR is in NP ∩ co-NP.
- Pf.
  - Certificate: a factor p of x that is less than y.
  - Disqualifier: the prime factorization of x (where each prime factor is less than y), along with a certificate that each factor is prime.

Primality Testing and Factoring

- We established: PRIMES $\le_P$ COMPOSITES $\le_P$ FACTOR.

- Natural question: Does FACTOR $\le_P$ PRIMES ?
- Consensus opinion. No.

- State-of-the-art.
  - PRIMES is in P. ← proved in 2001
  - FACTOR not believed to be in P.

- RSA cryptosystem.
  - Based on dichotomy between complexity of two problems.
  - To use RSA, must generate large primes efficiently.
  - To break RSA, suffices to find efficient factoring algorithm.
  - The first Real Quantum machine will break most Crypto around!