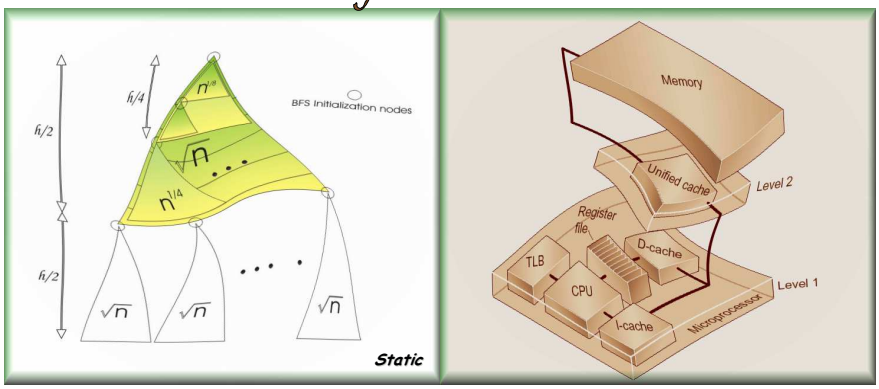
Cache Oblivious Algorithms
Theory & Practice



Research Proficiency Examination

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CO Algorithms: Brief History

- Frigo, Leiserson, Prokop, Ramachandran (FOCS 99)
  Cache Oblivious Algorithms
- Harold Prokop's **Thesis**
- Bender, Demaine, Farch-Coltun (FOCS 00)
  Cache Oblivious B-Trees
- …
- Arge, Bender, Demaine et.al. (STOC02)
  CO Priority Queue

Talk Outline…

- **Motivation**
  - ④ Matrix Multiplication/Transposition
  - ④ Static Searches in Bal. Bin. Trees
- **The Model**
- **CO-Sorting**
- **Some Analysis**
- CO-Sorting **Experiments**
- Do's and Don'ts of the model
- Future work

Workstations

SUN UltraSparc 2:
UltraSparc 16kB L1, 512kB L2.

SGI Visual Workstation 540:
Quad-Pentium III 32kB L1, 1024kB L2.

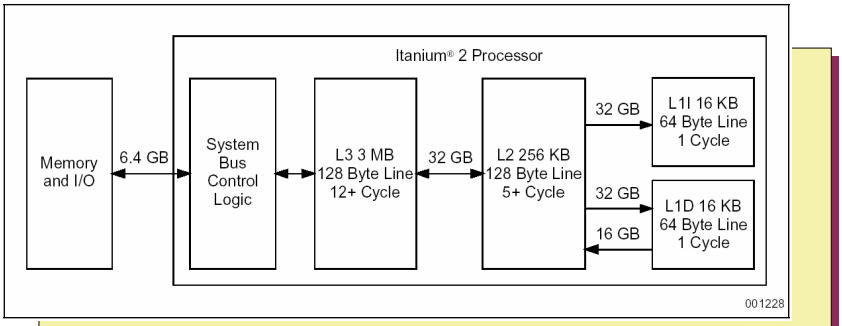
Dell Precision:
Dual-Pentium III 32kB L1 512kB L2.

IBM ThinkPad 600:
Pentium II 32kB L1, 256kB L2.

Compaq Presario:
AMD K6-III 64kB L1, 256kB L2, 1024kB L3.

How can we write portable code that runs efficiently on different multilevel caching architectures?

Intel Itaniums



Matrix Multiplication (MM)

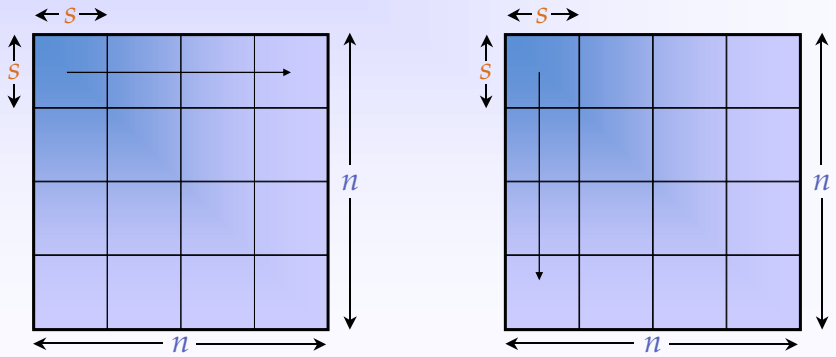
$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

$$\begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \times \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix}$$

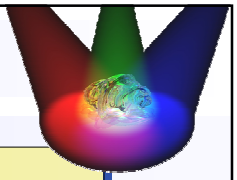
C A B

Cache-Aware MM

```
BLOCK-MULT (A,B,C,n)
1  for i ← 1 to n/s
2      do for j ← 1 to n/s
3          do for k ← 1 to n/s
4              do ORD-MULT (A_ik,B_kj,C_ij,s)
```

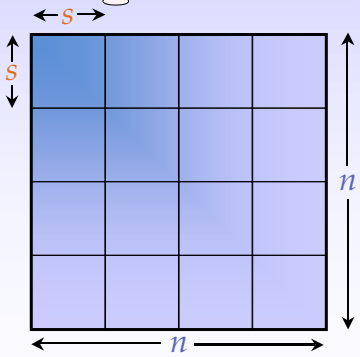


Cache-Aware MM



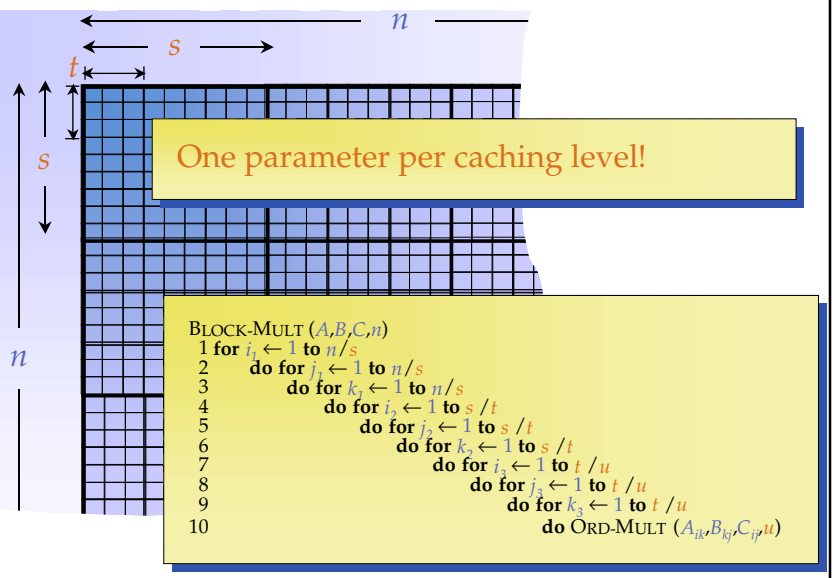
```
BLOCK-MULT (A,B,C,n)
1  fo
2
3
4                    ULT (A_ik,B_kj,C_ij,s)
```

Oracle?!



- Tune s so that A_{ik}, B_{kj}, and C_{ij} just fit into cache? $s = \Theta(\sqrt{Z})$
- If $n > s$, then
$$Q(n) = \Theta((n/s)^3(s^2/L)) = \Theta(n^3/L\sqrt{Z}).$$
- Optimal [HK81].

Three-Level Cache




```
BLOCK-MULT (A,B,C,n)
 1 for i1 ← 1 to n/s
 2     do for j1 ← 1 to n/s
 3         do for k1 ← 1 to n/s
 4             do for i2 ← 1 to s/t
 5                 do for j2 ← 1 to s/t
 6                     do for k2 ← 1 to s/t
 7                         do for i3 ← 1 to t/u
 8                             do for j3 ← 1 to t/u
 9                                 do for k3 ← 1 to t/u
10                                     do ORD-MULT (Aik,Bkj,Cij,u)
```

Recursive Matrix Multiplication

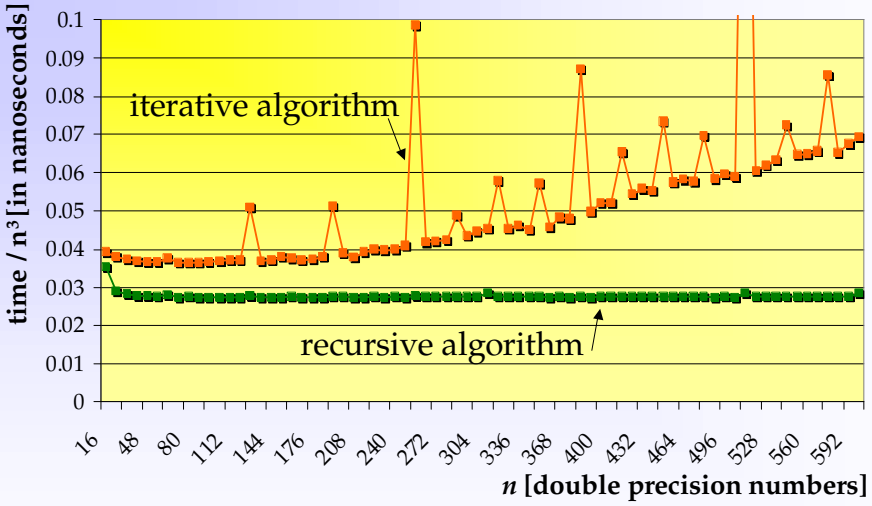
Divide and conquer on $n \times n$ matrices.

$$\begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \times \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$$

$$= \begin{pmatrix} A_{11}B_{11} & A_{11}B_{12} \\ A_{21}B_{11} & A_{21}B_{12} \end{pmatrix} + \begin{pmatrix} A_{12}B_{21} & A_{12}B_{22} \\ A_{22}B_{21} & A_{22}B_{22} \end{pmatrix}$$

8 multiplications of $(n/2) \times (n/2)$ matrices.
1 addition of $n \times n$ matrices.

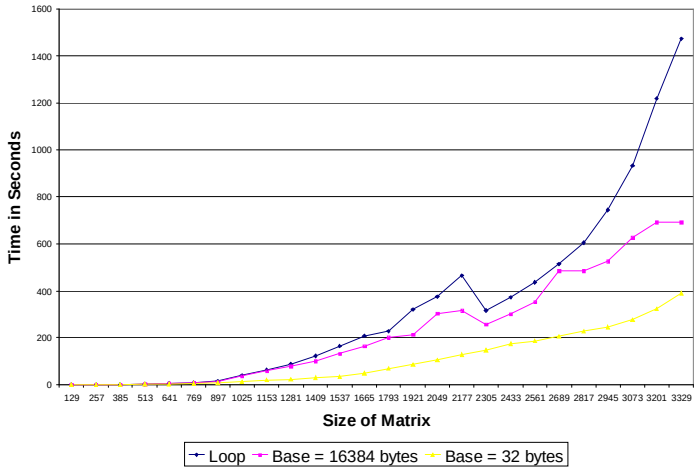
$\Theta(n^3)$-*Matrix Multiplication (n,n) × (n,n)*



450-MHz AMD K6-III processor with 32kB L1-cache, 64kB L2-cache, and 1MB L3-cache.

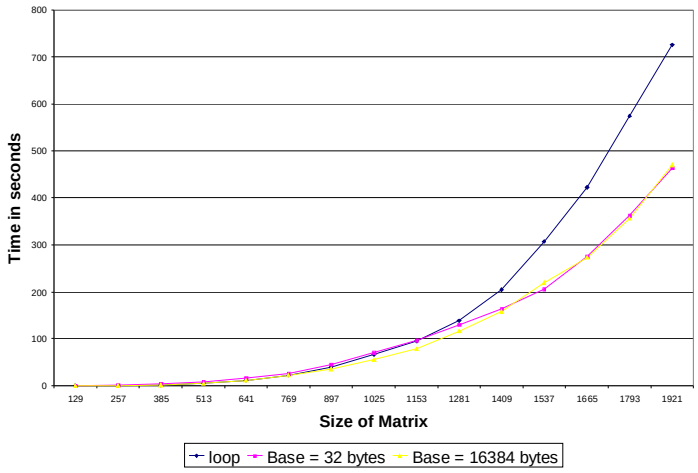
Experiments: MM

- Linux Athlon 1Ghz/1Gb/g++ -O3



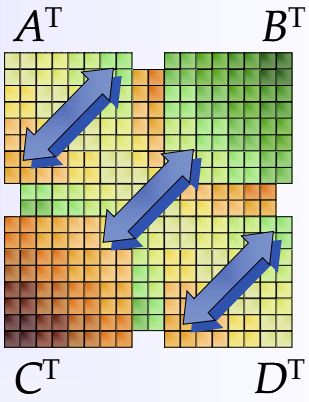
Experiments: MM

- Linux/Itanium/2GB/g++ -O3



Recursive Transpose

1. Partition matrix in 4 submatrices A, B, C, and D
2. Recursively transpose A.
3. Recursively transpose and swap B and C.
4. Recursively transpose D.



$\Theta(n^2 / L)$ cache misses, which is optimal. Used as a subroutine in our optimal cache-oblivious FFT [HK81].

Code: The InPlace Loop

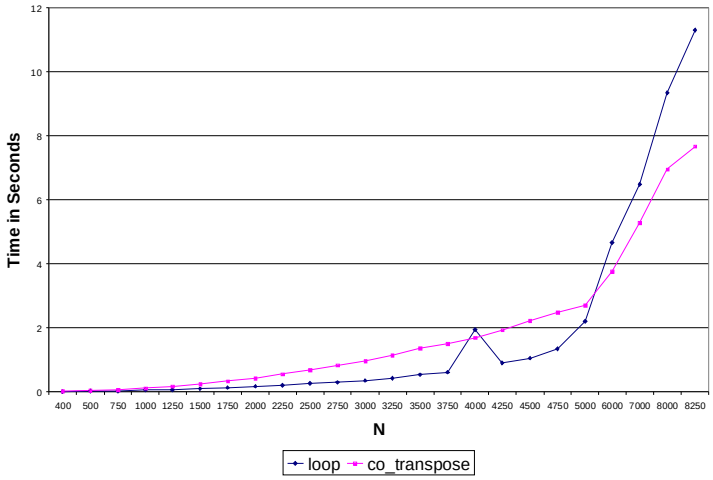
```
for (i = 0; i < N; i++)
    for (j = i+1; j < N; j++)
        swap(A[i][j], A[j][i])
```

Code : Co Transpose

```
void transpose(int x, int delx, int y, int dely,
               ElementType I[N][P], ElementType O[P][N]){
        // Base Case of recursion
        // Should be tailored for specific machines if one wants
        // this code to perform better.
        if((delx == 1) && (dely == 1)) {
                O[y][x] = I[x][y];
                return;
        }
        // Divide the transposition into two sub transposition
        // problems, depending upon which side of the matrix is
        // bigger.
        if(delx >= dely){
                int xmid = delx / 2;
                transpose(x,xmid,y,dely,I,O);
                transpose(x+xmid,delx-xmid,y,dely,I,O);
                return;
        }
        // Similarly cut from ymid into two subproblems
        ...
}
```

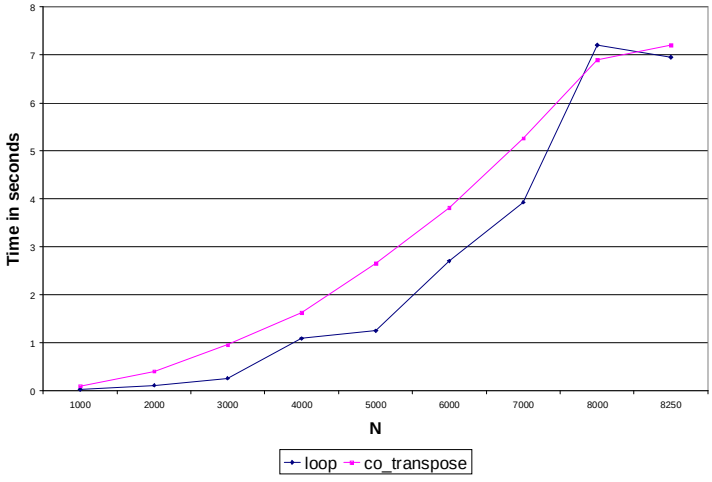
Experiments: MT

- Notebook, Windows 2k/512Mb/PIII 1GHz/g++ -O3



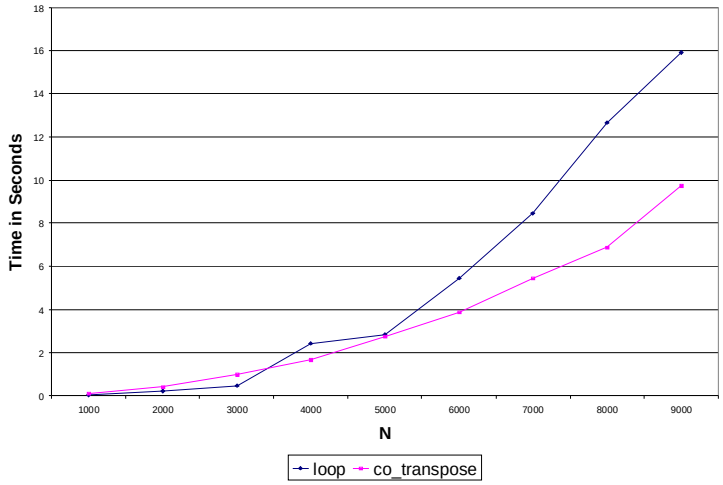
Experiments: MT

- Notebook, Windows 2k/512Mb/PIII 1GHz/g++ -O3



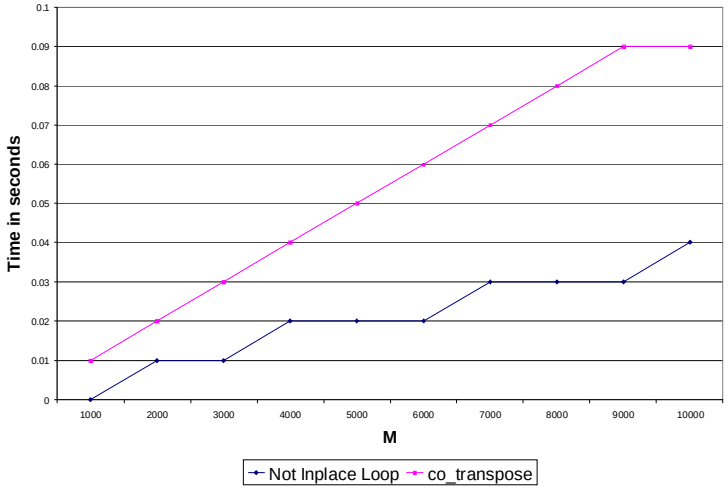
Experiments: MT

- Linux Athlon 1Ghz/1Gb/g++ -O3



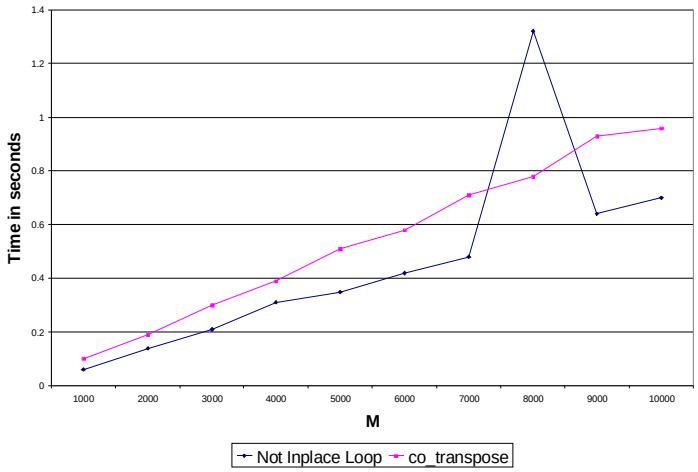
Experiments: MT

- Linux Athlon 1Ghz/1Gb/g++ -O3/ Size =N x (P =100) , tall matrices

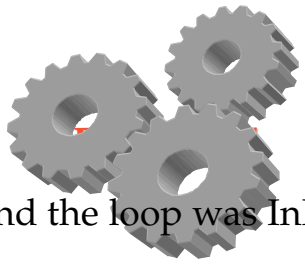


Experiments: MT

- Linux Athlon 1Ghz/1Gb/g++ -O3/ Size = N x (P =1000)



What went Wrong?

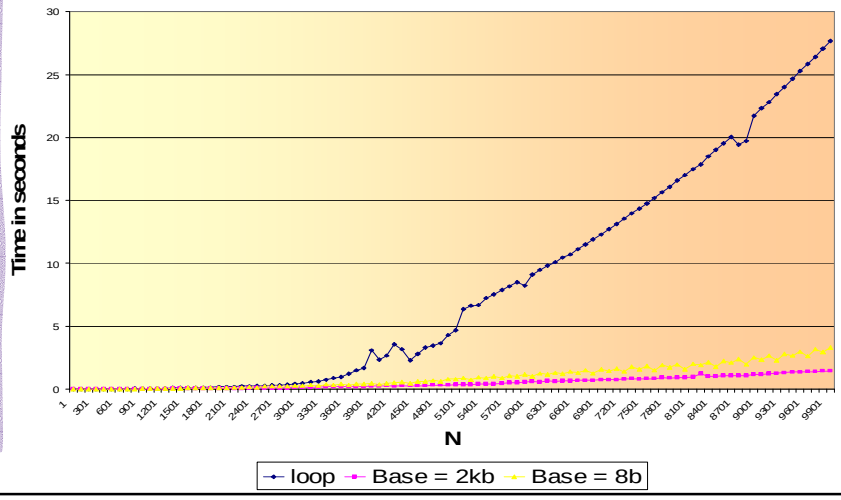


And the loop was InPlace!

Experiments: MT

Loop not Inplace

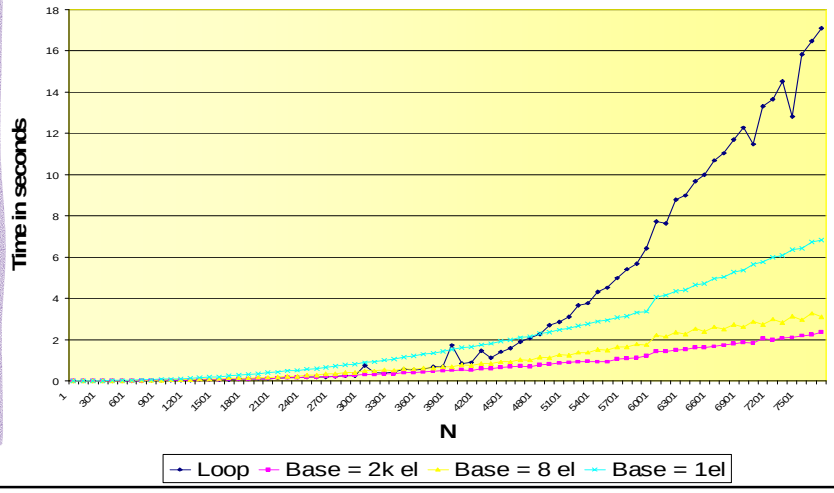
- Linux Athlon 1Ghz/1Gb/g++ -O3/ Size = N x N



Experiments: MT

Loop not Inplace

- Notebook, Windows 2k/512Mb/PIII 1GHz/g++ -O3



Did we miss something?

- Alg 1: Naïve Algorithm
- Alg 2: Simple blocking using fixed B
- Alg 3: Half Copy
- Alg 4: Full Copy
- Alg 5: CO
- Alg 6: Morton Ordering

Chatterjee & Sen HPCA 00

Did we miss something?

Running time (seconds), block size = 2^5						
$\log_2 N$	Alg.1	Alg. 2	Alg. 3	Alg. 4	Alg. 5	Alg. 6
10	0.21	0.10	0.06	0.06	0.08	0.03
11	0.86	0.49	0.39	0.35	0.45	0.14
12	3.37	1.63	1.05	1.14	2.16	0.54
13	13.56	6.38	4.55	4.99	6.69	2.13

Running time (seconds), block size = 2^6						
$\log_2 N$	Alg.1	Alg. 2	Alg. 3	Alg. 4	Alg. 5	Alg. 6
10	0.13	0.08	0.06	0.05	0.08	0.03
11	0.85	0.42	0.34	0.28	0.45	0.13
12	3.38	1.58	0.89	0.97	1.97	0.52
13	13.51	5.99	3.58	3.91	7.00	2.09

Running time (seconds), block size = 2^7						
$\log_2 N$	Alg.1	Alg. 2	Alg. 3	Alg. 4	Alg. 5	Alg. 6
10	0.14	0.12	0.05	0.05	0.09	0.03
11	0.87	0.42	0.36	0.24	0.47	0.20
12	3.36	1.46	0.85	0.88	2.03	0.59
13	13.46	5.74	3.12	3.35	6.86	2.35

Static Searches

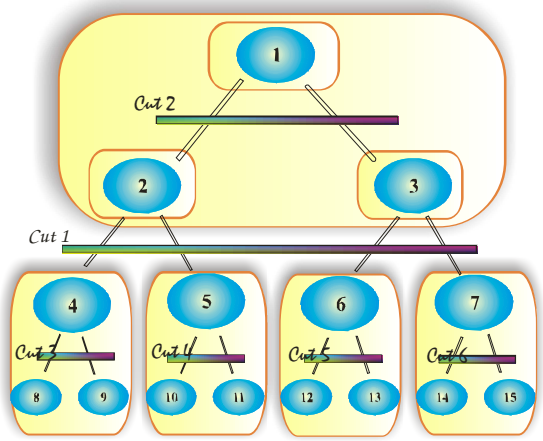
- Only for balanced binary trees
- Assume there are no insertions and deletions
- Only searches

Can we speed it up?

What is a layout?

- Mapping of nodes of a tree to the Memory
- Different kinds of layouts
  - In-order
  - Post-order
  - Pre-order
  - Van Emde Boas
- *Main Idea* : Store Recursive subtrees in contiguous memory

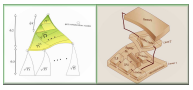
Example of Van Emde Boas

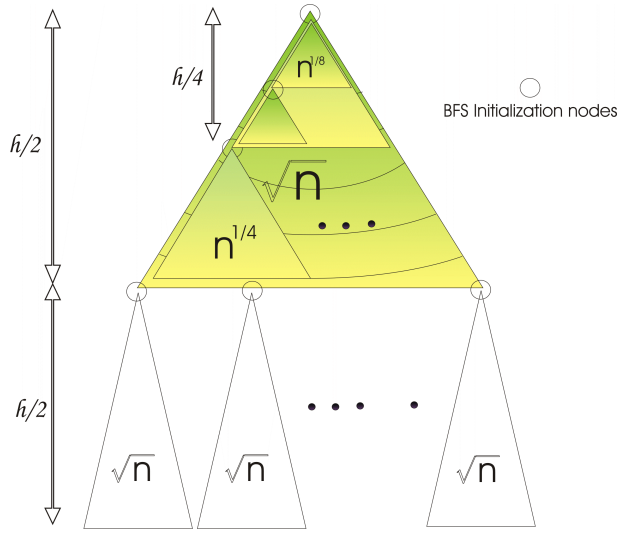


ACTUAL LAYOUT OF TREE IN MEMORY:

1, 2, 3, 4, 8, 9, 5, 10, 11, 6, 12, 13, 7, 14, 15

Another View





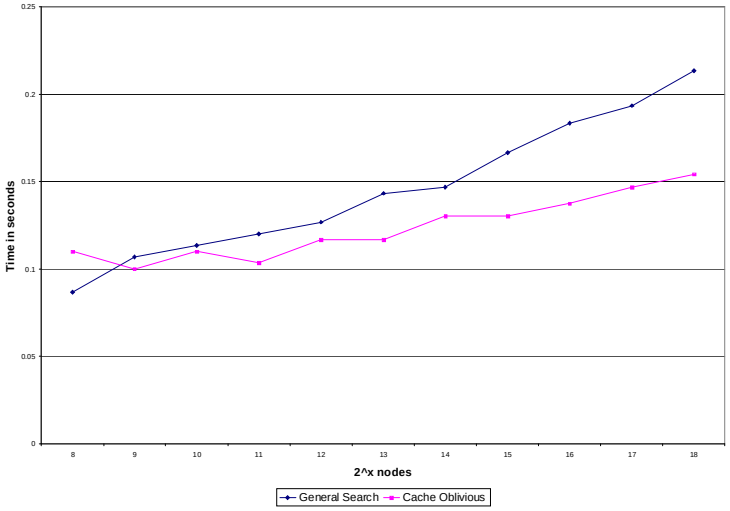
Theoretical Guarantees?

- Cache Complexity Q(n) = $O(\log_L n)$
- Work Complexity W(n) = $O(\log\ n)$

From Prokop's Thesis

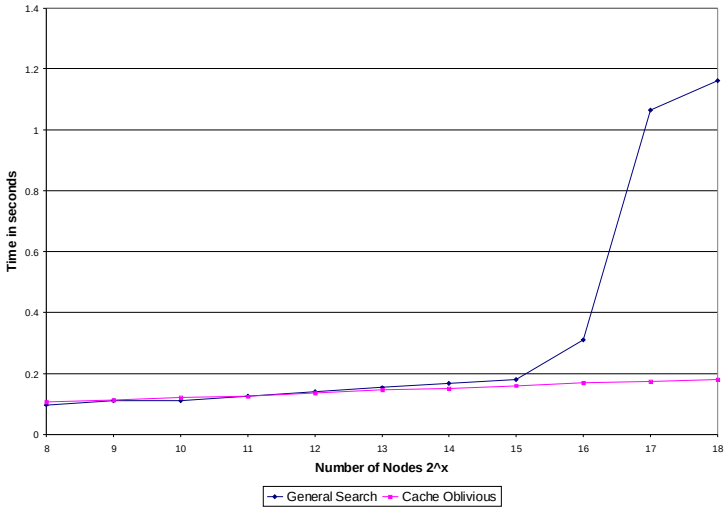
In Practice??

- Windows notebook/512MB/PIII 1Gz/256 byte nodes



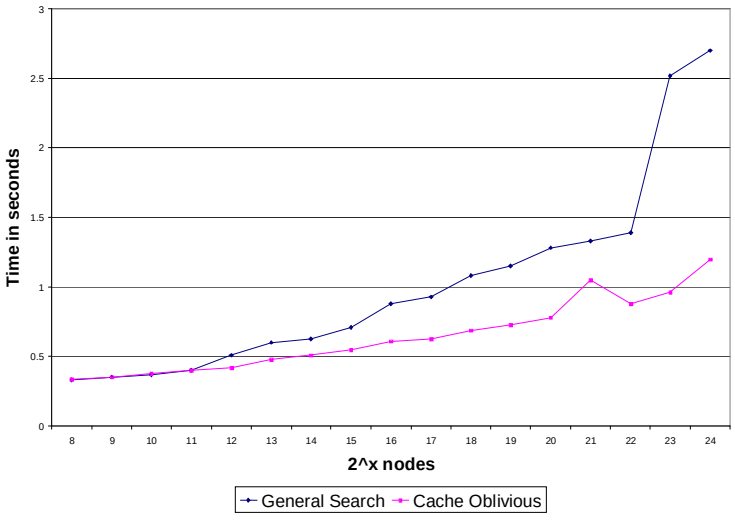
In Practice II

- Windows notebook/512MB/PIII 1Gz/32 byte nodes



In Practice III

- Linux/Itanium/2GB/g++ -O3/ 48 byte nodes



In Practice!

- Matrix Operations by Morton Ordering By David S. Wise

  (Cache oblivious Practical Matrix operation results)
- Bender, Duan, Wu

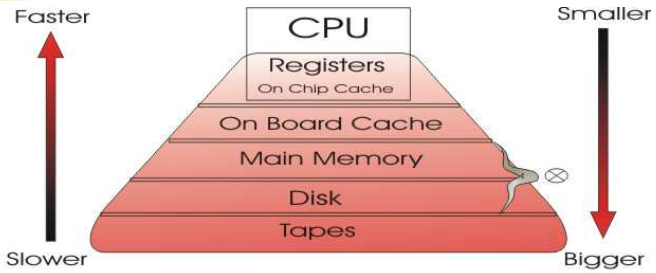
  (Cache oblivious dictionaries)
- Rahman, Cole, Raman

  (CO B-Trees)

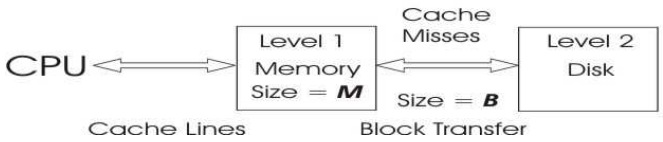
Talk outline…

- Motivation (Searching BBT)
- *The Model*
- CO-Sorting
- …

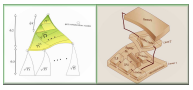
(M,B) Ideal Cache Model



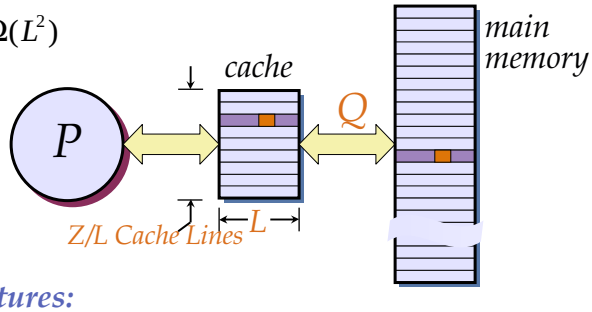
⊗ Two consecutive levels of the hierarchy



(Z,L) Ideal Cache Model



$Z = \Omega(L^2)$



Features:

- Two-level hierarchy.
- Cache of size Z.
- Cache-line length L.
- Fully associative.
- Optimal, omniscient replacement.

Measures:

- *Work W.*
- *Cache misses Q.*

Assumptions?

- Two Levels of Memory
- Tall Cache Assumption
- Optimal Cache Replacement "No Asymptotic loss"
- Fully-associative LRU can be used instead of optimal replacement with no asymptotic loss of performance [ST85].
- Fully-associative LRU caches can be maintained in ordinary memory with constant slowdown in expected performance.

Cache Obliviousness

- Cache-oblivious algorithms naturally tune for
  - varying cache sizes.
  - multiple levels of cache.

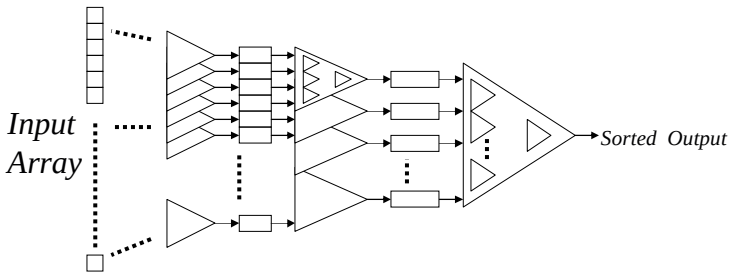
    When a subproblem fits into a given level of cache, no further cache misses are incurred beyond those required to bring the subproblem itself into the cache.
- An optimal cache-oblivious algorithm can be made to run optimally in the HMM [AACN87] and SUMH [VN93] models

CO-Sorting!

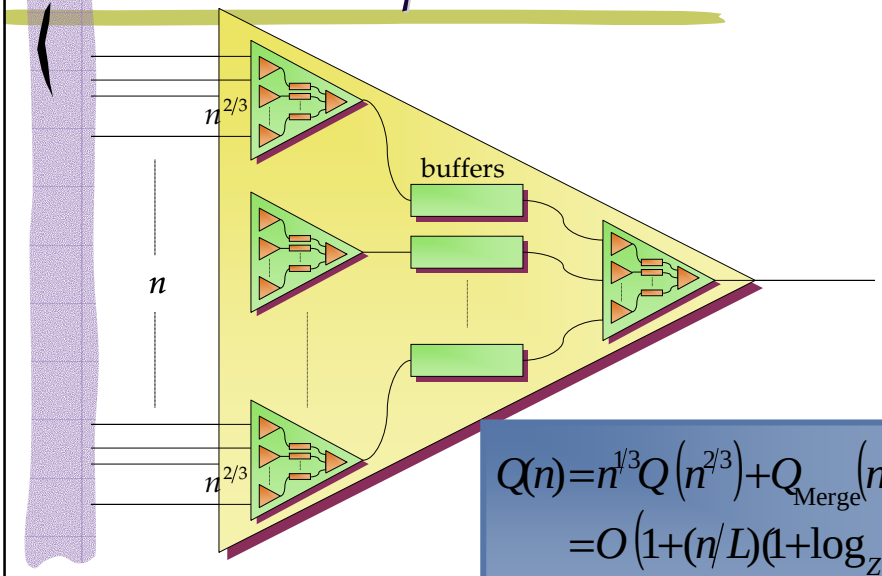
- Only two methods known
  - **Funnel Sort** (Modified Merge Sort)
  - **Distribution Sort** ⑥
    (Modified Sample Sort, We implement a randomized version)
  - **Column Sort**

Funnel Sort

- Partition Input into $N^{1/3}$ pieces of size $N^{2/3}$ each.
- Sort each piece Recursively
- Merge sorted pieces using a $N^{1/3}$ -merger



Harold's slide picture

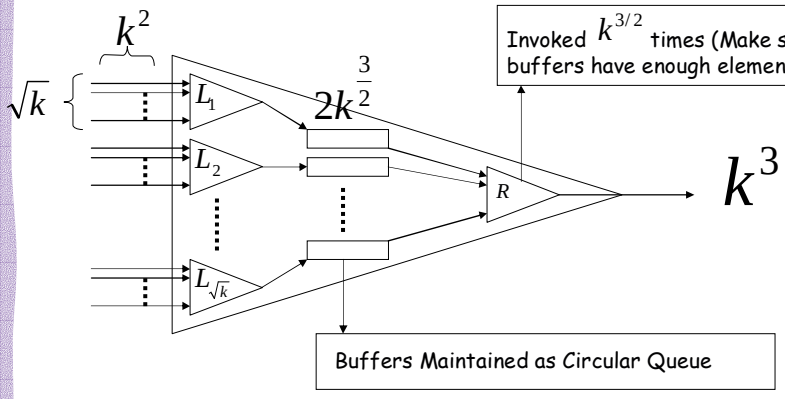


$$Q(n) = n^{1/3} Q\left(n^{2/3}\right) + Q_{\text{Merge}}\left(n^{1/3}\right)$$
$$= O\left(1 + (n/L)(1 + \log_Z n)\right)$$

Funnel Sort: k-mergers

- Takes input k sorted sequences
- Outputs k^3 elements!
- It's a clever scheduling of mergers!
- Keeps work complexity O(nlogn)

Funnel Sort: k-Merger



One invocation of R outputs $k^{3/2}$ elements

Funnel Sort : Optimality

- Work Complexity
  - $O(n \log n)$
- Cache Complexity
  - $O(1+\frac{n}{L}(1+\log_Z n))$

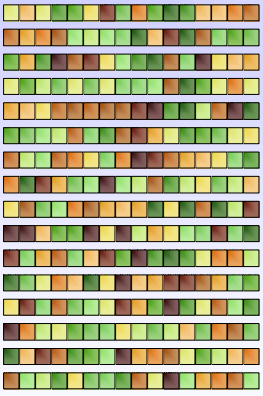
Agarwal and Vitter show that there is an $\Omega(\frac{n}{L}\log_{\frac{Z}{L}}(\frac{n}{L}))$ Bound on the number of cache misses.

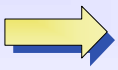
Distribution Sort

- Partition A into $\sqrt{n}$ sub-arrays each of size $\sqrt{n}$; Sort Recursively
- Distribute into buckets
- Sort Buckets Recursively
- Copy Buckets to output

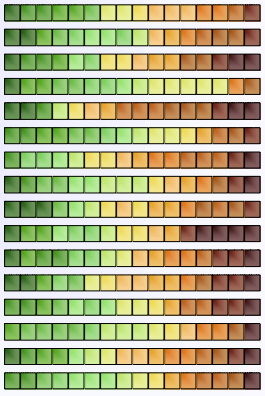
Recursive Sorting of Subarrays

n input elements, partitioned into $\sqrt{n}$ contiguous subarrays of size $\sqrt{n}$:



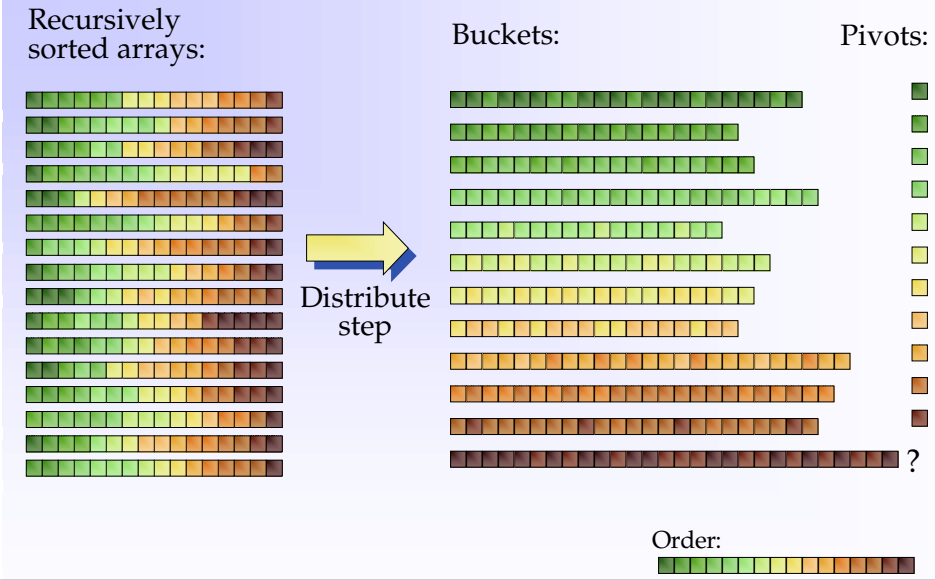


Recursively sorted arrays:



Order:

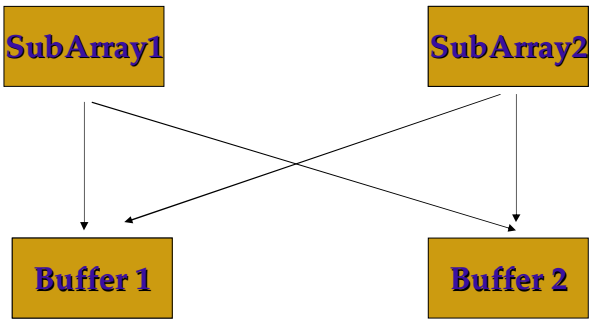
Distribution Step



The Distribution Step

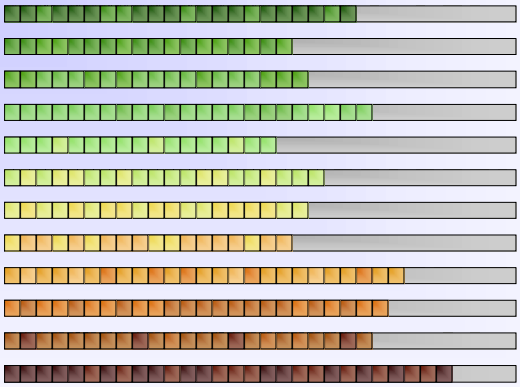
- Has to distribute subarrays into buckets $B_1, B_2, B_3 \ldots B_q$
- Not In-Place
- Similar to recursive Sample-Sort without doing Binary Search on pivots

The Recursive Bucketing used



Recursive Sorting of Buckets

After distribution step:



Recursively sort each bucket.

Some Analysis now

- Matrix Transposition
- Matrix Multiplication(Strassen)
- Column Sort ⑥

Talk outline…

- Motivation (Searching BBT)
- The Model
- CO-Sorting
  - ④ Funnel Sort
  - ④ Distribution Sort
- Some Analysis
- CO-Sorting Experiments

Randomized CO Sorting

Algorithm 9.7.1 procedure `Sort`(A)

Require: An input array A of Size N

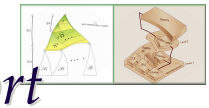
1: Partition A into $\sqrt{N}$ sub arrays of size $\sqrt{N}$. Recursively sort each subarray.
2: $R \leftarrow$ `ComputeSplitters`$(A, \sqrt{N})$
3: Sort(R) recursively. $R = \{r_0, r_1, ..., r_{\sqrt{N}}\}$.
4: Calculate counts c_i such that $c_i = |\{x \mid x \in A \text{ and } r_i \leq x < r_{i+1}\}|$
5: $c_{1+\sqrt{N}} = |A| - \Sigma_i c_i$.
6: Distribute A into buckets $B_0, B_1, ...B_{1+\sqrt{N}}$ where last element of each bucket is r_i except the last bucket. For the last bucket, the last element is maximum element of A. Note that $|B_i| = c_i$.
7: Recursively sort each B_i
8: Copy sorted buckets back to A.

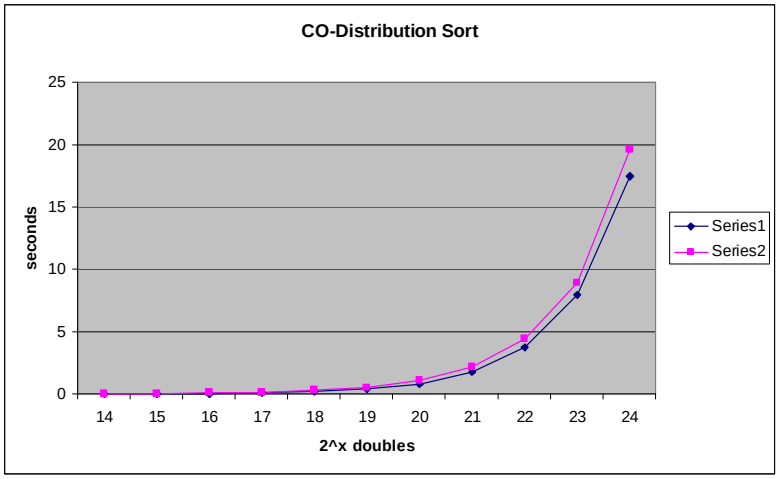
CO-Sorting Experiments

- Results for a 2-Pass Distribution Sort
- Does not use Bucket Splitting
- Uses a CO-Counting Phase
- Uses random sampling for splitters

  (That makes the implementation slightly suboptimal)

In Practice: 2 Pass CO-Dissort





Base = 2^14 = 16384

Does this imply anything?*

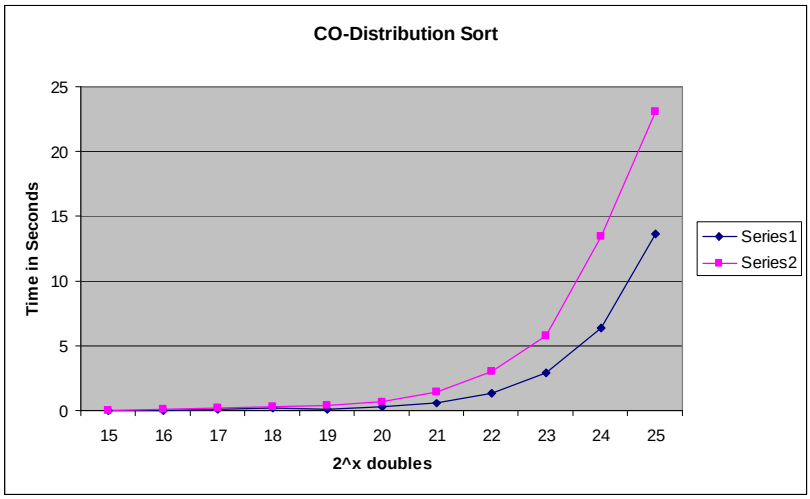
If you have 1Gb of Memory ⌘

You can sort, I/O efficiently,

1000 Gigabytes

just using one level of Recursion.

Two Levels of Recursion

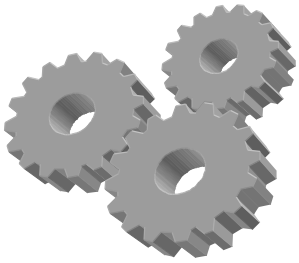


Base = 2^10 = 1024

Talk outline…

- **Motivation**
  - ④ Matrix Multiplication/Transposition
  - ④ Static Searches in Bal. Bin. Trees
- **The Model**
- **CO-Sorting**
- **Some Analysis**
- CO-Sorting **Experiments**
- *Is the model an oversimplification ?*

What went Wrong?



What went Wrong?

Assumptions!

Is the model oversimplified?

In Practice

- Associativity (Not fully associative)
- Complicated Algorithms
  (Asymptotics hides disasters!)
- Unified Caches (Instruction Caches)
- TLB (not tall)
- Concurrency (Coherence misses: Xeon)
- Replacement Policy (4Gb Limit)
- Multiple disks (Can increse I/O speed)
- Write Through Caches
  (Causes misses even if problem fits into cache)

Elaborated Here

What did I learn from it?

DO's
- Scans
- D&C, Rec
- Blocking
- Inline ju
- Local acc
- Experime differen algorithn

Don't's
- Non-Loc
- Believe (Optima
- Ignore platforr
- Belive m
- Believe results.

Sometimes
- Sub-optimal can be better than optimal
- Cache Aware might be the way to go.
- more than one CO optimal algorithm is available and only one is good among them.

Known Optimal Results

- Matrix Multiplication
- Matrix Transpose
- n-point FFT
- LUP Decomposition
- Sorting
- Searching

Results Known

Priority Q (Ins/Del/Del Min)	$O(\frac{1}{B}\log_{\frac{M}{B}}\frac{N}{B})$
List Ranking	$O(sort(V))$
Tree Algos	$O(sort(V))$
Directed BFS/DFS	$O((V+\frac{E}{B})\log_2 V + sort(E))$
Undirected BFS	$O(V + sort(E))$
MSF	$O(sort(E)+\log_2\log_2 V)$

Results Known

Array Reversal	$O(scan(V))$
LU Decomposition	$\Theta(1 + \frac{N^2}{B} + \frac{N^3}{B\sqrt{M}})$
FFT	$O(sort(V))$
B-Trees (Insertions/Del)*	$O(\log_B n)$
Tree Layout	$OPT\ O(\log(B))$
Convex Hulls in 3D	$O(sort(N))$

New Result

Cache oblivious (With Edgar Ramos,UIUC)
Voronoi Diagrams in 2D/3D.

Applications:
- Delaunay Triangulations
- Curve/Surface Reconstruction
- Meshing

Publications

- Book : Algorithms for Memory Hierarchies
  Chapter: Cache Oblivious Algorithms
  (Editors Meyer et.al.)
- *Cache Oblivious Voronoi diagrams*
  (with Edgar Ramos, In Progress)
- *Minimum Enclosing Balls*
  (with Joe Mitchell, Alper Yildirim)
  (in Alenex 03) < Also cache oblivious >

Other Introductions

- Chapter by
  Eric Demaine

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Erez Zadok

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