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SOME CONTINUOUS MONTE CARLO METHODS FOR THE DIRICHLET PROBLEM[1]

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0. Summary. Monte Carlo techniques are introduced, using stochastic models which are Markov processes. This material includes the N-dimensional Spherical, General Spherical, and General Dirichlet Domain processes. These processes are proved to converge with probability 1, and thus to yield direct statistical estimates of the solution to the N-dimensional Dirichlet problem. The results are obtained without requiring any further restrictions on the boundary or the function defined on the boundary, in addition to those required for the existence and uniqueness of the solution to the Dirichlet problem. A detailed study is made for the N-dimensional Spherical process; this includes a study of the order of the average number of steps required for convergence. Asymptotic confidence intervals are obtained. When computing effort is measured in terms of the order of the average number of steps required for convergence, the often-made conjecture that the computing effort of a Monte Carlo procedure should be a linear function of the dimensionality of the problem is shown to be true for the cases considered. Comments are included regarding the application of these processes on digital computers, and truncation methods are suggested.

1. Introduction. Throughout this paper, D will denote a bounded finitely connected N-dimensional domain in a Euclidean space. Further, $\Gamma[D]$ will denote the boundary points of D. A point in space will be denoted by x, where x has coordinates $(x_1, x_2, \cdots, x_N)$.

The N-dimensional Dirichlet problem. Given the domain D and a continuous function $f(x)$ defined on the boundary $\Gamma[D]$, the N-dimensional Dirichlet problem consists in finding a function $u(x)$ continuous in $D + \Gamma[D]$, reducing to $f(x)$ on $\Gamma[D]$, and having in D continuous partial derivatives of second order which satisfy Laplace's differential equation, i.e., in finding $u(x)$ such that

$$\Delta^2 u(x) = \sum_{i=1}^{N} \frac{\partial^2 u(x)}{\partial x_i^2} = 0, \qquad x \varepsilon D, \tag{1.1}$$

$$u(x) = f(x), \qquad x \varepsilon \Gamma[D]. \tag{1.2}$$

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Some writers have generalized the Dirichlet problem to allow the following: In place of (1.1), the general linear elliptic operator

$$L(u) = \sum_{i,j=1}^{N} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{j=1}^{N} b_j(x)$$

is introduced, where $a_{ij}(x)$, $b_j(x)$ possess continuous second-order derivatives in D; while in place of (1.2), they have $u(x) = g(x)$, $x \varepsilon \Gamma[D]$, where $g(x)$ is permitted to have points of discontinuity on $\Gamma[D]$. These generalizations of the Dirichlet problem will not be considered in this paper, even though some of the material developed is applicable to the more general problem.

In essence, the Monte Carlo method is experimental. The tempting title "Monte Carlo" is being used here as it has been by others, e.g., by Metropolis and Ulam [17], in order to convey that an unknown solution to a given physical problem is being estimated by a method which essentially depends on a statistical sampling technique. This approach requires the utilization of random variables of an appropriate stochastic process such that samples of the process yield valid statistical estimates of the desired unkown quantities.

The previous Monte Carlo studies on the Dirichlet problem (for a review of the literature see Curtiss [2]) have given estimators for the solution to discrete replacements of the Dirichlet problem, since they initially replaced the given domain D by a network of points and replaced the differential operator by a difference operator. Further, the previous studies needed to impose assumptions in addition to those made here concerning the regularity of the boundary of D. It will be seen that the processes considered here yield direct statistical estimates of the solution to the Dirichlet problem. Consequently, the only inherent possible source of error with these processes results from the statistical fluctuation of the estimators. Naturally, if these methods are utilized on a digital computer there will be an additional "round-off" error due to the replacement of a continuous variable by a variable which possesses only a discrete number of digits.

Since the one-dimensional Dirichlet problem reduces to finding a straight line through two points, it will be completely omitted from subsequent consideration in this paper.

We next consider the question of the nature of $\Gamma[D]$. It has already been demonstrated that the theory of probability, as such, can be used as a rigorous mathematical tool in the study of differential and integral equations (see, for example, Feller [7] and Kac [9]). However, the contribution that Monte Carlo studies might make to boundary value problems, when the analytical questions of existence and uniqueness of a solution have not been settled, is a moot question. Consequently, we make the following assumption:

Fundamental Assumption. Throughout this paper, the boundary $\Gamma[D]$ will be assumed to be of sufficient regularity to ensure that the Dirichlet problem has a solution and that it is unique. Kellogg [12] and [13] has a detailed discussion and bibliography concerning regularity conditions on $\Gamma[D]$; see also de La Vallée Poussin [22].

2. Definition of the Spherical process. First consider a preliminary definition.

DEFINITION 2.1. *Maximum N-sphere*: $K(x)$. Given a domain D with boundary $\Gamma[D]$, and any point x belonging to $D + \Gamma[D]$, then $K(x)$ is the maximum N-dimensional sphere with center x and radius r if $r = \inf_{x' \varepsilon \Gamma[D]} \|x' - x\|$; $\mathbf{K}(x)$ denotes the surface of $K(x)$, where $\mathbf{K}(x)$ is empty if $x \varepsilon \Gamma[D]$.

DEFINITION 2.2. *The Spherical process.* Given a domain D of N dimensions with boundary $\Gamma[D]$, and any point x belonging to $D + \Gamma[D]$, then the N-dimensional Spherical process originating from x is $\Phi(x)$, where

(A) $\Phi(x) = \{S(x, \phi), 0 \leqq \phi \leqq 1\}$; i.e., $\Phi(x)$ is the totality of all sequences of points $S(x, \phi)$, $0 \leqq \phi \leqq 1$, where

(B) Each value of ϕ specifies a sequence of points $S(x, \phi) = \{P_{i+1}(x, \phi), i = 0, 1, \cdots\}$ generated according to the following stipulations:

(1) About the point $P_0(x, \phi) = x$, determine the maximum N-sphere $K(P_0)$;

(2) Select the point $P_1(x, \phi)$ uniformly at random on $\mathbf{K}(P_0)$;

(3) The point $P_{i+1}(x, \phi)$ is determined recursively from $P_i(x, \phi)$ and $K(P_i)$ in the same manner as $P_1(x, \phi)$ was determined from $P_0(x, \phi)$.

In introducing the N-dimensional Spherical process, we have actually set up a probability space whose underlying points are sequences of directions picked uniformly at random and picked so as to be mutually independent. Probabilities are defined as follows. Corresponding to the jth direction, i.e., the direction from the point $P_{j-1}(x, \phi)$ to the point $P_j(x, \phi)$, there is a point Q_j on the surface $\mathbf{F}$ of the unit N-sphere F. If $\mathbf{F}_j$ is any Lebesgue measurable subset of $\mathbf{F}$ of measure $m(\mathbf{F}_j)$, then picking the jth direction *uniformly at random* is equivalent to setting $\Pr\{Q_j \varepsilon \mathbf{F}_j\} = m(\mathbf{F}_j)/m(\mathbf{F})$. Picking the directions to be *mutually independent* is equivalent to setting

$$\Pr\{Q_1 \varepsilon \mathbf{F}_1, Q_2 \varepsilon \mathbf{F}_2, \cdots, Q_s \varepsilon \mathbf{F}_s\} = \prod_{j=1}^{s} \Pr\{Q_j \varepsilon \mathbf{F}_j\}$$

for each s. It then follows, by the Extension Theorem due to Kolmogorov [14] (see page 29), that the probability distribution on the space of infinite sequences will be properly defined. Further, it will be seen that the Spherical process is a Markov process with a discrete parameter.

We next consider the Einstein-von Smoluchowski stochastic model of Brownian motion, since this model is of help in showing that the Spherical process furnishes a valid direct statistical estimate of the solution to the Dirichlet problem.

3. The Brownian motion process. Let $(\Omega, \mathcal{E}, \Pr)$ be a probability space—i.e., $\Omega = \{\omega\}$ is a set of elements ω, $\mathcal{E} = \{E\}$ is a Borel field of subsets E of Ω, and $\Pr(E)$ is a countably additive measure defined on $\mathcal{E}$ with normalization $\Pr(\Omega) = 1$.

Throughout this paper $X(t, \omega)$ will denote the well-known N-dimensional Brownian motion process starting from x, i.e.,

$$X(t, \omega) = \{(x^1(t, \omega), x^2(t, \omega), \cdots, x^N(t, \omega)) \mid 0 \leqq t < \infty, \omega \varepsilon \Omega\}$$

is the Cartesian product of N mutually independent one-dimensional Brownian motion processes satisfying the condition that

$$X(0, \omega) = x = (x^1(0, \omega), x^2(0, \omega), \cdots, x^N(0, \omega)).$$

For a detailed definition of this process, see, for example, Doob [4], page 97, or Dvoretzky, Erdös, and Kakutani [6].

It is assumed that the basic probability measure is completed in such a way that Theorem 3.1, to be introduced, is valid (see Doob [3] or [4]).

A *sample function* or *path* of the process $X(t, \omega)$ is a function of t, defined for $0 \leqq t < \infty$ and obtained by fixing ω. When speaking of "almost all sample functions," this is to be understood to mean "almost all ω."

We have defined the particular Brownian motion process $X(0, \omega) = x$, where x is a point in N space, so that in the sequel we can speak of a sample function originating from a point x which is of interest. We shall now consider some of the known and, in particular, new properties of the Brownian motion process that will be pertinent. It will be seen that all subsequent material of this section will rest on the following important theorem due to Wiener. A recent version is found in Paley and Wiener [19]. A proof may also be found in Doob [3] or [4] or in Lévy [15] or [16].

THEOREM [Wiener] 3.1. *Almost all sample functions of the Brownian motion process are continuous; i.e., the subset* Ω_0 *of* Ω, *consisting of all* ω *for which* $X(t, \omega)$ *is a continuous function of* t *for* $0 \leqq t < \infty$, *is* $\mathcal{E}$*-measurable and* $\Pr(\Omega_0) = 1$.

The first-passage time of the process is defined in the following manner. Given any point x belonging to a domain D with boundary $\Gamma[D]$ and any ω belonging to Ω, consider the sample function of the process $X(t, \omega)$ originating from x. If there exists a positive number $\tau = \tau(x, \Gamma[D], \omega)$ such that $X(\tau, \omega) \in \Gamma[D]$ and $X(t, \omega) \notin \Gamma[D]$ for any t with $0 \leqq t < \tau$, then $\tau(x, \Gamma[D], \omega)$ is called the *first-passage time* to the boundary $\Gamma[D]$ for the sample function originating at the point x.

If $\tau(x, \Gamma[D], \omega)$ exists, then $P(x, \Gamma[D], \omega)$ denotes the point at which the sample function $X(t, \omega)$, originating from x, intersects $\Gamma[D]$ for the first time after $t = 0$. $P(x, \Gamma[D], \omega)$ is called the *point of first intersection.*

Following Kakutani [10] and [11], let $\Omega(x, \Gamma[D])$ denote the set of $\omega \in \Omega_0$ such that $\tau(x, \Gamma[D], \omega)$ exists. His results imply that $\Omega(x, \Gamma[D])$ is a measurable subset of Ω and that $\tau(x, \Gamma[D], \omega)$ is a real-valued measurable function of ω on

$$\Omega(x, \Gamma[D]).$$

His material yields the following theorem, which in the form presented here can also be found in earlier studies by Bachelier, Lévy, and Wiener.

THEOREM [Kakutani] 3.2. *Given any point* x *belonging to a domain* D *with boundary* $\Gamma[D]$, *then almost all sample functions of the Brownian motion process on* D *originating at* x *intersect the boundary* $\Gamma[D]$, *i.e.,* $\Pr\{\Omega(x, \Gamma[D])\} = 1$.

The next two theorems are due to Kakutani [11]. The only conditions to be imposed on the domain D and its boundary $\Gamma[D]$ are that they be regular for the Direchlet problem (see the Introduction). A subset E of the boundary $\Gamma[D]$

will be called an *elementary set* if it consists of a finite number of mutually disjoint nonabutting simple surfaces on the boundary $\Gamma[D]$, including or excluding their closures, where a simple surface is a homeomorphism of the surface of an N-sphere.

THEOREM [Kakutani] 3.3. *Given a domain D and its boundary $\Gamma[D]$, and E an elementary set on the boundary $\Gamma[D]$, then the probability,* $\Pr(x, E, D)$, *that the Brownian motion process originating from a point x belonging to D will intersect the set E for some $t > 0$, without intersecting $\Gamma[D] - E$ before it, is a harmonic function of x in D, and* $\lim_{x \varepsilon D, x \to x_0} \Pr(x, E, D) = 1$ or 0 *according as x_0 is an inner point of E or of $\Gamma[D] - E$.*

THEOREM [Kakutani] 3.4. *With the same conditions as given in Theorem* 3.3, *let $f(x)$ be a real-valued continuous function defined on the boundary $\Gamma[D]$. Then, for any point x_0 belonging to D, the value $u(x_0)$ of the solution $u(x)$ of the Dirichlet problem for the domain D and the boundary value function $f(x)$ is obtained by taking the integral of a Poisson type of $f(x)$ with respect to the kernel* $\Pr(x_0, E, D)$ *on $\Gamma[D]$, or by taking the mathematical expectation of the composed function*

$$f(P(x_0, \Gamma[D], \omega)):$$

$$u(x_0) = \int_{\Gamma[D]} \Pr(x_0, dx, D) f(x) = \int_\Omega f(P(x_0, \Gamma[D], \omega))\, d\omega.$$

The details of the proofs of Kakutani's theorems, and certain generalizations, may be found in the recent paper by Doob [5].

The following definition of successive first intersections of the Brownian motion process will be useful.

DEFINITION 3.1. *Successive first intersections.* Given any point x belonging to a domain D with boundary $\Gamma[D]$ and any ω belonging to Ω, consider the sample function of the Brownian motion process $X(t, \omega)$ originating from x. If this sample function has a first intersection on the surface $\mathbf{K}(P_0 = x)$, denote this point as $P_1(x, \mathbf{K}(P_0), \omega)$. Successive points $P_{i+1}(x, \mathbf{K}(P_i), \omega)$, $i = 1, 2, \cdots$, of first intersection on successive surfaces $\mathbf{K}(P_i)$, if these points exist, will be defined recursively as was done for the Spherical process (Definition 2.2). A *sequence of successive first intersections* associated with a particular sample function exists if $P_{i+1}(x, \mathbf{K}(P_i), \omega)$ exists for each i, $i = 0, 1, \cdots$. Let $T(x, \omega)$ denote this sequence, i.e., $T(x, \omega) = \{P_{i+1}(x, \mathbf{K}(P_i), \omega), \quad i = 0, 1, 2, \cdots\}$.

Before making use of Definition 3.1, the following remarks are in order. (1) If x is the center of any N-dimensional sphere, say $S(x)$, then the probability distribution of points of first intersection, $\boldsymbol{F}(x, \mathbf{S}(x), \omega)$, on the surface $\mathbf{S}(x)$, for the sample functions of the Brownian motion process originating from x, is uniformly distributed on the surface $\mathbf{S}(x)$ for all $\omega \varepsilon \Omega(x, \mathbf{S}(x))$. (2) If the Brownian motion process has a first intersection, say for $t = \tau(\omega) = \tau(x, \mathbf{F}, \omega)$, with a closed boundary $\mathbf{F}$ (of a specified type), with probability 1, then the process $X(s, + \tau(\omega), \omega)$ in the new parameter variable s is again a Brownian motion process. Moreover, the difference, $X(s + \tau(\omega), \omega) - X(\tau(\omega), \omega)$, process is quite independent of the original process, $X(t, \omega)$, for $0 \leqq t \leqq \tau(\omega)$.

As intuitively obvious as these statements are, their proofs require measure-theoretic considerations involving function spaces. Since the literature does not include detailed proofs of the above remarks, the author wishes to express his appreciation to Professor G. Hunt for demonstrating their proofs in a private communication.

By Theorems 3.1 and 3.2, the totality of ω's having continuous sample functions $X(t, \omega)$ originating from x and intersecting $\Gamma[D]$ has measure 1; i.e.,

$$\Pr \{\Omega(x, \Gamma[D])\} = 1.$$

Consequently, the proof of the following theorem can be obtained by restricting $\omega \varepsilon \Omega(x, \Gamma[D])$ and by completing an induction argument which makes use of the facts mentioned in remark (2).

THEOREM 3.5. *Given any point x belonging to a domain D with boundary $\Gamma[D]$, a sequence $T(x, \omega)$ of successive first intersections exists for almost all sample functions of the Brownian motion process originating from x.*

THEOREM 3.6. *Given any point x belonging to a domain D with boundary $\Gamma[D]$, then with probability 1, the sequence $T(x, \omega) = \{P_{i+1}(x, \mathbf{K}(P_i), \omega), i = 0, 1, 2, \cdots\}$ of successive first intersections corresponding to the sample function $X(t, \omega)$ of the Brownian motion process originating from x converges to a point of the boundary $\Gamma[D]$. This point of convergence coincides with the point where the sample function intersects the boundary.*

PROOF. We restrict ω to that subset of Ω for which the sample functions originating from the given point x are continuous, intersect the boundary $\Gamma[D]$ for finite values of t, and have sequences $T(x, \omega)$. From Theorem 3.5, this subset of Ω is $\Omega(x, \Gamma[D])$ and it has measure 1. Hence, to prove the theorem, we need only to prove it to be true for all $\omega \varepsilon \Omega(x, \Gamma[D])$. So we now select any $\omega \varepsilon \Omega(x, \Gamma[D])$ and let t_0 be the finite value of t for which the sample function first intersected the boundary $\Gamma[D]$, say at the point z. Using three major steps, we will now prove that the corresponding sequence of points $T(x, \omega) = \{P_{i+1}(x, \mathbf{K}(P_i), \omega), i = 0, 1, \cdots\}$ converges to the point z.

1. The sequence $T(x, \omega) = \{P_{i+1}(x, \mathbf{K}(P_i), \omega), i = 0, 1, \cdots\}$ converges. In $T(x, \omega)$, the t_i's, $i = 1, 2, \cdots$, of the successive first intersections are monotone increasing and bounded by t_0. Hence, the t_i's are convergent, and $T(x, \omega)$ converges by continuity, since $X(t, \omega)$ is continuous.

2. The limit point of the sequence $T(x, \omega)$ is a boundary point. Assume the contrary, i.e., that the limit point by Step 1, say x_0, is an interior point of the domain D. Then there exists a $d > 0$, where $d = \inf_{x' \varepsilon \Gamma[D]} \|x' - x_0\|$. Since x_0 is the limit point for the sequence, there exists an i_0 such that for all $i > i_0$, $\|P_{i+1}(x, \mathbf{K}(P_i), \omega) - x_0\| < d/4$. Consider any $i > i_0 + 1$. Then

$$\|P_i(x, \mathbf{K}(P_{i-1}), \omega) - x_0\| < d/4),$$

and the maximum N-sphere about $P_i(x, \mathbf{K}(P_{i-1}), \omega)$ i.e., the sphere $K(P_i)$, has a radius $r_i = \inf_{x' \varepsilon \Gamma[D]} \|P_i(x, \mathbf{K}(P_{i-1}), \omega) - x'\| > 3d/4$. But by Definition

3.1, the point $P_{i+1}(x, \mathbf{K}(P_i), \omega)$ must lie on the surface $\mathbf{K}(P_i)$, and since $i + 1 > i_0$, the point $P_{i+1}(x, \mathbf{K}(P_i), \omega)$ must also satisfy the condition

$$\|P_{i+1}(x, \mathbf{K}(P_i), \omega) - x_0\| < d/4.$$

This is impossible. Hence, x_0 cannot be an interior point; therefore, the limit point lies on the boundary $\Gamma[D]$.

3. To prove that the limit point, x_0, of the sequence $T(x, \omega)$ must be the point z where the sample function $X(t, \omega)$ first intersects $\Gamma[D]$, i.e., $X(t_0, \omega) = z$, we will assume the contrary. By Step 2, we know $x_0 \varepsilon \Gamma[D]$. Thus, we assume $x_0 \neq z$ and $x_0 \varepsilon \Gamma[D]$; further, let $\bar{t}$ be the corresponding limiting value of the t_i's in $T(x, \omega)$. If $x_0 \neq z$, then $\bar{t} < t_0$. When the successive t_i's of $T(x, \omega)$ converge to $\bar{t}$, the corresponding $X(t_i, \omega)$'s converge to x_0, and by continuity, $x_0 = X(\bar{t}, \omega)$. But by assumption, z was the point of first intersection of $X(t, \omega)$ with the boundary $\Gamma[D]$. Thus, this contradicts that $\bar{t} < t_0$, and hence $x_0 = z$. Therefore, $T(x, \omega)$ must converge to z, and the proof is completed.

With the relevant properties of the Brownian motion process taken into account, we now exhibit the relationship between the Brownian motion process and the Spherical process that will be useful. We need only consider the mapping in one direction—namely, that to each sequence $T(x, \omega)$ of the Brownian motion process, there is a sequence $S(x, \phi)$ of the Spherical process.

Using Theorem 3.5, remarks (1) and (2) concerning the Brownian motion process, and the definition of the Spherical process, an induction arguments yields the following desired result:

THEOREM 3.7. *Given any point x belonging to a domain D with boundary $\Gamma[D]$, then the probability distribution of the sequences of successive first intersections $T(x, \omega) = \{P_{i+1}(x, \mathbf{K}(P_i), \omega), i = 0, 1, 2, \cdots\}$, of the sample functions of the Brownian motion process originating from x, is the same as the probability distribution of the sequences of the successive points $S(x, \phi) = \{P_{i+1}(x, \phi), i = 0, 1, 2, \cdots\}$ of the Spherical process originating from x.*

Thus the results developed for the sequences $T(x, \omega)$ can be reinterpreted for the Spherical process. Consequently, we shall next show that the Spherical process converges with probability 1 and yields the solution for any given Dirichlet problem.

4. Solution of the Dirichlet problem by the Spherical process.

THEOREM 4.1. *Given any point x belonging to a domain D with boundary $\Gamma[D]$, then with probability 1, the Spherical process originating from x converges to a point of the boundary $\Gamma[D]$.*

PROOF. The proof will follow by applying Theorems 3.6 and 3.7. By Theorem 3.7 we know that to each sequence $T(x, \omega)$ of successive first intersections of the Brownian motion process, we can associate a sequence $S(x, \phi)$ of the Spherical process. Hence, with probability 1, the sequence $S(x, \phi)$ must converge to the boundary $\Gamma[D]$, since by Theorem 3.6, the sequences $T(x, \omega)$ converge with probability 1 to the boundary $\Gamma[D]$.

THEOREM 4.2. *Given a domain D with boundary $\Gamma[D]$ and an elementary set E of the boundary $\Gamma[D]$, then the probability,* $\Pr(S(x, \phi), E, D)$, *that the Spherical process originating from a point x of the domain D will converge to the set E, without having converged to $\Gamma[D] - E$ before it, is a harmonic function of x in D and* $\lim_{x \in D, x \to x_0} \Pr(S(x, \phi)E, D) = 1$ *or* 0 *according as x_0 is an inner point of E or of $\Gamma[D] - E$.*

PROOF. It is correct to speak of the distribution of the points of convergence on $\Gamma[D]$ of the Spherical process, since by Theorem 4.1, the Spherical process converges to $\Gamma[D]$ with probability 1. By Theorem 3.7 we know that the distribution of these points of convergence must be the same as those for the sequences $\{T(x, \omega), \omega \varepsilon \Omega(x, \Gamma[D])\}$. By use of Theorem 3.6, the distribution of the points of convergence of the sequences $\{T(x, \omega), \omega \varepsilon \Omega(x, \Gamma[D])\}$ to $\Gamma[D]$ is the same as the distribution of points of first intersection on $\Gamma[D]$ for the Brownian motion process. Hence, Theorem 3.3 then yields this theorem immediately.

Exactly as Theorem 3.4 follows from Theorem 3.3, a similar result in light of Theorem 4.2 can be stated for the Spherical process.

THEOREM 4.3. *With the same conditions as given in Theorem 4.2, let $f(x)$ be a real-valued continuous function defined on the boundary $\Gamma[D]$. Then for any point x_0 belonging to D, the value $u(x_0)$ of the solution $u(x)$ of the Dirichlet problem for the domain D and the boundary value function $f(x)$ is obtained by taking the integral, of a Poisson type, of $f(x)$ with respect to the kernel* $\Pr(X(x_0, \phi), E, D)$ *on $\Gamma[D]$ or by taking the mathematical expectation of the composed function*

$$f(P(x_0, \Gamma[D], \omega)):$$

$$u(x_0) = \int_{\Gamma[D]} \Pr(S(x_0, \phi), dx, D) f(x) = \int_{\Omega} f(P(x_0, \Gamma[D], \omega))\, d\omega.$$

5. Generalizations of the Spherical process.

5.1. *The Generalized Spherical process.* An immediate generalization of the Spherical process would be an attempt to use spheres whose radii are not necessarily maximum. Specifically, the generalization of the Spherical process found in Section 2 is as follows.

DEFINITION 5.1. *The Generalized Spherical process.* Given a domain D of N dimensions with boundary $\Gamma[D]$, and any point x belonging to $D + \Gamma[D]$, then the Generalized N-dimensional Spherical process originating from x is $\hat{\Phi}(x)$, where

(A) $\hat{\Phi}(x) = \{\hat{S}(x, \phi), 0 \leqq \phi \leqq 1\}$, i.e., $\hat{\Phi}(x)$ is the totality of all sequences of points $\hat{S}(x, \phi)$, $0 \leqq \phi \leqq 1$, where

(B) Each value of ϕ specifies a sequence of points $\hat{S}(x, \phi) = \{\hat{P}_{i+1}(x, \phi), i = 0, 1, 2, \cdots,\}$ generated according to the following stipulations:

(1) About the point $\hat{P}_0(x, \phi) = x$, determine an N-sphere $\hat{K}(\hat{P}_0, e_0)$, where e_0 is the radius of the sphere and $e_0 = \lambda_0(\phi) r_0$, $\epsilon < \lambda_0(\phi) \leqq 1$, for some positive $\epsilon > 0$, and r_0 is the radius of $K(\hat{P}_0)$ of Definition 2.2;

(2) Select the point $\hat{P}_1(x, \phi)$ uniformly at random on $\hat{\mathbf{K}}(\hat{P}_0, e_0)$;

(3) The point $\hat{P}_{i+1}(x, \phi)$ is determined recursively from $\hat{P}_i(x, \phi)$, and $\hat{K}(\hat{P}_i, e_i)$ in the same manner as $\hat{P}_1(x, \phi)$ was determined from $\hat{P}_0(x, \phi)$.

To show that the Generalized Spherical process furnishes a method of obtaining the solution for the Dirichlet problem requires only the most obvious restatement of the material developed for the Spherical process. The introduction of the requirement that $\epsilon < \lambda_i(\phi) \leqq 1$, $i = 0, 1, 2, \cdots$, $0 \leqq \phi \leqq 1$, is to ensure that the process will not degenerate, i.e., converge to an interior point of the domain D. With this requirement, and making the obvious changes, an inspection of the proof of Theorem 3.6 shows the theorem to be valid in the present situation.

We next consider the generalization of allowing the transitions to take place on surfaces other than spheres.

5.2. *The General Dirichlet Domain Process.* We shall now specify the N-dimensional domains that will be acceptable in place of the N-dimensional spheres.

DEFINITION 5.2. *An admissible domain.* Given an N-dimensional domain D with boundary $\Gamma[D]$, then an N-dimensional domain, say D_i, with boundary $\Gamma[D_i]$ is admissible with respect to any point, say P, if $P \varepsilon D$ and if the following conditions are satisfied:

(A) $P \varepsilon D_i$ and $D_i \subset D$.

(B) The normal derivative of the Green's function for the domain D_i is known on $\Gamma[D_i]$.

(C) The domain D_i, with respect to the point P, has the property that, for some $\epsilon > 0$, for every ray originating from the point P, the ratio of the distances along the ray from the point P to the points of intersection of the ray with $\Gamma[D_i]$ and $\Gamma[D]$, respectively, is greater than ϵ.

DEFINITION 5.3. *General Dirichlet Domain process.* Given an N-dimensional domain D with boundary $\Gamma[D]$, and any point x belonging to $D + \Gamma[D]$, then the N-dimensional General Dirichlet Domain process originating from x is $\bar{\Phi}(x)$, where

(A) $\bar{\Phi}(x) = \{\bar{S}(x, \phi), 0 \leqq \phi \leqq 1\}$; i.e., $\bar{\Phi}(x)$ is the totality of all sequences of points $\bar{S}(x, \phi)$, $0 \leqq \phi \leqq 1$, where

(B) Each value of ϕ specifies a sequence of points $\bar{S}(x, \phi) = \{\bar{P}_{i+1}(x, \phi), i = 0, 1, 2, \cdots\}$ generated according to the following stipulations:

(1) With respect to the point $\bar{P}_0(x, \phi) = x$, select any admissible domain, say $D_1(\phi)$;

(2) Select a point $\bar{P}_1(x, \phi)$ on the boundary $\Gamma[D_1(\phi)]$ of $D_1(\phi)$ from the probability distribution satisfying the condition that the probability of $\bar{P}_1(x, \phi)$'s being in any Lebesgue measurable subset on $\Gamma[D_1(\phi)]$ is equal to the Lebesgue integral of the normalized normal derivative of the Green's function of $\Gamma[D_1(\phi)]$ over the subset in question;

(3) The point $\bar{P}_{i+1}(x, \phi)$ is determined recursively from $\bar{P}_i(x, \phi)$, and an admissible domain $D_i(\phi)$ in the same manner as $\bar{P}_1(x, \phi)$ was determined from $\bar{P}_0(x, \phi)$.

The General Dirichlet Domain process furnishes a method for solving the Dirichlet problem. This is seen by reviewing the material developed for the Spherical process and making the necessary minor modifications. For example, formerly we used the fact that the probability distribution of first intersections on the surface of $K(\bar{P}_i(\phi))$ for the sample functions of the Brownian motion process originating from $\bar{P}_i(\phi)$ was uniformly distributed on the surface $\mathbf{K}(\bar{P}_i(\phi))$. We now would use the fact that the probability distribution of first intersections on the surface $\Gamma[D_i(\phi)]$ of an admissible domain $D_i(\phi)$ for sample functions of the Brownian motion process originating from $\bar{P}_i(\phi)\ \varepsilon\ D_i(\phi)$ is the same as the probability distribution specified in Definition 5.3 for the given point $\bar{P}_i(\phi)$ and domain $D_i(\phi)$. That the sample functions have this distribution follows from Theorem 3.3. With this result, and making the necessary minor modifications in the material given for the Spherical process, the development would be similar to that already presented. However, in order to ensure that the process will not degenerate and to ensure that a result comparable to Theorem 3.6 is attainable, requirement (C) of Definition 5.2 is used.

6. Order of the number of steps. It is of interest to have some indication as to the order of the number of steps required for convergence of the N-dimensional Spherical process. A conservative indication can be obtained by studying first the convergence of the process to an infinite $(N - 1)$-dimensional hyperplane in N-space. The study will also be useful for showing that the computing effort, when measured in terms of the order of the number of steps required, increases approximately as a linear function of the dimensionality of the problem.

In this section, $y_i = P_i(x, \phi)$, $i = 0, 1, 2, \cdots$, will denote a point in N-space generated by the Spherical process with $\Gamma[D]$ an infinite $(N - 1)$-dimensional hyperplane, where $y_0 = P_0(x, \phi) = x_0$ is the initial starting point of the process. For convenience, it will be assumed that the coordinate system is so oriented with respect to the given hyperplane that y_i also denotes the distance along a normal from the hyperplane to the point. From the definition of the Spherical process, $y_{i+1} = P_{i+1}(x, \phi)$ lies on the surface $\mathbf{K}(P_i)$. Thus, with the assumed orientation of the hyperplane and the coordinate system, θ_{i+1} will denote the direction angle between the normal of the hyperplane passing through the point $y_i = P_i(x, \phi)$ and the radius vector of the N-sphere $K(P_i)$ at the point y_{i+1}.

We then have

$$\frac{y_{i+1}}{y_i} = 1 \cos \theta_{i+1}\,, \tag{6.1}$$

where $0 \leqq \theta_i \leqq 2\pi$, $i = 0, 1, 2, \cdots$.

From symmetry, and since we shall only be interested in the distance from the given hyperplane, we can restrict the subsequent discussion to θ in the range $0 \leqq \theta \leqq \pi$.

THEOREM 6.1. *Let $y_i = P(x, \phi)$, $i = 0, 1, 2, \cdots$, be subject to the formulation leading to condition* (6.1). *Then the N-dimensional Spherical process has the property that with the boundary $\Gamma[D]$ being an infinite $(N - 1)$-dimensional hyperplane*

in N-space, the expected value, $E\{\log (y_{i+1} / y_i) \mid N\}$, $i = 0, 1, 2, \cdots$, *is negative for any value N and is given as follows:*

$$E\left\{\log \frac{y_{i+1}}{y_i} \,\middle|\, N\right\} = \log 2 + \Psi\left(\frac{N-1}{2}\right) - \Psi(N-1) \quad \text{for } i = 0, 1, 2, \cdots, \tag{6.2}$$

where $\Psi(z)$ *is the psi, or digamma, function, i.e.,* $\Psi(z) = d / dz \log \Gamma(z)$, $z > 0$, *and* $\Gamma(z)$ *is the gamma function. In particular,*

$$N = 2, E\left\{\log \frac{y_{i+1}}{y_i} \,\middle|\, N\right\} = -\log 2, \tag{6.3}$$

$$N \text{ odd}, N \geqq 3, E\left\{\log \frac{y_{i+1}}{y_i} \,\middle|\, N\right\} = \log 2 - \sum_{j=(N-1)/2}^{N-2} \frac{1}{j}, \tag{6.4}$$

$$N \text{ even}, N \geqq 4, E\left\{\log \frac{y_{i+1}}{y_i} \,\middle|\, N\right\} = -\log 2 + \sum_{j=N/2}^{N-2} \frac{1}{j}. \tag{6.5}$$

Proof. From the definition of the Spherical process, it is a Markov process where at each stage i of the process the next point $P_{i+1}(x, \phi)$ is selected independently and uniformly at random on $\mathbf{K}(P_i)$, $i = 0, 1, 2, \cdots$. Thus

$$E\{\log (y_{i+1} / y_i) \mid N\} = E\{\log (y_1 / y_0) \mid N\}$$

for $i = 0, 1, 2, \cdots$. Hence, it is sufficient to consider $E\{\log (y_1 / y_0) \mid N\}$. Owing to the earlier-mentioned symmetry for θ_i, we need only to consider selecting points uniformly at random on the surface of the appropriate N-dimensional hemisphere. Thus, with respect to θ_i, this implies that the probability measure defined on the surface of the N-dimensional hemisphere is

$$\frac{\sin^{N-2} \theta \, d\theta}{\int_0^\pi \sin^{N-2} \theta \, d\theta},$$

where $0 \leqq \theta \leqq \pi$. By (6.1),

$$E\left\{\log \frac{y_1}{y_0} \,\middle|\, N\right\} = \frac{\int_0^\pi \log (1 - \cos \theta) \sin^{N-2} \theta \, d\theta}{\int_0^\pi \sin^{N-2} \theta \, d\theta}.$$

Using $1 - \cos \theta = 2[\sin \theta/2]^2$ and then letting

$$u = \theta/2, \qquad \sin 2u = 2 \sin u \cos u,$$

and using formula 483 in Peirce [20] and formula 6(c) of Table 338 in Gröbner and Hofreiter [8], we obtain

$$E\left\{\log \frac{y_{i+1}}{y_i} \,\middle|\, N\right\} = \log 2 + \frac{2^{N-2}\left\{\left[\Gamma\left(\frac{N-1}{2}\right)\right]^2 / \Gamma(N-1)\right\}\left\{\Psi\left(\frac{N-1}{2}\right) - \Psi(N-1)\right\}}{\sqrt{\pi}\, \Gamma\left(\frac{N-1}{2}\right) \Big/ \Gamma\left(\frac{N}{2}\right)}.$$

Upon simplification we have the desired conclusion (6.2). We then obtain (6.3) from (6.2) by direct evaluation, e.g., by using Table 411 in Gröbner and Hofreiter [8].

When N is odd and $N \geqq 3$, we can use the result that

$$\Psi(t) - \Psi(z) = \int_0^1 (u^{z-1} - u^{t-1}) / (1 - u)\, du$$

(formula 8(b) of Table 411, Gröbner and Hofreiter [8]). From formula 1(a) of Table 161, Gröbner and Hofreiter [8], we obtain upon simplification the desired conclusion for (6.4).

(6.5) is obtained by a direct inductive argument which uses that $\Psi(z + 1) = \Psi(z) + 1/z$ and $\Psi(\frac{1}{2}) = -\varepsilon - \log 4$, where ε is Euler's constant. With the explicit form of the expectation being given by either (6.3), (6.4), or (6.5), we shall now show that the expectation is negative. We proceed as follows: First, we note that

$$\log \frac{q+1}{p} < \sum_{j=p}^{q} \frac{1}{j} < \log \frac{q}{p-1} \tag{6.6}$$

for p and q integers, with $p < q$.

For N odd, $N \geqq 3$, we have from (6.4) and (6.6), with $p = (N-1)/2$ and $q = N - 2$, that

$$\log \frac{N-3}{N-2} < E\left\{\log \frac{y_{i+1}}{y_i} \,\middle|\, N\right\} < 0, \qquad i = 0, 1, 2, \cdots. \tag{6.7}$$

Likewise, for N even, $N \geqq 4$, we have from (6.5) and (6.6), with $p = N/2$ and $q = N - 2$, that

$$\log \frac{N-1}{N} < E\left\{\log \frac{y_{i+1}}{y_i} \,\middle|\, N\right\} < 0, \qquad i = 0, 1, 2, \cdots. \tag{6.8}$$

Hence, by (6.3), (6.7), and (6.8), the expectation is negative for any finite $N \geqq 2$. Thus the proof is completed.

For large values of N, the following asymptotic result will be useful.

THEOREM 6.2. *Subject to the conditions imposed on the N-dimensional Spherical process in Theorem* 6.1, $E\{\log (y_{i+1}/y_i) \mid N\}$, $i = 0, 1, 2, \cdots$, *is asymptotically* $-\frac{1}{2}N$ *for large* N, *i.e.*, $2N(E\{\log (y_{i+1}/y_i) \mid N\}) \sim -1$.

PROOF. From Theorem 6.1,

$$E\left\{\log \frac{y_{i+1}}{y_i} \,\middle|\, N+1\right\} = \log 2 + \Psi\left(\frac{N}{2}\right) - \Psi(N), \qquad i = 0, 1, 2, \cdots.$$

From Nörlund [18], page 106, we have that

$$\Psi(z) \sim \log z - \frac{1}{2z} - \sum_{k=1}^{m} \frac{B_{2k}}{2kz^{2k}} + R_{2m+1},$$

where

$$R_{2m+1} = -\frac{B_{2m+2}}{(2m+2)(z+\lambda)^{2m+2}}, \qquad 0 < \lambda < \tfrac{1}{2},$$

and B_m is the mth Bernoulli number. Since $B_2 = \frac{1}{6}$, we obtain

$$E\left\{\log \frac{y_{i+1}}{y_i} \,\middle|\, N+1\right\} \sim \log 2 + \left[\log \frac{N}{2} - \frac{1}{N} - \frac{1}{12\left(\frac{N}{2}+\lambda_1\right)^2}\right] - \left[\log N - \frac{1}{2N} - \frac{1}{12(N+\lambda_2)^2}\right] \sim -\frac{1}{2N}.$$

Consequently, for large N, we have the desired conclusion, namely,

$$E\left\{\log \frac{y_{i+1}}{y_i} \,\middle|\, N\right\} \sim -\frac{1}{2N}, \qquad i = 0, 1, 2, \cdots.$$

We will now use Kolmogorov's form of the strong law of large numbers to obtain the following result concerning convergence.

THEOREM 6.3. *Given any $\epsilon > 0$, $\eta > 0$ there exists an n_0 such that, subject to the conditions placed on the N-dimensional Spherical process in Theorem* 6.1,

$$\Pr\left\{\frac{1}{n}\log \frac{y_n}{y_0} + \left| E\left\{\log \frac{y_1}{y_0} \,\middle|\, N\right\}\right| < \epsilon \text{ for all } n > n_0\right\} > 1 - \eta,$$

or equivalently,

$$\Pr\left\{\frac{y_n}{y_0} < \exp\left[-n\left(\left| E\left\{\log \frac{y_1}{y_0} \,\middle|\, N\right\}\right| - \epsilon\right)\right] \text{for all } n > n_0\right\} > 1 - \eta.$$

PROOF. By successive use of (6.1), we obtain that $y_n/y_0 = \prod_{i=1}^{n}(1 - \cos\theta_i)$, or that

$$\log \frac{y_n}{y_0} = \sum_{i=1}^{n} \log (1 - \cos\theta_i).$$

Using (6.1) and dividing both sides by n yields

$$\frac{1}{n}\log \frac{y_n}{y_0} = \frac{1}{n}\left[\log \frac{y_1}{y_0} + \log \frac{y_2}{y_1} + \cdots + \log \frac{y_n}{y_{n-1}}\right]. \tag{6.9}$$

Consider the right-hand side of (6.9). From the definition of the Spherical process and the remarks made in the proof of Theorem 6.1, the quantities $\log (y_{i+1}/y_i) = \log (1 - \cos\theta_{i+1}), i = 0, 1, 2, \cdots, n-1$, are mutually independent and identically distributed. Further, from Theorem 6.1 we know

$$E\{\log (y_{i+1})/y_i \mid N\}$$

for $i = 0, 1, 2, \cdots, n-1$. Then, since the expectation of the sum is the sum of the expectations (see for example Cramér [1], p. 173), condition (6.9) yields that

$$E\left\{\frac{1}{n}\log \frac{y_n}{y_0} \,\middle|\, N\right\} = E\left\{\log \frac{y_{i+1}}{y_i} \,\middle|\, N\right\}.$$

Further, the expectation is bounded and nonpositive. Since $\log (y_{i+1}/y_i)$, $i = 0, 1, 2, \cdots, n-1$, are mutually independent and identically distributed

with bounded and equal first moments, we are able to use Kolmogorov's strong law of large numbers, (see for example Doob [4], Theorem 5.1, page 142). Hence, the right side of (6.9) converges to $E\{\log y_1 / y_0 \mid N\}$ with probability 1. Consequently, $1/n \log y_n / y_0$ converges with probability 1 to $E\{\log y_1 / y_0 \mid N\}$. Representing the strong law of large numbers in the more expressive form of the ϵ, η notation, we have the desired conclusions of the theorem.

We next consider finding the variance of the statistic $1/n \log y_n / y_0$.

THEOREM 6.4. *Subject to the conditions imposed on the N-dimensional Spherical process in Theorem* 6.1, *the variance of* $1/n \log y_n / y_0$ *for fixed n is a monotone decreasing function of N and the variance is given as follows*:

$$\sigma^2\left\{\frac{1}{n}\log\frac{y_n}{y_0}\,\middle|\,N\right\} = \frac{1}{n}\left[\Psi'\left(\frac{N-1}{2}\right) - \Psi'(N-1)\right], \tag{6.10}$$

where $\Psi'(z)$ *is the trigamma function, i.e., the derivative of the digamma function.*

Proof. As mentioned in the proofs of Theorems 6.1 and 6.3, the quantities $\log (y_{i+1} / y_i)$, $i = 0, 1, 2, \cdots, n-1$, are mutually independent and identically distributed. Hence, using (6.9), we obtain

$$\sigma^2\left\{\frac{1}{n}\log\frac{y_n}{y_0}\,\middle|\,N\right\} = \frac{1}{n}E\left\{\left(\log\frac{y_1}{y_0}\right)^2\,\middle|\,N\right\} - \frac{1}{n}\left[E\left\{\log\frac{y_1}{y_0}\,\middle|\,N\right\}\right]^2. \tag{6.11}$$

We already know $E\{\log (y_1 / y_0) \mid N\}$ from Theorem 6.1. Since

$$E\{(\log y_1 / y_0)^2 \mid N\} = E\{(\log (1 - \cos\theta_1))^2 \mid N\},$$

the problem is reduced to finding this latter expected value, i.e.,

$$E\{[\log (1 - \cos\theta)]^2 \mid N\} = \frac{\int_0^\pi [\log (1 - \cos\theta)]^2 \sin^{N-2}\theta\, d\theta}{\int_0^\pi \sin^{N-2}\theta\, d\theta}.$$

Letting $1 - \cos\theta = 2[\sin\theta/2]^2$ and expanding, the above equals

$$(\log 2)^2 + \frac{(2\log 2)\cdot 2\int_0^\pi \log\left(\sin\frac{\theta}{2}\right)\sin^{N-2}\theta\, d\theta}{\int_0^\pi \sin^{N-2}\theta\, d\theta} + \frac{4\int_0^\pi \left[\log\left(\sin\frac{\theta}{2}\right)\right]^2 \sin^{N-2}\theta\, d\theta}{\int_0^\pi \sin^{N-2}\theta\, d\theta}.$$

From the proof of Theorem 6.1, we know that

$$\frac{2\int_0^\pi \log\left(\sin\frac{\theta}{2}\right)\sin^{N-2}\theta\, d\theta}{\int_0^\pi \sin^{N-2}\theta\, d\theta} = \Psi\left(\frac{N-1}{2}\right) - \Psi(N-1).$$

Letting $u = \sin \theta/2$ and $\sin \theta = 2 \sin \theta/2 \cos \theta/2$ and then using formula 59, Table 324, of Gröbner and Hofreiter [8], we find that

$$\int_0^\pi \left[\log\left(\sin\frac{\theta}{2}\right)\right]^2 \sin^{N-2}\theta\, d\theta = 2^{N-4}\,\frac{\Gamma\left(\frac{N-1}{2}\right)\Gamma\left(\frac{N-1}{2}\right)}{\Gamma(N-1)} \cdot\left\{\Psi'\left(\frac{N-1}{2}\right) - \Psi'(N-1) + \left[\Psi\left(\frac{N-1}{2}\right) - \Psi(N-1)\right]^2\right\}.$$

We obtain upon simplification that

$$E\{[\log(1-\cos\theta)]^2 \mid N\} = (\log 2)^2 + 2\log 2\left\{\Psi\left(\frac{N-1}{2}\right) + \Psi(N-1)\right\} + \left\{\Psi'\left(\frac{N-1}{2}\right) - \Psi'(N-1) + \left[\Psi\left(\frac{N-1}{2}\right) - \Psi(N-1)\right]^2\right\}.$$

Thus, using (6.11) and making straightforward simplifications yields (6.10).

We next show that the variance for fixed n is a monotone decreasing function of N, i.e., that $\sigma^2\{1/n \log y_n / y_0 \mid N\} \geqq \sigma^2\{1/n \log y_n / y_0 \mid N+1\}$ for any $N \geqq 2$. We shall have the desired result if we show that

$$\frac{\Psi'\left(\frac{N-1}{2}\right) - \Psi'(N-1)}{\Psi'\left(\frac{N}{2}\right) - \Psi'(N)} \geqq 1 \qquad \text{for any } N \geqq 2. \tag{6.12}$$

Since $\Psi'(z) = \sum_{r=0}^{\infty} 1/(z+r)^2$, (6.12) is equivalent to showing that

$$\frac{\sum_{r=0}^{\infty}\frac{4}{(N-1+2r)^2} - \sum_{r=0}^{\infty}\frac{1}{(N-1+r)^2}}{\sum_{r=0}^{\infty}\frac{4}{(N+2r)^2} - \sum_{r=0}^{\infty}\frac{1}{(N+r)^2}} \geqq 1,$$

or that

$$\frac{3\sum_{r=0}^{\infty}\frac{1}{(N-1+2r)^2} - \sum_{r=0}^{\infty}\frac{1}{(N-1+2r+1)^2}}{3\sum_{r=0}^{\infty}\frac{1}{(N+2r)^2} - \sum_{r=0}^{\infty}\frac{1}{(N+2r+1)^2}} \geqq 1,$$

or equivalently,

$$3\sum_{r=0}^{\infty}\frac{1}{(N-1+2r)^2} + \sum_{r=0}^{\infty}\frac{1}{(N+1+2r)^2} \geqq 4\sum_{r=0}^{\infty}\frac{1}{(N+2r)^2}.$$

But since each term of each series is positive, the question is reduced to asking whether

$$\frac{3}{(t-1)^2} + \frac{1}{(t+1)^2} \geqq \frac{4}{t^2} \qquad \text{for } t > 1.$$

Straightforward algebraic calculations show this latter condition to be true. Hence,

$$\sigma^2\left\{\frac{1}{n}\log\frac{y_n}{y_0}\,\middle|\,N\right\} \geqq \sigma^2\left\{\frac{1}{n}\log\frac{y_n}{y_0}\,\middle|\,N+1\right\} \qquad \text{for any } N \geqq 2.$$

With this result the proof is completed.

We will now use Lindeberg's form of the central limit theorem to obtain the following result, which is useful for determining asymptotic confidence intervals for the statistic $1/n \log y_n / y_0$.

THEOREM 6.5. *Subject to the conditions imposed on the N-dimensional Spherical process in Theorem* 6.1, *the statistic* $1/n \log y_n / y_0 + |E\{\log y_1 / y_0 \mid N\}|$ *is asymptotically normally distributed with mean* 0 *and variance*

$$\sigma^2 = \sigma^2\{1/n \log y_n / y_0 \mid N\},$$

i.e.,

$$\lim_{n\to\infty} \Pr\left[\frac{1}{n}\log\frac{y_n}{y_0} + \left|E\left\{\log\frac{y_1}{y_0}\,\middle|\,N\right\}\right| < \sigma\lambda\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\lambda} e^{-t^2/2}\,dt, \quad \textit{uniformly in } \lambda.$$

PROOF. As in the proof of Theorem 6.3, we have that

$$\frac{1}{n}\log\frac{y_n}{y_0} = \frac{1}{n}\left[\log\frac{y_1}{y_0} + \log\frac{y_2}{y_1} + \cdots + \log\frac{y_n}{y_{n-1}}\right].$$

Likewise, the quantities $\log y_{i+1} / y_i$, $i = 0, 1, \cdots, n-1$, are mutually independent and identically distributed. From Theorems 6.1 and 6.4, we know the mean and variance of the quantities $\log y_{i+1} / y_i$ for $i = 0, 1, \cdots, n-1$. Since the variance is finite, we can appeal to Lindeberg's form of the central limit theorem for the desired conclusion of the theorem (see for example Doob [4], Theorem 4.3, page 140).

Hence, by applying Theorem 6.5, we can obtain asymptotic confidence intervals for y_n / y_0 of the form

$$\lim_{n\to\infty} \Pr\left\{\exp\left[-n\left(\left|E\left\{\log\frac{y_1}{y_0}\,\middle|\,N\right\}\right| - \lambda_1\sigma\right)\right] \leqq \frac{y_n}{y_0} \leqq \exp\left[-n\left(\left|E\left\{\log\frac{y_1}{y_0}\,\middle|\,N\right\}\right| - \lambda_2\sigma\right)\right]\right\} = \frac{1}{\sqrt{2\pi}}\int_{\lambda_1}^{\lambda_2} e^{-t^2/2}\,dt. \tag{6.13}$$

A popular conjecture exists to the effect that a Monte Carlo procedure has its utility increased as the dimensionality of the problem is increased. The basis for this belief has been that while other numerical techniques usually require an increase in computing labor which is an exponential function of N, a Monte Carlo technique should only require an increase in computing labor which is a linear function of N. By appealing to the material developed in this section,

we see that in this given situation the conjecture concerning the nature of the increase in computing labor when using a Monte Carlo technique is true. Clearly, if the amount of computing labor is measured in terms of the average number of steps required to be arbitrarily near the boundary, then in the present situation the average number of steps required depends on $1 / (|E\{1/n \log y_n / y_0 \mid N\}|) = 1 / (|E\{\log y_1 / y_0 \mid N\}|)$. But by Theorem 6.2, $1/(|E\{\log y_1 / y_0 \mid N\}|)$ is asymptotically $2N$. Hence, the increase in computing labor as a function of N is approximately linear in N.

Using the material of this section, we obtain the following numerical results:

N	Expectation $E\left\{\frac{1}{n} \log \frac{y_n}{y_0} \mid N\right\}$	Asymptotic estimate for expectation	n times the variance $\sigma^2\left\{\frac{1}{n} \log \frac{y_n}{y_0} \mid N\right\}$	Appraisal of linearity $\frac{1}{E\left\{\frac{1}{n} \log \frac{y_n}{y_0} \mid N\right\}} - \frac{1}{E\left\{\frac{1}{n} \log \frac{y_n}{y_0} \mid N+1\right\}}$
2	−.6932	−.2500	3.289	−1.817
3	−.3068	−.1667	1.000	−1.917
4	−.1932	−.1250	0.540	−1.962
5	−.1401	−.1000	0.361	−1.961
6	−.1099	−.0833	0.269	−2.000
7	−.0901	−.0743	0.213	−1.973
8	−.0765	−.0625	0.177	−2.011
9	−.0663	−.0556	0.151	−1.961
10	−.0587	−.0500	0.131	
$\sim$	0	0	0	−2

Thus, as N increases, we see that in the given situation the process will require, on the average, more steps to get arbitrarily near the boundary, since $|E\{1/n \log y_n / y_0 \mid N\}|$ decreases with increasing N. However, the process has the interesting compensating feature that the variance $\sigma^2\{1/n \log y_n / y_0 \mid N\}$ decreases with increasing N so as to give rise to at least a more stable statistic for fixed n.

We will now see that the material developed in this section for the N-dimensional Spherical process, with $\Gamma[D]$ being an infinite $(N-1)$-dimensional hyperplane, can be used to yield an upper bound for the order of the average number of steps required when $D + \Gamma[D]$ is any N-dimensional convex set.

THEOREM 6.6. *Let x_0 be any point belonging to an N-dimensional convex set $D + \Gamma[D]$ and at distance d_0 from $\Gamma[D]$. Then, the order of the average number of steps required for convergence to $\Gamma[D]$ by the N-dimensional Spherical process, defined on $D + \Gamma[D]$ and originating at x_0, is equal to or less than the order of the average number of steps required for convergence to any infinite $(N-1)$-dimensional hyperplane by the N-dimensional Spherical process, defined with respect to this infinite hyperplane and originating at distance d_0 from this hyperplane, i.e.,*

$$E\left\{\frac{1}{n} \log \frac{r_n}{r_0} \middle| N\right\} \leqq E\left\{\log \frac{y_1}{y_0} \middle| N\right\},$$

where r_i, $i = 0, 1, 2, \cdots, n-1$, *denotes the successive radii of the* N*-dimensional Spherical process with respect to* $D + \Gamma[D]$, *and* $y_0 = d_0$ *denotes the distance to a properly oriented* $(N-1)$*-dimensional hyperplane.*

PROOF. Clearly, $r_n / r_0 = \prod_{i=0}^{n-1} r_{i+1} / r_i$. Hence, we have

$$\frac{1}{n} \log \frac{r_n}{r_0} = \frac{1}{n}\left[\log \frac{r_1}{r_0} + \log \frac{r_2}{r_1} + \cdots + \log \frac{r_n}{r_{n-1}}\right]. \tag{6.14}$$

Let the Spherical process defined on $D + \Gamma[D]$ be at the point x_i at the ith stage of the process, where i is arbitrary. Then, r_i denotes the radius of the maximum N-sphere $K(x_i)$ of the Spherical process defined on $D + \Gamma[D]$. Since $D + \Gamma[D]$ is convex, it has an $(N-1)$-dimensional supporting hyperplane at each point of $\Gamma[D]$. Consequently, r_i also denotes the distance from x_i along a normal to an $(N-1)$-dimensional supporting hyperplane of $\Gamma[D]$ which is at a minimum distance from x_i. With reference to this hyperplane, consider the Spherical process, such as discussed in Theorem 6.1, where $y_i = d_i$ has the meaning attributed to it in Theorem 6.1. From Theorem 6.1, $E\{\log y_{i+1} / y_i \mid N\}$ is known. Further, since $D + \Gamma[D]$ is convex, $r_{i+1} \leqq y_{i+1}$. Thus, $\log r_{i+1} / r_i \leqq \log y_{i+1} / r_i = \log y_{i+1} / y_i$. Hence, $E\{\log r_{i+1} / r_i \mid N\}$ exists and

$$E\{\log r_{i+1} / r_i \mid N\} \leqq E\{\log y_{i+1} / y_i \mid N\} = E\{\log y_1 / y_0 \mid N\}$$

for $i = 0, 1, \cdots, n-1$. It is therefore permissible to apply the mathematical expectation operator to both sides of condition (6.13) (see Cramér [1], page 173).

Taking the expected value of both sides of (6.14) yields that

$$E\{1/n \log r_n / r_0 \mid N\} \leqq E\{\log y_1 / y_0 \mid N\}.$$

Thus, the proof is completed.

From Theorem 6.6 we are able to use (6.13) to obtain upper bounds for the asymptotic confidence intervals for the average distance of the N-dimensional Spherical process from the boundary $\Gamma[D]$ when $D + \Gamma[D]$ is any convex set.

7. Machine techniques.

7.1. *Generation of positions.* In order to utilize either the N-dimensional Spherical or Generalized Spherical process on a computer, it will be necessary to be able to generate positions uniformly at random on the surface of any given N-sphere. Actually, it is the ease with which one will be able to carry out this operation that makes these two classes of processes desirable. We shall restrict the subsequent discussion to the unit N-sphere, since there will be no resulting loss in generality.

The following procedure is one way to generate points on the surface of the unit N-sphere uniformly at random.

Generate N independent normal deviates x_i, $i = 1, 2, \cdots, N$, then corresponding to the point $x = (x_1, x_2, \cdots, x_N)$, locate a point y on the unit N-sphere having the N-direction cosines, $x_i / \sqrt{x_1^2 + x_2^2 + \cdots + x_N^2}$, $i = 1, 2,$

$\cdots$, N. The points y obtained this way will be uniformly distributed on the surface of the unit N-sphere. This follows by known properties of the normal distribution (see for example Cramér [1], Chap. 24). The essential details are that the density function of x, say $g(x) = g(x_1, x_2, \cdots, x_N)$, is an N-dimensional canonical normal density function, and it is known that the N-dimensional canonical normal density function has constant probability on surfaces of N-dimensional spheres. It is not necessary to go into the details of how to generate normal deviates, since Teichroew [21] gives detailed procedures for generating normal deviates on high-speed computers.

For reasons of practical feasibility and economy when using a computer, any of the processes discussed in this paper will require that they be modified by the introduction of truncation procedures. By a *truncation procedure* for a process, we mean that when the particular process being used has come within a prescribed distance, say δ, of the given boundary $\Gamma[D]$, the process is terminated. We next give attention to the question of truncation.

7.2. *δ-truncation of first order.* With respect to any of the three given classes of processes, a δ-truncation procedure of first order is given as follows: The first time the given process is within δ of the boundary $\Gamma[D]$, the process is terminated and the nearest point, say y, of the boundary $\Gamma[D]$ is recorded as the point to which the process would have converged and the value $f(y)$ is tallied.

The question then arises as to how this truncation procedure affects the solution of the Dirichlet problem, say at the point x_0. For the sake of brevity, and without loss of generality, we consider only the Spherical process for the present. If we use the δ-truncation procedure of first order, we shall be changing the estimate of Pr $(S(x_0, \phi), dx, D)$. The error introduced by this change will be studied from a heuristic point of view. However, we know from Section 4 that Pr $(S(x_0, \phi), dx, D)$ is a harmonic function in D and $\lim_{x_0 \epsilon D, x_0 \to x}$ Pr $(S(x_0, \phi), dx, D) = 1$ or 0, according as x is an inner point of dx or of $\Gamma[D] - dx$, where dx represents an elementary set of $\Gamma[D]$. Thus, Pr $(S(x_0, \phi), dx, D)$ is, except for the end points of dx, a continuous function on $D + \Gamma[D]$. Further, $f(x)$ was given to be continuous, and thus bounded, on $\Gamma[D]$. Let $M = \max_{x \epsilon \Gamma[D]} |f(x)|$. Further, the process is a Markov process with stationary independent increments. Consequently, given an $\epsilon > 0$, it follows that there exists a δ such that the maximum error in Pr $(S(x_0, \phi), dx, D)$ from δ-truncation of the first order is $< \epsilon/M$. Thus, the absolute value of the difference between $u(x_0)$ and the value obtained by using the δ-truncation process of first order can be made less than ϵ by selecting δ small enough.

7.3. *δ-truncation of higher order.* The following type of δ-truncation is suggested as a tentative procedure. This type of procedure represents an interesting field for further investigation. This procedure differs from the δ-truncation procedure of first order in that once having arrived at the point, say z, within distance δ of the boundary, a restricted N-dimensional solid angle is selected instead of the closest boundary point. The method then consists in proceeding in the direction selected until $\Gamma[D]$ is intersected. The value of $f(x)$ at this point of

intersection is used as the tally. For the Spherical or the General Spherical process, the solid angle is selected uniformly at random in a restricted region. The General Dirichlet Domain process would select the angle according to the normalized normal derivative of the Green's function of the particular admissible domain being utilized. Regardless of which process is being used, the solid angle is restricted to lie within the N-dimensional hemisphere determined by an infinite $(N - 1)$-dimensional hyperplane parallel to the supporting hyperplane of the boundary point nearest to the given interior point z and passing through z. This procedure seems worthy of study for the following reasons.

For the infinite $(N - 1)$-dimensional hyperplane in N-space the δ-truncation procedure of first order still produces an error. However the δ-truncation procedure suggested here provides an exact solution of this problem. This follows from an N-dimensional generalization of the material given in Kellogg [13], pp. 66–69. Consequently, for a well-behaved boundary $\Gamma[D]$ and any point sufficiently close to $\Gamma[D]$, it seems intuitively evident that the portion of the boundary near the point in question can be approximated reasonably well by an $(N - 1)$-dimensional hyperplane. If this is true, then clearly the δ-truncation procedure of higher order will contribute a smaller error than the δ-truncation process of first order. Other higher-order truncation procedures could utilize knowledge of the curvature of $\Gamma[D]$ when the direction is selected. Likewise, discrete analogs of these truncation procedures could be adopted in place of the classical interpolation methods being used near the boundary for discrete approximations to the Dirichlet problem.

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