Let $\sigma = \max |\sigma_i|$, then

$$|\sigma'| < \frac{(2^n - 1)\sigma[(Y + 1)^n - 1]}{(1 - Y)^n}.$$

COROLLARY. *The error $e(t, \sigma)$ of integrating a function with a maximum value of σ is bounded by*

$$(31) \qquad |e(t, \sigma)| \le \sigma h\left[2^n + \frac{(Y + 1)^n - 1}{(1 - Y)^n}(2^n - 1)\right].$$

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[Editorial Note: A second part of this paper, to appear in the next issue of *MTAC*, will contain illustrative numerical examples of the application of the above ideas.]

[1] W. E. MILNE, "The remainder in linear methods of approximation," NBS, *Jn. Research*, v. 43, 1949, p. 501–511. This gives a more general approach to step errors of integration formulas based upon approximation by sets of functions.

[2] R. E. GREENWOOD, "Numerical integration of linear sums of exponential functions," *Ann. Math. Stat.*, v. 20, 1949, p. 608–611.

[3] P. BROCK & F. J. MURRAY, "Planning and error analysis for the numerical solution of a test system of differential equations on the IBM sequence calculator," *Cyclone Report*, Reeves Instrument Corp., New York 28. See also F. J. MURRAY, "Planning and error considerations for the numerical solution of a system of differential equations on a sequence calculator," *MTAC*, v. 4, p. 133–144.

[4] F. J. MURRAY, "Linear equation solvers," *Quart. Appl. Math.*, v. 7, 1948, p. 263–274.

[5] L. H. THOMAS of the Watson Scientific Computing Laboratory indicated this formula for A_n to the authors. He also indicated that the A_n are equal in absolute value to the coefficients of the Adams-Bashforth method of step by step numerical integration.

[6] W. FELLER, *Probability Theory*. New York, 1950, v. 1, p. 52.

[7] G. BIRKHOFF & S. MACLANE, *A Survey of Modern Algebra*. New York, 1948, p. 424.

A Note on the Inversion of Matrices by Random Walks

In a recent note, FORSYTHE & LEIBLER[1] described a method (first suggested by J. v. NEUMANN and S. M. ULAM) for the inversion of certain types of matrices by a "Monte Carlo" sampling procedure. The authors explain their scheme in terms of drawing balls from an urn, but the procedure might, of course, be just as well described as a random walk.

A boundary value problem involving a difference equation in a bounded domain is equivalent to a system of linear algebraic equations in as many unknowns as there are lattice points in the domain. It is therefore to be expected that the sampling methods for the solution of such difference equations as explained in CURTISS[2] and WASOW[3] are closely related to the method of Forsythe & Leibler.[1]

In order to study this relation we rephrase the latter method in the language of random walks. We consider a set of m points $P_1, \cdots, P_m$ and introduce a moving particle which, starting from P_i, jumps from point to point in such a way that the probability of going from P_ν to P_μ in one jump is $p_{\nu\mu}$. Also at each point P_ν there is a probability $p_\nu = 1 - \sum_{\mu=1}^{m} p_{\nu\mu}$ of the random walk ending there.

Furthermore, the moving particle possesses a variable "mass" V which, at a step from, say, P_ν to P_μ is multiplied by a factor $v_{\nu\mu}$. The initial mass at P_i is one. Our procedure consists in estimating the expected value of the random variable G_{ij} defined as follows. The moving point is known to start from P_i:

$$G_{ij} = \begin{cases} 0, \text{ if the walk ends at } k \neq j \\ Vp_j^{-1}, \text{ if the walk ends at } j \end{cases}$$

Observe that G_{ij} is defined only for points where $p_j \neq 0$.

Let A be the matrix with elements

$$a_{ij} = p_{ij}v_{ij}$$

(no summation is implied), denote by b_{ij} the elements of the matrix

$$B = I - A$$

and by β_{ij} the elements of B^{-1}. Then the following theorem is proved in Forsythe & Leibler.[1]

Theorem: If and only if the eigenvalues of the matrix $\{|a_{ij}|\}$ *are less than one in absolute value, then the mathematical expectation* $E[G_{ij}]$ *exists and*

$$E[G_{ij}] = \beta_{ij}.$$

Thus, an experimental estimate of the expectation of G_{ij} yields a numerical value for one element of the inverse matrix of B.

The procedures followed in Curtiss[2] and Wasow[3] to find Green's function for difference equations, when properly worded, are special cases of a scheme for matrix inversion which differs from the random walk just described only in that the random variable G_{ij} is replaced by the random variable M_{ij} which is by definition equal to *the total amount of mass carried through the point P_j on the several visits in the course of a random walk starting from P_i*. (If the point stays put at P_j, this is to be counted as a new visit.) It is very easy to show directly that

$$E[G_{ij}] = E[M_{ij}], \text{ when } p_j \neq 0.$$

For let V be the mass of the particle when it passes through P_j at the end of a path from P_i to P_j whose probability of being taken is q, then

$$E[G_{ij}] = p_j \sum qVp_j^{-1} = \sum qV = E[M_{ij}],$$

the summation being extended over all possible paths connecting P_i and P_j. Observe that the word "path" is used here in a somewhat generalized sense, referring not to a geometric configuration but to an ordered sequence of points P_k beginning with P_i and ending with P_j.

If $p_j = 0$, then G_{ij} is not defined. But $E[M_{ij}]$ exists and is equal to β_{ij}, as can be proved exactly as in Forsythe & Leibler.[1] This is one advantage of using the random variable M_{ij} instead of G_{ij}. Apart from this remark it is not easy to decide in advance which of the two methods is preferable in a given problem, since this requires a comparison of the variances. The decision is made more complicated by the fact that the factorization of the given number a_{ij} into the product $p_{ij}v_{ij}$ is, to a large extent, arbitrary. If the original problem is a boundary value problem for a difference equation, M_{ij} is, to say the least, the intuitively more natural random variable, since one would like to associate the end of a random walk with the first crossing of the boundary of the given domain. With this interpretation, p_j is zero for all points from which the boundary cannot be reached in one step, and the random variable G_{ij} is unsuitable.

In the light of the present discussion some of the proofs in Wasow[3] can be modified—but not substantially shortened—by making use of the criterion for existence of $E[G_{ij}]$, and hence of $E[M_{ij}]$, which is stated in the theorem of this section.

In the especially simple case that all v_{ij} are equal to one, some additional information concerning the respective advantages of using the random variables M_{ij} or G_{ij} is contained in the following theorem, valid in this special case. *Let ν_j be the probability that a particle known to start from P_j will never return to P_j. Then*

$$\sigma[M_{ij}] \leq \sigma[G_{ij}] \tag{1}$$

if and only if

$$p_j \leq \frac{\nu_j}{2 - \nu_j}.$$

In order to prove this inequality, we denote by λ_{ij} the probability of going from P_i to P_j without passing through P_j on the way; i.e., λ_{ij} is the total probability associated with all paths connecting P_i and P_j, all intermediate points being different from P_j. In our special case the random variable M_{ij} is the number N of visits at P_j during a random walk starting at P_i. Then, for $k \geq 1$, $\Pr\{N = k\} = \lambda_{ij}\lambda_{jj}^{k-1}\nu_j$. A short calculation yields

$$E[M_{ij}^2] = E[N^2] = \lambda_{ij}\nu_j \sum_{k=1}^{\infty} k^2 \lambda_{jj}^{k-1} = \lambda_{ij}\nu_j \frac{1 + \lambda_{jj}}{(1 - \lambda_{jj})^3}. \tag{2}$$

But since the total probability of the walk ending eventually is 1 we have

$$\nu_j + \lambda_{jj}\nu_j + \lambda_{jj}^2\nu_j + \cdots = 1,$$

i.e.,

$$\nu_j = 1 - \lambda_{jj}, \tag{3}$$

so that (2) simplifies into

$$E[M_{ij}^2] = \lambda_{ij} \frac{1 + \lambda_{jj}}{(1 - \lambda_{jj})^2}. \tag{4}$$

Next, let X be the random variable which is 1 if the last point of the random walk is P_j, and zero otherwise. Then

$$\Pr\{X = 1\} = \lambda_{ij}p_j + \lambda_{ij}\lambda_{jj}p_j + \lambda_{ij}\lambda_{jj}^2 p_j + \cdots = \lambda_{ij}p_j/(1 - \lambda_{jj}).$$

Hence

$$E[X^2] = \Pr\{X^2 = 1\} = \Pr\{X = 1\} = \lambda_{ij}p_j/(1 - \lambda_{jj}).$$

Since, in our special case,

$$G_{ij} = X/p_j,$$

it follows that

$$E[G_{ij}^2] = \frac{\lambda_{ij}}{p_j(1 - \lambda_{jj})}. \tag{5}$$

Now, as the means of M_{ij} and G_{ij} are the same, the inequality (1) is equivalent to $E[M_{ij}^2] \leq E[G_{ij}^2]$, i.e., by formulas (4) and (5) to

$$\frac{1 + \lambda_{jj}}{1 - \lambda_{jj}} \leq \frac{1}{p_j}.$$

Application of formula (3) transforms this into $p_j \leq \nu_j/(2 - \nu_j)$, which proves our statement.

The practical value of this theorem is limited, because ν_j is, in general, not known. But it shows that, at least in this special case, the answer to the question which method is preferable for the calculation of the element β_{ij} does not depend on the subscript i. It also confirms the intuitively plausible conjecture that M_{ij} is the better random variable to use whenever p_j is comparatively small.

W. R. WASOW

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[1] G. E. FORSYTHE & R. A. LEIBLER, "Matrix inversion by a Monte Carlo method," *MTAC*, v. 4, 1950, p. 127–129.

[2] J. H. CURTISS, "Sampling methods applied to differential and difference equations," Seminar on Scientific Computation, *Proc.*, Nov. 1949, p. 87–109. IBM, New York.

[3] W. WASOW, "Random walks and the eigenvalues of elliptic difference equations," NBS, *Jn. Research*, v. 46, 1951, p. 65–73.

RECENT MATHEMATICAL TABLES

961[B, E].—R. C. SPENCER & G. E. REYNOLDS, *A Table of Normalized Parabolic Coordinates and Arc Lengths.* Air Force Cambridge Research Laboratories, publication E 4083, Cambridge 1951, 9 p., 21.9 × 27.3 cm.

The functions

$$y = \tfrac{1}{2}x^2, \quad s = \tfrac{1}{2}(\operatorname{arc\,sinh} x + x(1 + x^2)^{\frac{1}{2}})$$

are given to 5D for $x = 0(.01)2$. These give the ordinates and arc distances of points on the "normalized" parabola $y = \frac{1}{2}x^2$. From these tables two graphs are derived showing s as a function of x and s, x as functions of y. From the latter graph the excess of arc length over abscissa may be read off to facilitate the laying out of a parabolic antenna.

D. H. L.

962[F].—J. P. KULIK, L. POLETTI & R. J. PORTER, *Liste des Nombres Premiers du Onzième Million* (*plus précisément de* 10006741 *à* 10999997). [Amsterdam 1951, published by N. G. W. H. Beeger, Nicolaas Witsenkade 10, Amsterdam]. ii + 25 p., 20.3 × 27.6 cm. Price 3 Dutch florins.

This table is the result of collating three of the four existing manuscript factor tables of the eleventh million.[1] The arrangement is essentially that of LEHMER'S[2] list of primes of which this table is a natural extension. Thus the rank of a prime occupying page P, column C, and line L is given by

$$2500\text{P} + 100\text{C} + \text{L} + 662400.$$

The number of primes in this million is 61938.

Much of the credit for the successful completion of this table goes to BEEGER and GLODEN who were responsible for collating and reconciling the