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Application of Brownian Motion to the Equation of Kolmogorov-Petrovskii-Piskunov*

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1. Introduction

The content of this paper is a simplified proof of the theorem of Kolmogorov-Petrovskii-Piskunov [5] to the effect that *if $u = u(t, x)$ is the solution of*[1]

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} + u^2 - u \tag{1}$$

with initial datum

$$f(x) = \begin{cases} 1 & \text{if} \quad x > 0\,, \\ 0 & \text{if} \quad x < 0\,, \end{cases} \tag{2}$$

and if the number m is the median of $u[u(t, m) = \frac{1}{2}]$, then

$$\lim_{t\uparrow\infty} u(t, x + m) = w_{\sqrt{2}}(x) \tag{3}$$

exists and is a "wave" solution of (1) *travelling at speed $\sqrt{2}$*, i.e., $w_{\sqrt{2}}(x - \sqrt{2}\,t)$ solves (1), or, what is the same,

$$0 = \tfrac{1}{2} w''_{\sqrt{2}} + \sqrt{2}\, w'_{\sqrt{2}} + w^2_{\sqrt{2}} - w_{\sqrt{2}}\,. \tag{4}$$

Kolmogorov-Petrovskii-Piskunov proved that $m \sim \sqrt{2}\, t$. The estimate

$$m \leqq 2^{1/2} t - 2^{-3/2} \log t\,, \qquad t \uparrow \infty\,, \tag{5}$$

will emerge from the present proof. The precise comportment of m is unknown. The method of proof will make plain that *if the datum $u(0+, \cdot) = f$*

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[1] Kolmogorov-Petrovskii-Piskunov have $u - u^2$ in place of $u^2 - u$; the two problems are related by the substitution $u \to 1 - u$.

satisfies $0 \leqq f \leqq 1$ *and if, for fixed* $0 < b \leqq \sqrt{2}$,

$$\lim_{x \uparrow \infty} e^{bx}[1 - f(x)] = a \tag{2'}$$

exists, then

$$\lim_{t \uparrow \infty} u(t, x + ct) = w_c(x) \tag{3'}$$

exists and is a wave solution of (1) *travelling at speed* $c = 1/b + \frac{1}{2}b$, i.e., $w_c(x - ct)$ solves (1), or, what is the same,

$$0 = \tfrac{1}{2}w_c'' + cw_c' + w_c^2 - w_c\,. \tag{4'}$$

The gap between (3) and (3′), corresponding to data f with tails as in (2′) but for $\sqrt{2} < b < \infty$, is left open, though it will be clear that for the analogue of (3′) to hold you will have to travel along with the solution in a style intermediate between $\sqrt{2}\,t$ and m, i.e., you will have to look at $u(t, x + \sqrt{2}\,t - l)$ with $l \uparrow \infty$ more slowly than $\sqrt{2}\,t - m$. A nice problem is to confirm (3) for solutions of (1) in case the datum (2) is modified by permitting f to increase from 0 to 1 in $0 \leqq x \leqq 1$, say. This has been accomplished by Kanel [2], [3], [4] by the method of Kolmogorov-Petrovskii-Piskunov [5] for a wide class of equations

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} + c(u)$$

in place of (1). The case $c(u) = u(1-u)(u-\varepsilon)$, $0 < \varepsilon < \frac{1}{2}$, is of special interest in neuro-physiology; see Cohen [1] and Nagasawa [6]. The present method is easily extended (for what it is worth) to cover $c(u) = a[b_2 u^2 + b_3 u^3 + \cdots - u]$ with $u > 0$ and $0 \leqq b_2, b_3, \cdots$ summing to 1. The case $u(1-u)(u-\varepsilon)$ with $a = \varepsilon$, $b_2 = \varepsilon^{-1}(1+\varepsilon)$, $b_3 = -\varepsilon^{-1}$ is *not* included.

2. Branching

The basic model employed to deal with (1) is a simple branching process, defined as follows: At time $t = 0$, a single particle commences a standard Brownian motion $\mathfrak{x}$, starting from the origin and continuing for an exponential holding time T independent of $\mathfrak{x}$ with $P(T > t) = e^{-t}$. At this moment, the particle splits in two, the new particles continuing along independent Brownian paths starting from $\mathfrak{x}(T)$. These particles, in turn, are subject to the same splitting rule, with the result that, after an elapsed time $t > 0$, you have n particles located at $\mathfrak{x}_1, \cdots, \mathfrak{x}_n$ with $P(n = k) = e^{-t}(1 - e^{-t})^{k-1}$, $k \geqq 1$. The connection with (1) comes about through the formula

$$u(t, x) = E[f(x+\mathfrak{x}_1) \cdots f(x+\mathfrak{x}_n)], \tag{6}$$

expressing the solution of (1) in terms of its datum f. The proof is easy. Let $0 \leqq f \leqq 1$ to ensure the existence of the expectation, let u be *defined* by (6), and let H_t be the Green operator $\exp\{\frac{1}{2}t\, \partial^2/\partial x^2\}$ for $\partial u/\partial t = \frac{1}{2}\partial^2 u/\partial x^2$. Then you may split the expectation into two pieces, according to whether the original particle splits at some time $T \leqq t$ or not, and obtain

$$\begin{aligned} u(t, x) &= P(T>t)\int_{-\infty}^{\infty} P[\mathfrak{x}(t)+x \in dy]f(y) \\ &\quad + \int_0^t P(T \in dt')\int_{-x}^{x} P[\mathfrak{x}(t')+x \in dy]u^2(t-t', y) \\ &= e^{-t}H_t f(x) + \int_0^t e^{-t'}H_{t'}u^2(t-t', x)\, dt' . \end{aligned}$$

Now an easy differentiation produces (1) after making the substitution $t' \to t-t'$ in the integral. The case $f=(2)$ of Kolmogorov-Petrovskii-Piskunov is of special interest: by a self-evident symmetry,

$$u(t, x) = P\left[\min_{i \leqq n} \mathfrak{x}_i(t) + x > 0\right] = P\left[\max_{i \leqq n} \mathfrak{x}_i(t) < x\right]. \tag{7}$$

3. Wave Solutions

The facts as regards solutions of (4′) are presented in Kolmogorov-Petrovskii-Piskunov [5]; (4′) may be presented in the phase plane of $w=\xi$, $w'=\eta$ by

$$\begin{aligned} \xi' &= \eta , \\ \eta' &= 2\xi(1-\xi) - 2c\eta , \end{aligned}$$

and you have a saddle point at $\xi=\eta=0$, with an out-solution issuing into the first quadrant, and an attractive singular point at $\xi=1$, $\eta=0$ about which the solution spirals if $0 \leqq c < \sqrt{2}$ but not if $c \geqq \sqrt{2}$. You require solutions of (4′) with $c \geqq 0$, $w(-\infty)=0$, $w(+\infty)=1$, and $0<w<1$ between, so the spiralling rules out $c<\sqrt{2}$, but it is found that the out-solution meets all requirements for any $c \geqq \sqrt{2}$, providing a *bona fide* wave solution travelling at that speed; see Figure 1. The latter is unique up to a translation; it is denoted by w_c. The right-hand tail of w_c will be wanted later on. The fact is that w_c satisfies

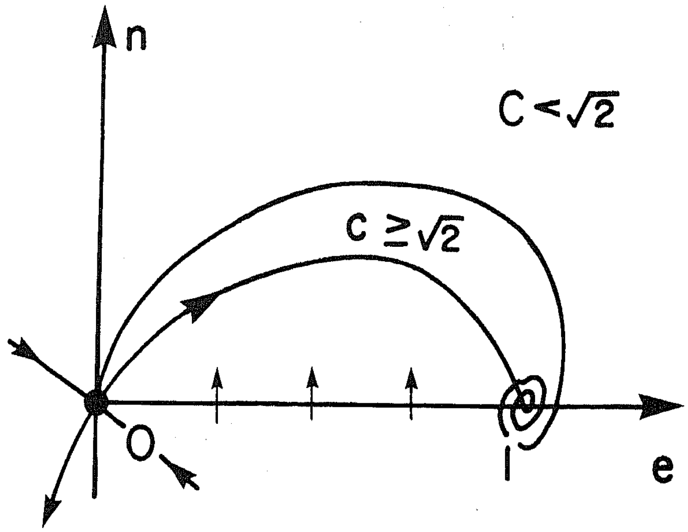

Figure 1

(2′) with $b = c - \sqrt{c^2 - 2}$ for any $c \geqq \sqrt{2}$, as you will easily check. Notice that this relation of b to c is inverted by $c = 1/b + \frac{1}{2}b$, and that as c runs from $\sqrt{2}$ to ∞, b runs from $\sqrt{2}$ to 0.

4. Lemma of Kolmogorov-Petrovskii-Piskunov

The main lemma used to prove (3) is as follows. Let u be the solution of (1) with datum $f = (2)$, let $0 < \varepsilon < 1$ be fixed, and let $\bar{x}$ be chosen as a function of $t > 0$ so as to make $u(t, \bar{x}) = \varepsilon$. It is plain from (6) that $\bar{x}$ is unique. The lemma states that *$u'(t, \bar{x})$ decreases with time.* For the proof, fix $t_0 > 0$ and $a > 0$, and let $v(t, x) = u(t + a, x + b) - u(t, x)$ with $b = \bar{x}(t_0 + a) - \bar{x}(t_0)$. Then

$$\frac{\partial v}{\partial t} = \frac{1}{2}\frac{\partial^2 v}{\partial x^2} + kv$$

with

$$k = u(t + a, x + b) + u(t, x) - 1,$$

and, by (2),

$$v(0+, x)\begin{cases} > 0 & \text{if} \quad x < 0, \\ < 0 & \text{if} \quad x > 0. \end{cases}$$

Besides, $v(t_0, x_0) = 0$ for $x_0 = \bar{x}(t_0)$. *It is to be proved* that $v(t_0, x) \leqq 0$ for $x > x_0$. Then you will have $v'(t_0, x_0) \leqq 0$, and the lemma will follow from that. Suppose, contrariwise, that $v(t_0, x_1) > 0$ for some $x_1 > x_0$. Then (t_0, x_1) *must be connected to $(t = 0) \times (x < 0)$ by a continuous curve C along which $v > 0$*, as in

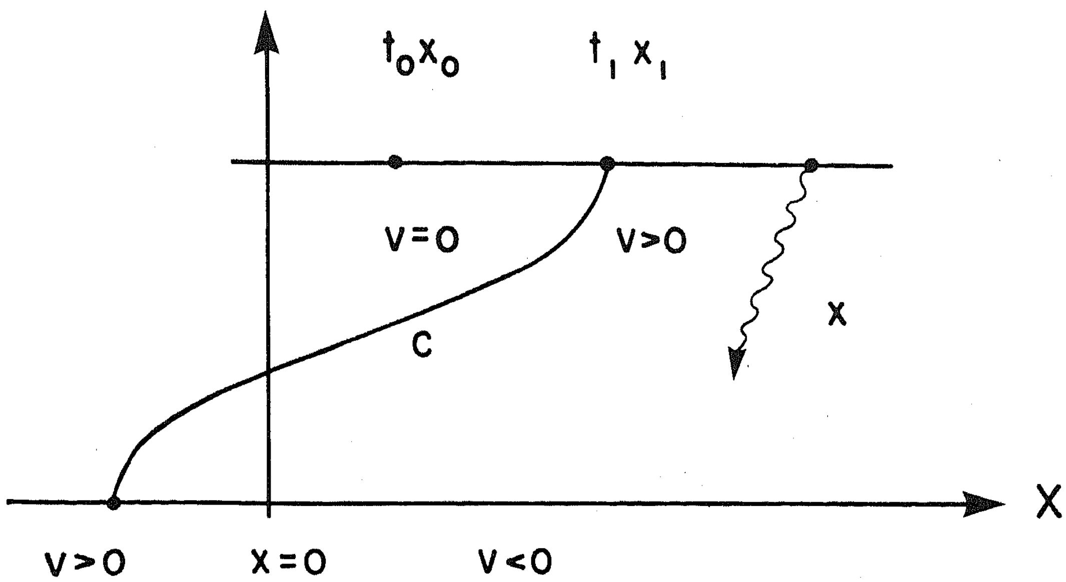

Figure 2

Figure 2. This is proved by writing $v(t_0, x)$ by means of Kac's formula:

$$v(t_0, x) = E \exp\left\{\int_{\mathfrak{t}}^{t_0} k[t_0 - t, \mathfrak{x}(t)]\, dt\right\} v[t_0 - \mathfrak{t}, \mathfrak{x}(\mathfrak{t})].$$

Here, $\mathfrak{x}$ is a standard Brownian motion starting at $\mathfrak{t}(0) = x$ running downwards as in Figure 2, and $0 \leqq \mathfrak{t} \leqq t_0$ is any Brownian stopping time. The desired contradiction is now obtained by assuming that the curve C of Figure 2 fails to exist. Fix $x = x_1$. Then, looking backwards from t_0, the first root $t \leqq t_0$ of $v[t_0 - \mathfrak{t}, \mathfrak{x}(t)] = 0$ defines a stopping time, and with that choice of $\mathfrak{t}$, the expectation vanishes, contradicting $v(t_0, x_1) > 0$. Now fix such a curve C and use the formula with $x = x_0$ and $\mathfrak{t}$ = the passage time to C. Then the expectation is positive, while the left-hand side vanishes, and the only way out is to admit that $v(t_0, x_1) > 0$ cannot be maintained. The proof is finished.

5. Proof of (3)

The proof of (3) now follows Kolmogorov-Petrovskii-Piskunov [5] with small improvements. By Section 4, $u'(t, \bar{x})$ decreases with time, so from

$$\int_{1/2}^{u(t,x+m)} \frac{d\varepsilon}{u'(t, \bar{x})} = x$$

with $m = \bar{x}$ for $\varepsilon = \frac{1}{2}$, you see that

$$\lim_{t \uparrow \infty} u(t, x + m) = w(x)$$

exists; in fact, $0 \leqq w \leqq 1$ is increasing with x, $w(0)=\frac{1}{2}$, and the tendency of $u(t, x+m)$ to $w(x)$ is by decrease (increase) if $x>0$ ($x<0$). The only point at issue is the identification of w as the wave solution for speed $\sqrt{2}$.

Step 1 is to prove (5): $m \leqq 2^{1/2}t - 2^{-3/2}\log t$ *for* $t \uparrow \infty$. By (7),

$$\begin{aligned}1-u(t,-x) &= P\left[\min_{i<n} \mathfrak{x}_i(t) < x\right] \\ &\leqq E[\text{the number of } i \leqq n \text{ for which } \mathfrak{x}_i(t) < x] \\ &= e^t P[\mathfrak{x}(t) < x] \\ &= e^t \int_{-\infty}^{x} \frac{e^{-y^2/2t}}{\sqrt{2\pi t}}\, dy ,\end{aligned}$$

as you may verify by use of (6) with $f = 1+\varepsilon$ (the indicator of $y \leqq x$) upon differentiating with regard to ε and putting $\varepsilon = 0$. Now a routine estimation confirms that

$$1-u(t, x+2^{1/2}t-2^{-3/2}\log t) = [1+o(1)]\frac{e^{-x/\sqrt{2}}}{2\sqrt{\pi}}$$

for $t \uparrow \infty$, and step 1 follows from the ensuing under-estimate

$$u(t, 2^{1/2}t-2^{-3/2}\log t) \geqq 1-\frac{1}{2\sqrt{\pi}}-o(1) > \frac{1}{2}.$$

Step 2 is to verify that w *is non-trivial*, i.e., $w \not\equiv \frac{1}{2}$. For $x<0$, $v = u(t, x+m)$ satisfies $\partial v/\partial t \geqq 0$. Now

$$\frac{\partial v}{\partial t} = \frac{1}{2}\frac{\partial^2 v}{\partial x^2} + m^{\bullet}\frac{\partial v}{\partial x} + v^2 - v , \tag{8}$$

so

$$0 \leqq \tfrac{1}{2}v'(t, o) + \tfrac{1}{2}m^{\bullet} - \int_{-x}^{0} v(1-v)\, dx ,$$

and $w \not\equiv \frac{1}{2}$ follows from the fact that $0 < v'(t, 0)$ is decreasing, $\lim m^{\bullet} \leqq \sqrt{2}$, and $v \uparrow w$ for $x<0$:

$$\int_{-\infty}^{0} w(1-w)\, dx \leqq \tfrac{1}{2}\lim_{t\uparrow\infty} v'(t, o) + \lim_{t\uparrow\infty} \tfrac{1}{2}m^{\bullet} < \infty . \tag{9}$$

Step 3. Equation (8) implies that, for $t\uparrow\infty$ *and any* $-\infty<x<\infty$,

$$\begin{aligned} o(1) &= \int_t^{t+1} dt' \int_0^x d\xi \int_0^\xi d\eta \left[\frac{1}{2}\frac{\partial^2 v}{\partial n^2} + m^\bullet \frac{\partial v}{\partial \eta} + v^2 - v\right] \\ &= \tfrac{1}{2}w(x) - \tfrac{1}{2} + xw'(0) \\ &\quad + [m(t+1) - m(t)] \times \left[\int_0^x (w - \tfrac{1}{2})\, d\xi + o(1)\right] \\ &\quad + \int_0^x d\xi \int_0^\xi (w^2 - w)\, d\eta + o(1)\,. \end{aligned}$$

The third line is justified by the mean-value theorem, keeping in mind that $m^\bullet \geqq 0$, as is plain from (6). Fix x so as to make $\int_0^x (w-\frac{1}{2})\, d\xi \neq 0$. You see at once that $\lim_{t\uparrow\infty}[m(t+1)-m(t)] = c$ exists, and it requires only two differentiations with regard to x to obtain (4′), proving that w is a (non-trivial) wave-form. Now c is necessarily at least $\sqrt{2}$, and to finish the proof, you have only to notice from (9) that

$$\frac{1}{\sqrt{2}} \geqq \underline{\lim}\, \tfrac{1}{2} m^\bullet \geqq \int_{-\infty}^0 w(1-w)\, dx - \tfrac{1}{2}w'(0)\,,$$

and from (4′) that

$$0 = \tfrac{1}{2}w'(0) + \tfrac{1}{2}c - \int_{-\infty}^0 w(1-w)\, dx\,,$$

whence $c=\sqrt{2}$.

A little variation of the proof confirms that

$$\lim_{t\uparrow\infty} T^{-1} m(t+T) - m(t) = c$$

for any T, i.e., $\sqrt{2}\,t - m(t)$ *is slowly varying*. More information about m would be desirable. It is easy to check from (7) that if $M = \max_{i\leqq n} \mathfrak{x}_i(t)$, then

$$E(M) = m + \int_{-\infty}^{\infty} x w'_{\sqrt{2}}(x)\, dx + o(1)$$

if $w_{\sqrt{2}}(0) = \frac{1}{2}$. $E(M)$ should be computable, though I do not know how to do it.

6. Proof of (3′)

The proof of (3′) is very easy: $w(x-ct)$ is a solution of (1) only if

$$w(x)=E[w(x+\mathfrak{x}_1+ct)\cdots w(x+\mathfrak{x}_n+ct)]\,.$$

Now if f satisfies (5), then with a suitable translate of w_c, you have

$$w_c[x(1-\delta)]\leqq f(x)\leqq w_c[x(1+\delta)]$$

for $\delta>0$ and $x\uparrow\infty$. But by (3), (5), and (7),

$$P\Big[\min_{i\leq n}\mathfrak{x}_i(t)+ct>\tfrac{1}{4}\log t\Big]=1-o(1)$$

for $t\uparrow\infty$, c being at least $\sqrt{2}$, so, with overwhelming probability, all the variables under the expectation sign in (6) are far to the right where f is comparable to w_c. The upshot is that

$$w_c[x(1-\delta)]+o(1)\leqq u(t,x+ct)\leqq w_c[x(1+\delta)]+o(1)$$

for $t\uparrow\infty$. The proof is finished.

7. A Martingale

The martingale

$$\mathfrak{z}(t)=e^{-t}\sum_{i=1}^{n}e^{-b\mathfrak{x}_i(t)-b^2t/2}$$

is closely related to Section 6. Fix $c=1/b+\frac{1}{2}b$. Then the expectation

$$u=E[e^{-\mathfrak{z}(t)}]=E[e^{-b(\mathfrak{x}_1(t)+ct}\cdots e^{-b(\mathfrak{x}_n(t)+ct)}]$$

is of the form (6) with $x=0$ and $f=\exp\{-e^{-bx}\}$, and if $b\leqq\sqrt{2}$, you have

$$\lim_{t\uparrow\infty}u=w_c(0)\,.$$

But also $\lim_{t\uparrow\infty}\mathfrak{z}(t)$ exists by the martingale convergence theorem, and this fact gives rise to an integral formula for the wave-form:

$$w_c(x)=E\exp\Big\{-\lim_{t\uparrow\infty}\mathfrak{z}(t)e^{-bx}\Big\}=\int_0^\infty e^{-ae^{-bx}}\,dP\Big[\lim_{t\uparrow\infty}\mathfrak{z}(t)<a\Big]\,.$$

For $b > \sqrt{2}$, the limit also exists, but now $\lim_{t \uparrow \infty} u = 1$, i.e., $P[\lim_{t \uparrow \infty} \mathfrak{z}(t) = 0] = 1$, since, in the opposite case, $w(x) = E \exp\{-\lim_{t \uparrow \infty} \mathfrak{z}(t) e^{-bx}\}$ would be a wave-form with tail $1 - w(x) = o[e^{-x/2}]$, and no such wave-form exists.

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