CAN ONE HEAR THE SHAPE OF A DRUM?

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To George Eugene Uhlenbeck on the occasion of his sixty-fifth birthday

> "La Physique ne nous donne pas seulement l'occasion de résoudre des problèmes . . . , elle nous fait presentir la solution." H. POINCARÉ.

Before I explain the title and introduce the theme of the lecture I should like to state that my presentation will be more in the nature of a leisurely excursion than of an organized tour. It will not be my purpose to reach a specified destination at a scheduled time. Rather I should like to allow myself on many occasions the luxury of stopping and looking around. So much effort is being spent on streamlining mathematics and in rendering it more efficient, that a solitary transgression against the trend could perhaps be forgiven.

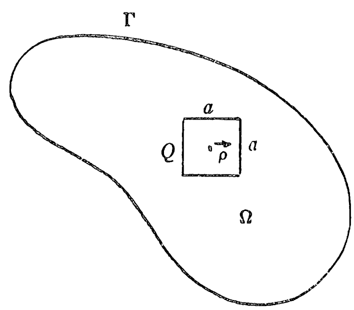

FIG. 1

1. And now to the theme and the title.

It has been known for well over a century that if a membrane Ω, held fixed along its boundary Γ (see Fig. 1), is set in motion its displacement (in the direction perpendicular to its original plane)

$$F(x, y; t) \equiv F(\vec{\rho}; t)$$

obeys the wave equation

$$\frac{\partial^2 F}{\partial t^2} = c^2 \nabla^2 F,$$

where c is a certain constant depending on the physical properties of the membrane and on the tension under which the membrane is held.

I shall choose units to make $c^2 = \frac{1}{2}$.

Of special interest (both to the mathematician and to the musician) are solutions of the form

$$F(\vec{\rho};t) = U(\vec{\rho})e^{i\omega t},$$

for, being harmonic in time, they represent the *pure tones* the membrane is capable of producing. These special solutions are also known as normal modes.

To find the normal modes we substitute $U(\vec{\rho})e^{i\omega t}$ into the wave equation and see that U must satisfy the equation $\frac{1}{2}\nabla^2 U + \omega^2 U = 0$ with the boundary condition $U=0$ on the boundary Γ of Ω, corresponding to the membrane being held fixed along its boundary.

The meaning of "$U=0$ on Γ" should be made clear; for sufficiently smooth boundaries it simply means that $U(\vec{\rho})\to 0$ as $\vec{\rho}$ approaches a point of Γ (from the inside). To show that a membrane is capable of producing a discrete spectrum of pure tones i.e. that there is a discrete sequence of ω's $\omega_1 \leqq \omega_2 \leqq \omega_3 \leqq \cdots$ for which nontrivial solutions of

$$\tfrac{1}{2}\nabla^2 U + \omega^2 U = 0, \qquad U = 0 \text{ on } \Gamma,$$

exist, was one of the great problems of 19th century mathematical physics. Poincaré struggled with it and so did many others.

The solution was finally achieved in the early years of our century by the use of the theory of integral equations.

We now know and I shall ask you to believe me if you do not, that for regions Ω bounded by a smooth curve Γ there is a sequence of numbers $\lambda_1 \leqq \lambda_2 \leqq \cdots$ called eigenvalues such that to each there corresponds a function $\psi(\vec{\rho})$, called an eigenfunction, such that

$$\tfrac{1}{2}\nabla^2 \psi_n + \lambda_n \psi_n = 0$$

and $\psi_n(\vec{\rho})\to 0$ as $\vec{\rho}\to$a point of Γ.

It is customary to normalize the ψ's so that

$$\iint_\Omega \psi_n^2(\vec{\rho})d\vec{\rho} = 1.$$

Note that I use $d\vec{\rho}$ to denote the element of integration (in Cartesian coordinates, e.g., $d\vec{\rho}\equiv dxdy$).

2. The focal point of my exposition is the following problem:

Let Ω_1 and Ω_2 be two plane regions bounded by curves Γ_1 and Γ_2 respectively, and consider the eigenvalue problems:

$\frac{1}{2}\nabla^2 U + \lambda U = 0$ in Ω_1	$\frac{1}{2}\nabla^2 V + \mu V = 0$ in Ω_2
with	with
$U = 0$ on Γ_1	$V = 0$ on Γ_2.

Assume that for each n the eigenvalue λ_n for Ω_1 is equal to the eigenvalue μ_n

for Ω_2. Question: Are the regions Ω_1 and Ω_2 congruent in the sense of Euclidean geometry?

I first heard the problem posed this way some ten years ago from Professor Bochner. Much more recently, when I mentioned it to Professor Bers, he said, almost at once: "You mean, if you had perfect pitch could you find the shape of a drum."

You can now see that the "drum" of my title is more like a tambourine (which really is a membrane) and that stripped of picturesque language the problem is whether we can determine Ω if we know all the eigenvalues of the eigenvalue problem

$$\tfrac{1}{2}\nabla^2 U + \lambda U = 0 \text{ in } \Omega,$$
$$U = 0 \text{ on } \Gamma.$$

3. Before I go any further let me say that as far as I know the problem is still unsolved. Personally, I believe that one cannot "hear" the shape of a tambourine but I may well be wrong and I am not prepared to bet large sums either way.

What I propose to do is to see how much about the shape can be inferred from the knowledge of all the eigenvalues, and to impress upon you the multitude of connections between our problem and various parts of mathematics and physics.

It should perhaps be stated at this point that throughout the paper only *asymptotic properties* of large eigenvalues will be used. This may represent, of course, a serious loss of information and it may perhaps be argued that *precise* knowledge of *all* the eigenvalues may be sufficient to determine the shape of the membrane. It should also be pointed out, however, that quite recently John Milnor constructed two noncongruent sixteen dimensional tori whose Laplace-Betrami operators have exactly the same eigenvalues (see his one page note "Eigenvalues of the Laplace operator on certain manifolds" Proc. Nat. Acad. Sc., 51 (1964) 542).

4. The first pertinent result is that one can "hear" the area of Ω. This is an old result with a fascinating history which I shall now relate briefly.

At the end of October 1910 the great Dutch physicist H. A. Lorentz was invited to Göttingen to deliver the Wolfskehl lectures. Wolfskehl, by the way, endowed a prize for proving, or disproving, Fermat's last theorem and stipulated that in case the prize is not awarded the proceeds from the principal be used to invite eminent scientists to lecture at Göttingen.

Lorentz gave five lectures under the overall title "Alte und neue Fragen der Physik"—Old and new problems of physics—and at the end of the fourth lecture he spoke as follows (in free translation from the original German): "In conclusion there is a mathematical problem which perhaps will arouse the interest of mathematicians who are present. It originates in the radiation theory of Jeans.

"In an enclosure with a perfectly reflecting surface there can form standing electromagnetic waves analogous to tones of an organ pipe; we shall confine our attention to very high overtones. Jeans asks for the energy in the frequency interval $d\nu$. To this end he calculates the number of overtones which lie between the frequencies ν and $\nu+d\nu$ and multiplies this number by the energy which belongs to the frequency ν, and which according to a theorem of statistical mechanics is the same for all frequencies.

"It is here that there arises the mathematical problem to prove that the number of sufficiently high overtones which lies between ν and $\nu+d\nu$ is independent of the shape of the enclosure and is simply proportional to its volume. For many simple shapes for which calculations can be carried out, this theorem has been verified in a Leiden dissertation. There is no doubt that it holds in general even for multiply connected regions. Similar theorems for other vibrating structures like membranes, air masses, etc. should also hold."

If one expresses this conjecture of Lorentz in terms of our membrane, it emerges in the form:

$$N(\lambda) = \sum_{\lambda_n < \lambda} 1 \sim \frac{|\Omega|}{2\pi} \lambda.$$

Here $N(\lambda)$ is the number of eigenvalues less than λ, $|\Omega|$ the area of Ω and $\sim$ means that

$$\lim_{\lambda \to \infty} \frac{N(\lambda)}{\lambda} = \frac{|\Omega|}{2\pi} \cdot$$

There is an apocryphal report that Hilbert predicted that the theorem would not be proved in his life time. Well, he was wrong by many, many years. For less than two years later Herman Weyl, who was present at the Lorentz' lecture and whose interest was aroused by the problem, proved the theorem in question, i.e. that as $\lambda \to \infty$

$$N(\lambda) \sim \frac{|\Omega|}{2\pi} \lambda.$$

Weyl used in a masterly way the theory of integral equations, which his teacher Hilbert developed only a few years before, and his proof was a crowning achievement of this beautiful theory. Many subsequent developments in the theory of differential and integral equations (especially the work of Courant and his school) can be traced directly to Weyl's memoir on the conjecture of Lorentz.

5. Let me now consider briefly a different physical problem which too is closely related to the problem of the distribution of eigenvalues of the Laplacian.

It can be taken as a basic postulate of classical statistical mechanics that if a system of M particles confined to a volume Ω is in equilibrium with a thermostat of temperature T the probability of finding specified particles at $\vec{r}_1, \vec{r}_2, \cdots, \vec{r}_M$ (within volume elements $\overrightarrow{dr_1}, \overrightarrow{dr_2}, \cdots, \overrightarrow{dr_M}$) is

$$\frac{\exp\left[-\frac{1}{kT}V(\vec{r}_1,\cdots,\vec{r}_M)\right]d\vec{r}_1\cdots d\vec{r}_M}{\int_\Omega\cdots\int_\Omega \exp\left[-\frac{1}{kT}V(\vec{r}_1\cdots\vec{r}_M)\right]d\vec{r}_1\cdots d\vec{r}_M},$$

where $V(\vec{r}_1, \cdots, \vec{r}_M)$ is the interaction potential of the particles and $k=R/N$ with R the "gas constant" and N the Avogadro number.

For identical particles each of mass m obeying the so called Boltzmann statistics the corresponding assumption in quantum statistical mechanics seems much more complicated. One starts with the Schrödinger equation

$$\frac{\hbar^2}{2m}\nabla^2\psi - V(\vec{r}_1,\cdots,\vec{r}_M)\psi = -E\psi \quad \left(\hbar = \frac{h}{2\pi}, \text{ where } h \text{ is the Planck constant}\right)$$

with the boundary condition $\lim \psi(\vec{r}_1. \cdots, \vec{r}_M)=0$, whenever at least one $\vec{r}_k$ approaches the boundary of Ω. (This boundary condition has the effect of confining the particles to Ω.) Let $E_1 \leqq E_2 \leqq E_3 \leqq \cdots$ be the eigenvalues and $\psi_1, \psi_2, \cdots$ the corresponding normalized eigenfunctions. Then the basic postulate is that the probability of finding specified particles at $\vec{r}_1, \vec{r}_2, \cdots, \vec{r}_M$ (within $d\vec{r}_1, \cdots, d\vec{r}_M$) is

$$\frac{\sum_{s=1}^{\infty} e^{-E_s/kT}\psi_s^2(\vec{r}_1,\cdots,\vec{r}_M)d\vec{r}_1\cdots d\vec{r}_M}{\sum_{s=1}^{\infty} e^{-E_s/kT}}.$$

There are actually no known particles obeying the Boltzmann statistics. But don't let this worry you—for our purposes this regrettable fact is immaterial.

Now, let us specialize our discussion to the case of an *ideal* gas which, by *definition*, means that $V(\vec{r}_1, \cdots, \vec{r}_M)\equiv 0$.

Classically, the probability of finding specified particles at $\vec{r}_1, \cdots, \vec{r}_M$ is clearly

$$\frac{d\vec{r}_1\cdots d\vec{r}_M}{|\Omega|^M},$$

where $|\Omega|$ is now the volume of Ω.

Quantum mechanically the answer is not nearly so explicit. The Schrödinger equation for an ideal gas is

$$\frac{\hbar^2}{2m}\nabla^2\psi = -E\psi$$

and the equation is obviously separable.

If I now consider the three-dimensional (rather than the $3M$-dimensional) eigenvalue problem

$$\tfrac{1}{2}\nabla^2\psi(\vec{r}) = -\lambda\psi(\vec{r}), \qquad \vec{r}\in\Omega,$$
$$\psi(\vec{r})\to 0 \quad \text{as} \quad \vec{r}\to \text{the boundary of } \Omega,$$

it is clear that the E_s as well as the $\psi_s(\vec{r}_1, \cdots, \vec{r}_M)$ are easily expressible in terms of the λ's and corresponding $\psi(r)$'s.

The formula for the probability of finding specified particles at $\vec{r}_1, \cdots, \vec{r}_M$ turns out to be

$$\prod_{k=1}^{M} \frac{\sum_{n=1}^{\infty} \exp\left[-\frac{\lambda_n \hbar^2}{mkT}\right]\psi_n^2(r_k)}{\sum_{n=1}^{\infty} \exp\left[-\frac{\lambda_n \hbar^2}{mkT}\right]}\, d\vec{r}_k.$$

Now, as $\hbar\to 0$ (or as $T\to\infty$) the quantum mechanical result should go over into the classical one and this immediately leads to the conjecture that as

$$\tau\to 0 \qquad \left[\tau = \frac{\hbar^2}{mkT}\right],$$
$$\sum_{n=1}^{\infty} e^{-\lambda_n\tau}\psi_n^2(\vec{r}) \sim \frac{1}{|\Omega|}\sum_{n=1}^{\infty} e^{-\lambda_n\tau}.$$

If instead of a realistic three-dimensional container Ω I consider a two-dimensional one, the result would still be the same

$$\sum_{n=1}^{\infty} e^{-\lambda_n\tau}\psi_n^2(\vec{r}) \sim \frac{1}{|\Omega|}\sum_{n=1}^{\infty} e^{-\lambda_n\tau}, \qquad \tau\to 0,$$

except that now $|\Omega|$ is the area of Ω rather than the volume.

Clearly the result is expected to hold only for $\vec{r}$ in the interior of Ω.

If we believe Weyl's result that (in the two-dimensional case)

$$N(\lambda) \sim \frac{|\Omega|}{2\pi}\lambda, \qquad \lambda\to\infty,$$

it follows immediately by an Abelian theorem that

$$\frac{1}{|\Omega|}\sum_{n=1}^{\infty} e^{-\lambda_n\tau} \sim \frac{1}{2\pi\tau}, \qquad \tau\to 0,$$

and hence that

$$\sum_{n=1}^{\infty} e^{-\lambda_n\tau}\psi_n^2(\vec{r}) \sim \frac{1}{2\pi\tau} = \frac{1}{2\pi}\int_0^{\infty} e^{-\lambda\tau}\,d\lambda.$$

Setting $A(\lambda) = \sum_{\lambda_n<\lambda}\psi_n^2(\vec{r})$, we can record the last result as

$$\int_0^\infty e^{-\lambda\tau}\, dA(\lambda) \sim \frac{1}{2\pi}\int_0^\infty e^{-\lambda\tau}\, d\lambda, \qquad \tau \to 0.$$

Since $A(\lambda)$ is nondecreasing we can apply the Hardy-Littlewood-Karamata Tauberian theorem and conclude what everyone would be tempted to conclude, namely that

$$A(\lambda) = \sum_{\lambda_n<\lambda} \psi_n^2(\vec{r}) \sim \frac{\lambda}{2\pi}, \qquad \lambda \to \infty,$$

for every $\vec{r}$ in the interior of Ω.

Though this asymptotic formula is thus nearly "obvious" on "physical grounds," it was not until 1934 that Carleman succeeded in supplying a rigorous proof.

In concluding this section it may be worthwhile to say a word about the "strategy" of our approach.

We are primarily interested, of course, in asymptotic properties of λ_n for large n. This can be approached by the device of studying the Dirichlet series

$$\sum_{n=1}^\infty e^{-\lambda_n t}$$

for small t. This in turn is most conveniently approached through the series

$$\sum_{n=1}^\infty e^{-\lambda_n t}\psi_n^2(\vec{\rho}) = \int_0^\infty e^{-\lambda t}\, dA(\lambda)$$

and thus we are led to the Abelian-Tauberian interplay described above.

6. It would seem that the physical intuition ought not only provide the mathematician with interesting and challenging conjectures, but also show him the way toward a proof and toward possible generalizations.

The context of the theory of black body radiation or that of quantum statistical mechanics, however, is too far removed from elementary intuition and too full of daring and complex physical extrapolations to be of much use even in seeking the kind of understanding that makes a mathematician comfortable, let alone in pointing toward a rigorous proof.

Fortunately, in a much more elementary context the problem of the distribution of eigenvalues of the Laplacian becomes quite tractable. Proofs emerge as natural extensions of physical intuition and interesting generalizations come within reach.

7. The physical context in question is that of *diffusion theory*, another branch of nineteenth century mathematical physics.

Imagine "stuff," initially concentrated at $\vec{\rho}(\equiv(x_0, y_0))$, diffusing through a plane region Ω bounded by Γ. Imagine furthermore that the stuff gets absorbed ("eaten") at the boundary.

The concentration $P_\Omega(\vec{\rho}\,|\,\vec{r};\,t)$ of matter at $\vec{r}(\equiv(x,\,y))$ at time t obeys the differential equation of diffusion

$$\text{(a)} \qquad \frac{\partial P_\Omega}{\partial t} = \frac{1}{2}\,\nabla^2 P_\Omega,$$

the boundary condition

$$\text{(b)} \qquad P_\Omega(\vec{\rho}\,|\,\vec{r};\,t) \to 0 \text{ as } \vec{r} \text{ approaches a boundary point,}$$

and the initial condition

$$\text{(c)} \qquad P_\Omega(\vec{\rho}\,|\,\vec{r};\,t) \to \delta(\vec{r} - \vec{\rho}) \quad \text{as} \quad t \to 0;$$

here $\delta(\vec{r}-\vec{\rho})$ is the Dirac "delta function," with "value" ∞ if $\vec{r} = \vec{\rho}$ and 0 if $\vec{r} \neq \vec{\rho}$.

The boundary condition (b) expresses the fact that the boundary is absorbing and the initial condition (c) the fact that initially all the "stuff" was concentrated at $\vec{\rho}$.

I have again chosen units so as to make the diffusion constant equal to $\frac{1}{2}$.

As is well known the concentration $P_\Omega(\vec{\rho}\,|\,\vec{r};\,t)$ can be expressed in terms of the eigenvalues λ_n and normalized eigenfunctions $\psi_n(\vec{r})$ of the problem

$$\tfrac{1}{2}\nabla^2\psi + \lambda\psi = 0 \text{ in } \Omega,$$
$$\psi = 0 \text{ on } \Gamma.$$

In fact, $P_\Omega(\vec{\rho}\,|\,\vec{r};\,t) = \sum_{n=1}^{\infty} e^{-\lambda_n t}\psi_n(\vec{\rho})\psi_n(\vec{r})$.

Now, for small t, it appears intuitively clear that particles of the diffusing stuff will not have had enough time to have felt the influence of the boundary Ω. As particles begin to diffuse they may not be aware, so to speak, of the disaster that awaits them when they reach the boundary.

We may thus expect that in some approximate sense

$$P_\Omega(\vec{\rho}\,|\,\vec{r};\,t) \sim P_0(\vec{\rho}\,|\,\vec{r};\,t), \quad \text{as} \quad t \to 0,$$

where $P_0(\vec{\rho}\,|\,\vec{r};\,t)$ still satisfies the same diffusion equation

$$\text{(a}'\text{)} \qquad \frac{\partial P_0}{\partial t} = \tfrac{1}{2}\,\nabla^2 P_0$$

and the same initial condition

$$\text{(c}'\text{)} \qquad P_0(\vec{\rho}\,|\,\vec{r};\,t) = \delta(\vec{r} - \vec{\rho}), \qquad t \to 0,$$

but is otherwise unrestricted.

Actually there is a slight additional restriction without which the solution is not unique (a remarkable fact discovered some years ago by D. V. Widder). The restriction is that $P_0 \geqq 0$ (or more generally that P_0 be bounded from below).

A similar restriction for P_Ω is not needed since for diffusion in a *bounded* region it follows automatically.

An explicit formula for P_0 is, of course, well known. It is

$$P_0(\vec{\rho}\,|\,\vec{r};t) = \frac{1}{2\pi t}\exp\left[-\frac{\|\vec{r}-\vec{\rho}\|^2}{2t}\right],$$

where $\|\vec{r}-\vec{\rho}\|$ denotes the Euclidean distance between $\vec{\rho}$ and $\vec{r}$.

I can now state a little more precisely the principle of "not feeling the boundary" explained a moment ago.

The statement is that as $t\to 0$

$$P_\Omega(\vec{\rho}\,|\,\vec{r};t) = \sum_{n=1}^{\infty} e^{-\lambda_n t}\psi_n(\vec{\rho})\psi_n(\vec{r}) \sim \frac{1}{2\pi t}\exp\left[-\frac{\|\vec{r}-\vec{\rho}\|^2}{2t}\right] = P_0(\vec{\rho}\,|\,\vec{r};t),$$

where $\sim$ stands here for "is approximately equal to." This is a bit vague but let it go at that for the moment.

If we can trust this formula even for $\vec{\rho}=\vec{r}$ we get

$$\sum_{n=1}^{\infty} e^{-\lambda_n t}\psi_n^2(\vec{r}) \sim \frac{1}{2\pi t}$$

and if we display still more optimism we can integrate the above and, making use of the normalization condition

$$\int_\Omega \psi_n^2(\vec{r})\,d\vec{r} = 1,$$

obtain

$$\sum_{n=1}^{\infty} e^{-\lambda_n t} \sim \frac{|\Omega|}{2\pi t}.$$

We recognize immediately the formulas discussed a while back in connection with the quantum-statistical-mechanical treatment of the ideal gas. If we apply the Hardy-Littlewood-Karamata theorem, alluded to before, we obtain as corollary the theorems of Carleman and Weyl.

To do this, however, we must be allowed to interpret $\sim$ as meaning "asymptotic to."

8. Now, a little mathematical soul-searching. Aren't we as far from a rigorous treatment as we were before? True, diffusion is more familiar than black body radiation or quantum statistics. But familiarity gives comfort, at best, and comfort may still be (and often is) miles away from the rigor demanded by mathematics.

Let us see then what we can do about tightening the loose talk.

First let me dispose of a few minor items which may cause you worry.

When I write $\psi=0$ on Γ or $P(\vec{\rho}\,|\,\vec{r};t)\to 0$ as $\vec{r}$ approaches a boundary point of Ω there is always a question of interpretation.

Let me assume that Γ is sufficiently regular so that no ambiguity arises i.e.

$$P(\vec{\rho} \mid \vec{r}; t) \to 0 \quad \text{as} \quad \vec{r} \to \text{a boundary point of } \Omega,$$

means exactly what it says, while $\psi = 0$ on Γ means

$$\psi \to 0 \quad \text{as} \quad \vec{r} \to \text{a boundary point of } \Omega.$$

Likewise, $P(\vec{\rho} \mid \vec{r}; t) \to \delta(\vec{r} - \vec{\rho})$ as $t \to 0$, has the obvious interpretation, i.e.

$$\lim_{t \to 0} \iint_A P(\vec{\rho} \mid \vec{r}; t)\, d\vec{r} = 1$$

for every open set A containing $\vec{\rho}$.

Now, to more pertinent items. If the mathematical theory of diffusion corresponds in any way to physical reality we should have the inequality

$$P_\Omega(\vec{\rho} \mid \vec{r}; t) \leqq P_0(\vec{\rho} \mid \vec{r}; t) = \frac{\exp\left[- \dfrac{\|\vec{\rho} - \vec{r}\|^2}{2t} \right]}{2\pi t}.$$

For surely less stuff will be found at $\vec{r}$ at time t if there is a possibility of matter being destroyed (on the boundary Γ of Ω) than if there were no possibility of such destruction.

Now let Q be a square with center at $\vec{\rho}$ totally contained in Ω. Let its boundary act as an absorbing barrier and denote by $P_Q(\vec{\rho} \mid \vec{r}; t)$, $\vec{r} \in Q$, the corresponding concentration at $\vec{r}$ at time t.

In other words, P_Q satisfies the differential equation

$$\text{(a}''\text{)} \qquad \frac{\partial P_Q}{\partial t} = \frac{1}{2} \nabla^2 P_Q$$

and the initial condition

$$\text{(c}''\text{)} \qquad P_Q(\vec{\rho} \mid \vec{r}; t) \to \delta(\vec{r} - \vec{\rho}) \quad \text{as} \quad t \to 0.$$

It also satisfies the boundary condition

$$\text{(b}''\text{)} \qquad P_Q(\vec{\rho} \mid \vec{r}; t) \to 0 \quad \text{as} \quad \vec{r} \to \text{a boundary point of } Q.$$

Again it appears obvious that

$$P_Q(\vec{\rho} \mid \vec{r}; t) \leqq P_\Omega(\vec{\rho} \mid \vec{r}; t), \qquad \vec{r} \in Q,$$

for the diffusing stuff which reaches the boundary of Q is lost as far as P_Q is concerned but *need not* be lost as a contribution to P_Ω.

Q has been chosen so simply because $P_Q(\vec{\rho} \mid \vec{r}; t)$ is known explicitly, and, in particular

$$P_Q(\vec{\rho}\mid\vec{\rho};t)=\frac{4}{a^2}\sum_{\substack{m,n\\ \text{odd integers}}}\exp\left[-\frac{(m^2+n^2)\pi^2}{2a^2}t\right],$$

where a is the side of the square.

The combined inequalities

$$P_Q(\vec{\rho}\mid\vec{r};t)\leqq P_\Omega(\vec{\rho}\mid\vec{r};t)\leqq\frac{\exp\left[-\dfrac{\|\vec{r}-\vec{\rho}\|^2}{2t}\right]}{2\pi t}$$

hold for all $\vec{r}\in Q$ and in particular for $\vec{r}=\vec{\rho}$. In this case we get

$$\frac{4}{a^2}\sum_{\substack{m,n\\ \text{odd integers}}}\exp\left[-\frac{(m^2+n^2)\pi^2}{2a^2}t\right]\leqq\sum_{n=1}^{\infty}e^{-\lambda_n t}\psi_n^2(\vec{\rho})\leqq\frac{1}{2\pi t}$$

and it is a simple matter to prove that as $t\to0$ we have *asymptotically*

$$\frac{4}{a^2}\sum_{\substack{m,n\\ \text{odd integers}}}\exp\left[-\frac{(m^2+n^2)\pi^2}{2a^2}t\right]\sim\frac{1}{2\pi t}\cdot$$

Thus asymptotically for $t\to0$ $\sum_{n=1}^{\infty}e^{-\lambda_n t}\psi_n^2(\vec{\rho})\sim1/2\pi t$ and Carleman's theorem follows.

It is only a little harder to prove Weyl's theorem.

If one integrates over Q the inequality

$$\frac{4}{a^2}\sum_{\substack{m,n\\ \text{odd}}}\exp\left[-\frac{(m^2+n^2)\pi^2}{2a^2}t\right]\leqq\sum_{n=1}^{\infty}e^{-\lambda_n t}\psi_n^2(\vec{\rho})$$

one obtains

$$4\sum_{\substack{m,n\\ \text{odd}}}\exp\left[-\frac{(m^2+n^2)\pi^2}{2a^2}t\right]\leqq\sum_{n=1}^{\infty}e^{-\lambda_n t}\iint_Q\psi_n^2(\vec{\rho})\,d\vec{\rho}.$$

We now cover Ω with a net of squares of side a, as shown in Fig. 2, and keep only those contained in Ω. Let $N(a)$ be the number of these squares and let $\Omega(a)$ be the union of all these squares. We have

$$\sum_{n=1}^{\infty}e^{-\lambda_n t}=\sum_{n=1}^{\infty}e^{-\lambda_n t}\iint_\Omega\psi^2(\vec{\rho})\,d\vec{\rho}\geqq\sum_{n=1}^{\infty}e^{-\lambda_n t}\iint_{\Omega(a)}\psi_n^2(\vec{\rho})\,d\vec{\rho}$$

$$\geqq4N(a)\sum_{\substack{m,n\\ \text{odd}}}\exp\left[-\frac{(m^2+n^2)\pi^2}{2a^2}t\right]$$

and, integrating the inequality $P_\Omega(\vec{\rho}\,|\,\vec{\rho};\, t) \leqq 1/2\pi t$ over Ω we get $\sum_{n=1}^{\infty} e^{-\lambda_n t} \leqq |\Omega|/2\pi t$.

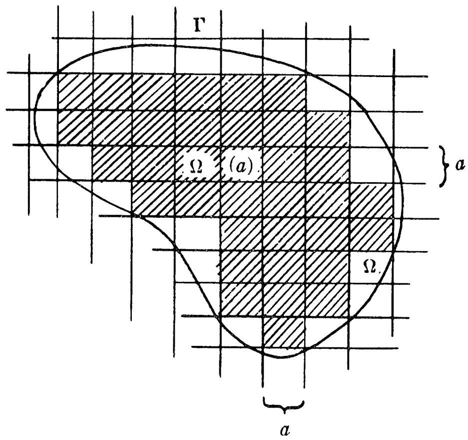

FIG. 2

Noting that $N(a)a^2 = |\Omega(a)|$ we record the fruits of our latest labor in the form of the inequality

$$|\,\Omega(a)\,|\, \frac{4}{a^2} \sum_{\substack{m,n \\ \text{odd}}} \exp\left[-\frac{(m^2+n^2)\pi^2}{2a^2} t \right] \leqq \sum_{n=1}^{\infty} e^{-\lambda_n t} \leqq \frac{|\,\Omega\,|}{2\pi t}\,.$$

From the fact (already noted above) that

$$\lim_{t \to 0} 2\pi t \frac{4}{a^2} \sum_{\substack{m,n \\ \text{odd}}} \exp\left[-\frac{(m^2+n^2)\pi^2}{2a^2} t \right] = 1$$

we conclude easily that

$$|\,\Omega(a)\,| \leqq \liminf_{t \to 0} 2\pi t \sum_{n=1}^{\infty} e^{-\lambda_n t} \leqq \limsup_{t \to 0} 2\pi t \sum_{n=1}^{\infty} e^{-\lambda_n t} \leqq |\,\Omega\,|\,;$$

and since, by choosing a sufficiently small, we can make $|\Omega(a)|$ arbitrarily close to $|\Omega|$, we must have $\lim_{t\to 0} 2\pi t \sum_{n=1}^{\infty} e^{-\lambda_n t} = |\Omega|$ or, in other words,

$$\sum_{n=1}^{\infty} e^{-\lambda_n t} \sim \frac{|\,\Omega\,|}{2\pi t}, \qquad t \to 0.$$

9. Are we now through with rigor? Not quite. For while the inequalities

$$P_\Omega(\vec{\rho}\,|\,\vec{r};\, t) \leqq \frac{\exp\left[-\dfrac{\|\vec{r}-\vec{\rho}\|^2}{2t} \right]}{2\pi t}$$

$$P_\Omega(\vec{\rho}\mid\vec{r};t) \geqq P_Q(\vec{\rho}\mid\vec{r};t), \qquad \vec{r}\in Q,$$

are utterly obvious on intuitive grounds they must be proved. Let me indicate a way of doing it which is probably by far not the simplest. I am choosing it to exhibit yet another physical context.

It has been known since the early days of this century, through the work of Einstein and Smoluchowski, that diffusion is but a macroscopic manifestation of microscopic Brownian motion.

Under suitable physical assumptions $P_\Omega(\vec{\rho}\mid\vec{r};\,t)$ can be interpreted as the probability density of finding a free Brownian particle at $\vec{r}$ at time t if it started on its erratic journey at $t=0$ from $\vec{\rho}$ and if it gets absorbed when it comes to the boundary of Ω.

If a large number N of independent free Brownian particles are started from $\vec{\rho}$ then

$$N\iint_A P(\vec{\rho}\mid\vec{r};t)\,d\vec{r}$$

is the average number of these particles which are found in A at time t. Since the statistical percentage error is of the order $1/\sqrt{N}$ continuous diffusion theory is an excellent approximation when N is large.

A significant deepening of this point of view was achieved in the early twenties by Norbert Wiener. Instead of viewing the problem as a problem in *statistics* of *particles* he viewed it as a problem in *statistics* of *paths*. Without entering into details let me review briefly what is involved here.

Consider the set of all continuous curves $\vec{r}(\tau)$, $0\leqq\tau<\infty$, starting from some arbitrarily chosen origin O. Let $\Omega_1, \Omega_2, \cdots, \Omega_n$ be open sets and $t_1<t_2<\cdots<t_n$ ordered instants of time. The Einstein-Smoluchowski theory required that (with suitable units)

$$\text{Prob.}\,\{\vec{\rho}+\vec{r}(t_1)\in\Omega_1, \vec{\rho}+\vec{r}(t_2)\in\Omega_2, \cdots, \vec{\rho}+\vec{r}(t_n)\in\Omega_n\}$$
$$=\int_{\Omega_1}\cdots\int_{\Omega_n} P_0(\vec{\rho}\mid\vec{r}_1;t_1)P_0(\vec{r}_1\mid\vec{r}_2;t_2-t_1)\cdots P_0(\vec{r}_{n-1}\mid\vec{r}_n;t_n-t_{n-1})\,d\vec{r}_1\cdots d\vec{r}_n$$

where, as before,

$$P_0(\vec{\rho}\mid\vec{r};t) = \frac{1}{2\pi t}\exp\left[-\frac{\|\vec{r}-\vec{\rho}\|^2}{2t}\right].$$

Wiener has shown that it is possible to construct a completely additive measure on the space of all continuous curves $\vec{r}(\tau)$ emanating from the origin such that the set of curves $\vec{\rho}+\vec{r}(\tau)$ which at times $t_1<t_2<\cdots<t_n$ find themselves in open sets $\Omega_1, \Omega_2, \cdots, \Omega_n$ respectively, has measure given by the Einstein-Smoluchowski formula above.

The set of curves such that $\vec{\rho}+\vec{r}(\tau)\in\Omega$, $0\leqq\tau\leqq t$, and $\vec{\rho}+\vec{r}(t)\in A$ (A—an open set) turns out to be measurable and it can be shown, if Ω has sufficiently smooth boundaries, that this measure is equal to

$$\int_A P_\Omega(\vec{\rho} \mid \vec{r}; t)\, d\vec{r}.$$

This is not a trivial statement and it should come as no surprise that it trivially implies the inequalities we needed a while back to make precise the principle of not feeling the boundary.

In fact, as the reader no doubt sees, the inequalities in question are simply a consequence of the fact that if sets $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ are such that

$$\mathcal{A} \subset \mathcal{B} \subset \mathcal{C}$$

then meas. $\mathcal{A} \leqq$ meas. $\mathcal{B} \leqq$ meas. $\mathcal{C}$.

One final remark before we go on. The set of curves for which

$$\vec{\rho} + \vec{r}(\tau) \in \Omega, \qquad 0 \leqq \tau \leqq t \quad \text{and} \quad \vec{\rho} + \vec{r}(t) \in A$$

is measurable even if the boundary of Ω is quite wild. The measure can still be written as $\int_A P_\Omega(\vec{\rho} \mid \vec{r}; t) d\vec{r}$ and it can be shown that in the interior of Ω, $P_\Omega(\vec{\rho} \mid \vec{r}; t)$ satisfies the diffusion equation $\partial P_\Omega / \partial t = \frac{1}{2} \nabla^2 P_\Omega$ as well as the initial condition

$$\lim_{t \to 0} \int_A P_\Omega(\vec{\rho}; \vec{r}; t)\, d\vec{r} = 1,$$

for all open sets A such that $\vec{\rho} \in A$.

It is, however, no longer clear how to interpret the boundary condition that

$$P_\Omega(\vec{\rho} \mid \vec{r}; t) \to 0 \quad \text{when} \quad \vec{r} \to \Gamma.$$

This difficulty forces the classical theory of diffusion to consider reasonably smooth boundaries. The probabilistic interpretation of $P_\Omega(\vec{\rho} \mid \vec{r};\, t)$ provides a natural definition of a *generalized solution* of the boundary value problem under consideration.

10. We are now sure that we can hear the area of a drum and it may seem that we spent a lot of effort to achieve so little.

Let me now show you that the approach we used can be extended to yield more, but to avoid certain purely geometrical complications I shall restrict myself to convex drums.

We have achieved our first success by introducing the principle of not feeling the boundary. But if $\vec{\rho}$ is close to the boundary Γ of Ω then the diffusing particles starting from $\vec{\rho}$ will, to some extent, begin to be influenced by Γ.

Let $\vec{q}$ be the point on Γ closest to $\vec{\rho}$ and let $l(\vec{\rho})$ be the straight line perpendicular to the line joining $\vec{\rho}$ and $\vec{q}$. (See Fig. 3.) Then a diffusing particle starting from ρ will see for a short time the boundary Γ as the straight line $l(\vec{\rho})$.

One may say, using again somewhat picturesque language, that, for small t, the particle has not had time to feel the curvature of the boundary.

If this principle is valid I should be allowed to approximate (for small t)

$$P_\Omega(\vec{\rho}\,|\,\vec{r};t) \quad \text{by} \quad P_{l(\vec{\rho})}(\vec{\rho}\,|\,\vec{r};t),$$

where $P_{l(\vec{\rho})}(\vec{\rho}\,|\vec{r};t)$ satisfies again the diffusion equation

$$\frac{\partial P}{\partial t} = \frac{1}{2}\nabla^2 P$$

with the initial condition $P\to\delta(\vec{\rho}-\vec{r})$ as $t\to 0$, but with the boundary condition

$$P_{l(\vec{\rho})}(\vec{\rho}\,|\,\vec{r};t)\to 0 \quad \text{as } \vec{r} \text{ approaches a point on } l(\vec{\rho}).$$

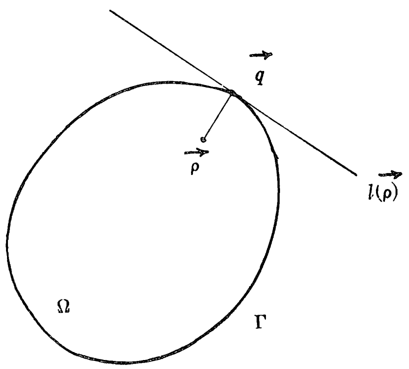

FIG. 3

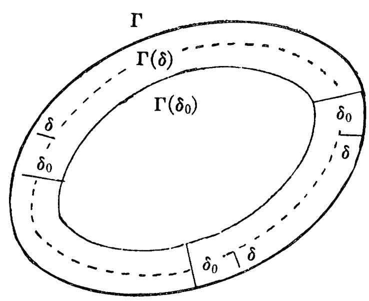

FIG. 4

Carrying this optimism as far as possible we would expect that to a good approximation

$$\int_\Omega P_\Omega(\vec{\rho}\,|\,\vec{\rho};t)\,d\vec{\rho} \sim \int_\Omega P_{l(\vec{\rho})}(\vec{\rho}\,|\,\vec{\rho};t)\,d\vec{\rho}.$$

It is well known that

$$P_{l(\vec{\rho})}(\vec{\rho}\,|\,\vec{\rho};t) = \frac{1-e^{-2\delta^2/t}}{2\pi t},$$

where $\delta=\|\vec{q}-\vec{\rho}\|$ = minimal distance from $\vec{\rho}$ to Γ. Thus (hopefully!)

$$\iint_\Omega P_\Omega(\vec{\rho}\,|\,\vec{\rho};t)\,d\vec{\rho} = \sum_{n=1} e^{-\lambda_n t} \sim \frac{|\,\Omega\,|}{2\pi t} - \frac{1}{2\pi t}\int_\Omega e^{-2\delta^2/t}\,d\vec{\rho}.$$

Here $|\Omega|/2\pi t$ is our old friend from before and it remains to calculate asymptotically (as $t\to 0$) the integral $\int_\Omega e^{-2\delta/t}d\vec{\rho}$. To do this consider the curve $\Gamma(\delta)$ of points in Ω whose "distance" from Γ is δ. (See Fig. 4.)

For small enough δ, $\Gamma(\delta)$ is well defined (and even convex) and the major contribution to our integral comes from small δ.

If $L(\delta)$ denotes the length of $\Gamma(\delta)$ we have

$$\int_{\Omega} e^{-2\delta^2/t}\, d\vec{\rho} = \int_0^{\delta_0} e^{-2\delta^2/t} L(\delta)\, d\delta + \text{something less than } |\Omega|\, e^{-2\delta_0^2/t}$$

and hence, neglecting an exponentially small term (as well as terms of order t)

$$\int_{\Omega} e^{-2\delta^2/t}\, d\vec{\rho} \sim \sqrt{t} \int_0^{\delta_0/\sqrt{t}} e^{-2x^2} L(x\sqrt{t})\, dx \sim \sqrt{t} L \int_0^{\infty} e^{-2x^2}\, dx = \frac{L}{4}\sqrt{2\pi t},$$

where $L = L(0)$ is the length of Γ.

We are finally led to the formula

$$\sum_{n=1}^{\infty} e^{-\lambda_n t} \sim \frac{|\Omega|}{2\pi t} - \frac{L}{4}\frac{1}{\sqrt{2\pi t}}, \quad \text{for } t \to 0,$$

and so we can also "hear" the length of the circumference of the drum!

The last asymptotic formula was proved only a few years ago by the Swedish mathematician Ake Pleijel [2] using an entirely different approach.

It is worth remarking that we can now prove that if all the frequencies of a drum are equal to those of a circular drum then the drum must itself be circular. This follows at once from the classical isoperimetric inequality which states that $L^2 \geqq 4\pi|\Omega|$, with equality occurring *only* for a *circle*.

By pitch alone one can thus determine whether a drum is circular or not!

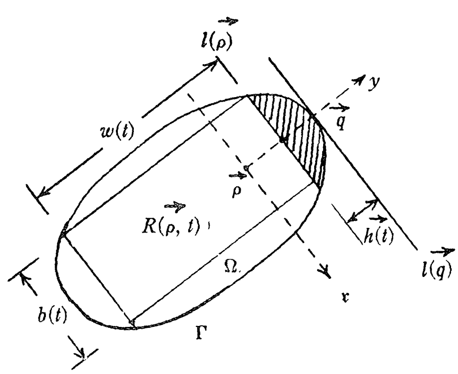

FIG. 5

11. Can the heuristic argument again be made rigorous? Indeed it can. First, we use the inequality

$$P_{\Omega}(\vec{\rho}\,|\,\vec{\rho};t) \leqq P_{l(\rho)}(\vec{\rho}\,|\,\vec{\rho};t) = \frac{1}{2\pi t} - \frac{1}{2\pi t} e^{-2\delta^2/t},$$

which is simply a refinement of the one used previously, namely, $P_\Omega(\vec{\rho}\,|\,\vec{\rho};\ t) \leqq 1/2\pi t$, and which can be proven the same way.

Next we need a precise lower estimate for $P_\Omega(\vec{\rho}\,|\,\vec{\rho};\ t)$ and this is a little more difficult. We "inscribe" the rectangle $R(\vec{\rho},\ t)$ as shown in Fig. 5, where $h(t)$, the height of the shaded segment, is to be determined a little later.

Let the side of R along the base of the segment be $b(t)$ and the other side be $w(t)$. It should be clear from the picture that the y-axis *bisects* the sides of the rectangle which are parallel to the x-axis.

Now consider $P_R(\vec{\rho}\,|\,\vec{\rho};\ t)$. This notation is perhaps confusing since it suggests that we are dealing with a boundary value problem in which the boundary varies with time. This is not the case. What we have in mind is the following: *fix* t, find $P_{R(t)}(\vec{\rho}\,|\,\vec{r};\ \tau)$ which is defined unambiguously, and finally set $\tau = t$. The result is $P_R(\vec{\rho}\,|\,\vec{r};\ t)$. A convenient expression is

$$P_R(\vec{\rho}\,|\,\vec{\rho};t) = \frac{1}{2\pi t}\left\{\sum_{-\infty}^{\infty}\left(\exp\left[-\frac{2b^2}{t}n^2\right] - \exp\left[-\frac{2b^2}{t}\left(n+\frac{1}{2}\right)^2\right]\right)\right\}$$
$$\times\left\{\sum_{-\infty}^{\infty}\left(\exp\left[-\frac{2w^2}{t}n^2\right] - \exp\left[-\frac{2w^2}{t}\left(n+\frac{\bar{\delta}}{w}\right)^2\right]\right)\right\}$$

where $\bar{\delta} = \delta - h(t) = \|\vec{q}-\vec{\rho}\| - h(t)$. Now let $h(t) = \epsilon\sqrt{t}$ and, assuming that $l(\vec{\rho})$ is actually *tangent* to the curve (which for a convex curve will happen with at most a denumerable number of exceptional points $\vec{q}$), we have

$$\lim_{t\to 0}\frac{b(t)}{h(t)} = \lim_{t\to 0}\frac{b(t)}{\epsilon\sqrt{t}} = \infty,$$

and consequently

$$\sum_{-\infty}^{\infty}\left(\exp\left[-\frac{2b^2}{t}n^2\right] - \exp\left[-\frac{2b^2}{t}\left(n+\frac{1}{2}\right)^2\right]\right) = 1 + o(1).$$

This is not quite enough, however, and one needs the stronger estimate

$$\sum_{-\infty}^{\infty}\left(\exp\left[-\frac{2b^2}{t}n^2\right] - \exp\left[-\frac{2b^2}{t}\left(n+\frac{1}{2}\right)^2\right]\right) = 1 + o(\sqrt{t}).$$

This will surely be the case, for example, if the curvature exists at $\vec{q}$, for this would imply that $h(t) \sim b^2(t)$ and the $o(\sqrt{t})$ term above would then be an enormous overestimate. Very mild additional regularity conditions at nearly all points $\vec{q}$ would insure $o(\sqrt{t})$. Without entering into a discussion of these conditions let us simply assume the boundary to be such as to guarantee at least $o(\sqrt{t})$.

Since $w(t)$ remains bounded from below as $t \to 0$, we also have

$$\sum_{-\infty}^{\infty}\left(\exp\left[-\frac{2w^2}{t}n^2\right] - \exp\left[-\frac{2w^2}{t}\left(n+\frac{\bar{\delta}}{w}\right)^2\right]\right)$$
$$= 1 - e^{-2\bar{\delta}^2/t} + \text{exponentially small terms.}$$

We are now almost through. We write (cf. Fig. 6)

$$\sum_{n=1}^{\infty} e^{-\lambda_n t} = \int_{\Omega} P_{\Omega}(\vec{\rho} \mid \vec{\rho}; t)\, d\vec{\rho} > \int_{\Omega(\epsilon\sqrt{t})} P_R(\vec{\rho} \mid \vec{\rho}; t)\, d\vec{\rho}$$

$$= \frac{(1 + o(\sqrt{t}))}{2\pi t} \int_{\Omega(\epsilon\sqrt{t})} (1 - e^{-2\bar{\delta}^2/t} + \text{exponentially small terms})\, d\vec{\rho}.$$

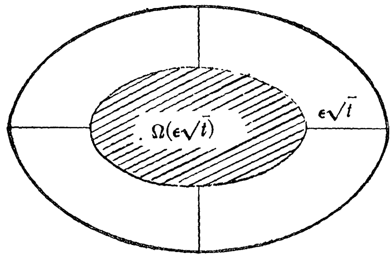

FIG. 6

Except then for exponentially small terms and the factor $1+o(\sqrt{t})$ in front we have the integral

$$\int_{\Omega(\epsilon\sqrt{t})} (1 - e^{-2\bar{\delta}^2/t})\, d\vec{\rho}$$

which, as before, can be seen to be asymptotically

$$|\, \Omega(\epsilon\sqrt{t}) \,| - \frac{L}{4} \sqrt{2\pi t},$$

where one neglects terms of order t and exponentially small terms. Since asymptotically $|\Omega(\epsilon\sqrt{t})| \sim |\Omega| - L\epsilon\sqrt{t}$ one can obtain the inequality

$$\sum_{n=1}^{\infty} e^{-\lambda_n t} > \frac{|\,\Omega\,|}{2\pi t} - \frac{(L + \epsilon')}{4} \frac{1}{\sqrt{2\pi t}},$$

where ϵ' is related in a simple way to ϵ. Since ϵ' can be made arbitrarily small, the asymptotic formula

$$\sum_{n=1}^{\infty} e^{-\lambda_n t} \sim \frac{|\,\Omega\,|}{2\pi t} - \frac{L}{4} \frac{1}{\sqrt{2\pi t}}$$

follows.

12. If our overall strategy of attack on the problem is right we should be able to go on and for points very close to a *smooth* boundary replace the boundary locally by suitable circles of curvature.

A result of Pleijel suggests strongly that for a simply connected drum with a smooth boundary (i.e. without corners and with curvature existing at every point) one has

$$\sum_{n=1}^{\infty} e^{-\lambda_n t} \sim \frac{|\Omega|}{2\pi t} - \frac{L}{4}\frac{1}{\sqrt{2\pi t}} + \frac{1}{6}.$$

Unfortunately I am unable to obtain this, for the exasperating reason that I am unable to get a workable expression for $P_\Omega(\vec{\rho}\,|\,\vec{\rho};\, t)$ if Ω is a circle.

Rather than yield to despair over this sad state of affairs let me devote the remainder of the lecture to *polygonal drums*, i.e. drums whose boundaries are polygons. This study will show beyond the shadow of a doubt that the constant term in our asymptotic expansion owes its existence to the overall curvature of the boundary.

13. Before I go on I need an expression for $P_{S(\theta_0)}(\vec{\rho}\,|\,\vec{r};\, t)$ where $S(\theta_0)$ is an infinite wedge of angle θ_0. In other words $P_{S(\theta_0)}$ is the solution of

$$\frac{\partial P}{\partial t} = \frac{1}{2}\nabla^2 P$$

subject to the usual initial condition $P_{S(\theta_0)}(\vec{\rho}\,|\,\vec{r};\, t) \to \delta(\vec{\rho}-\vec{r})$, $t \to 0$, and vanishing as $\vec{r}$ approaches a point on either side of the angle θ_0.

This is a very old, very classical, problem and if $\theta_0 = \pi/m$, with m an integer, it can be solved by the familiar method of images. For m not an integer, Sommerfeld invented a method which, so to speak, extends the method of images to a Riemann surface. A little later, in 1899 to be precise, H. S. Carslaw gave a more elementary approach in which $P_{S(\theta_0)}(\vec{\rho}\,|\,\vec{r};\, t)$ is represented by a suitable contour integral. Carslaw transforms the integral into an infinite series of Bessel functions but for our purposes it is best to resist the temptation of Bessel functions and to reduce the integral to a different form. I shall skip the details (though some are quite instructive) and simply reproduce the final result.

Set

$$v(\alpha) = (1/2\pi t) \sum_{\substack{\theta-\alpha-\pi<2k\theta_0 \\ <\theta-\alpha+\pi}} \exp\left[-\frac{r^2 - 2r\rho\cos(\theta-\alpha-2k\theta_0)+\rho^2}{2t}\right]$$

$$-\left(\sin\frac{\pi^2}{\theta_0}\right)\frac{\exp\left[-\dfrac{r^2+\rho^2}{2t}\right]}{4\pi\theta_0 t}\int_{-\infty}^{\infty}\frac{\exp\left[-\dfrac{r\rho}{t}\cosh y\right]}{\cosh\left\{\dfrac{\pi}{\theta_0}y + \dfrac{i\pi}{\theta_0}(\theta-\alpha)\right\} - \cos\dfrac{\pi^2}{\theta_0}}\,dy,$$

where the summation $\sum$ is extended over k's satisfying the inequality under the summation sign and $\vec{\rho} = (\rho, \alpha)$, $\vec{r} = (r, \theta)$.

Then

$$P_{S(\theta_0)}(\vec{\rho}\,|\,\vec{r};\,t) = v(\alpha) - v(-\alpha).$$

Note that if $\theta_0 = \pi/m$, with m an integer, the complicated integral is out, since the factor in front of it, to wit $\sin \pi^2/\theta_0 = \sin \pi m$, is zero; what remains in the resulting expression for $v(\alpha) - v(-\alpha)$ is a collection of terms easily identifiable with those obtained by the method of images.

Let us now assume that $\pi/2 < \theta_0 < \pi$ and see what $P_S(\vec{\rho}\,|\,\vec{\rho};\,t)$ is in this case. In the expression for $v(\alpha)$ when we set $\theta = \alpha$ the inequality under the $\sum$ sign becomes $-\pi < 2k\theta_0 < \pi$ and only $k=0$ is allowed. In $v(-\alpha)$ the inequality is $2\alpha - \pi < 2k\theta_0 < 2\alpha + \pi$ and what k's to take depends on α.

We see that:

$$0 < \alpha < \theta_0 - \frac{\pi}{2}, \text{ only } k = 0 \text{ is allowed,}$$

$$\frac{\pi}{2} < \alpha < \theta_0, \text{ only } k = 1 \text{ is allowed,}$$

but for $\theta_0 - \pi/2 < \alpha < \pi/2$ both $k=0$ and $k=1$ are allowed. (See Fig. 7.)

Let us now put $\vec{r} = \vec{\rho}$ (so that $\rho = r$) and write down in detail the expressions for $P_{S(\theta_0)}(\vec{\rho}\,|\,\vec{\rho};\,t)$ in the three sectors. For $0 < \alpha < \theta_0 - \pi/2$

$$P_S(\vec{\rho}\,|\,\vec{\rho};\,t) = \frac{1}{2\pi t} - \frac{\exp\left[-\dfrac{r^2}{t}(1-\cos 2\alpha)\right]}{2\pi t}$$
$$-\left(\sin\frac{\pi^2}{\theta_0}\right)\frac{\exp\left[-\dfrac{r^2}{t}\right]}{4\pi\theta_0 t}\int_{-\infty}^{\infty}\frac{\exp\left[-\dfrac{r^2}{t}\cosh y\right]}{\cosh\dfrac{\pi}{\theta_0}y - \cos\dfrac{\pi^2}{\theta_0}}\,dy$$
$$+\left(\sin\frac{\pi^2}{\theta_0}\right)\frac{\exp\left[-\dfrac{r^2}{t}\right]}{4\pi\theta_0 t}\int_{-\infty}^{\infty}\frac{\exp\left[-\dfrac{r^2}{t}\cosh y\right]}{\cosh\left\{\dfrac{\pi}{\theta_0}y + 2\pi i\dfrac{\alpha}{\theta_0}\right\} - \cos\dfrac{\pi^2}{\theta_0}}\,dy.$$

For $\pi/2 < \alpha < \theta_0$

$$P_S(\vec{\rho}\,|\,\vec{\rho};\,t) = \frac{1}{2\pi t} - \frac{\exp\left[-\dfrac{r^2}{t}(1-\cos 2(\theta_0-\alpha))\right]}{2\pi}$$
$$+ \text{ the same two integrals as above}$$

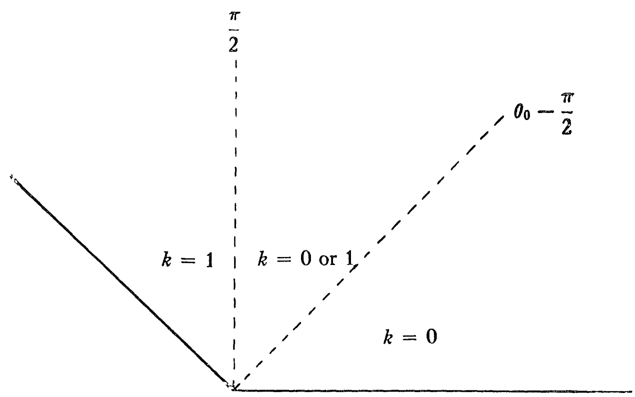

FIG. 7

and finally, for $\theta_0 - \pi/2 < \alpha < \pi/2$

$$P_S(\vec{\rho} \mid \vec{\rho}; t) = \frac{1}{2\pi t} - \frac{\exp\left[-\frac{r^2}{t}(1 - \cos 2\alpha)\right]}{2\pi t} - \frac{\exp\left[-\frac{r^2}{t}(1 - \cos 2(\theta_0 - \alpha))\right]}{2\pi t}$$

+ again the same two integrals.

We should recognize $r^2(1-\cos 2\alpha)$ (and $r^2(1-\cos 2(\theta_0-\alpha))$) as being $2\delta^2$ where δ is the distance from $\vec{\rho}$ to a side of the wedge.

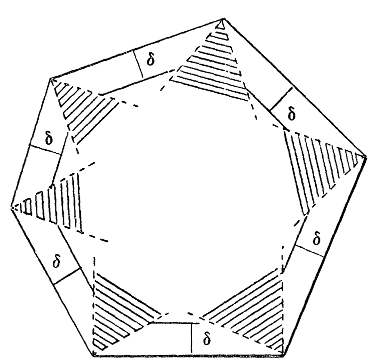

FIG. 8

14. To simplify matters somewhat let me assume that the polygonal drum is convex and that every angle is obtuse.

At each vertex we draw perpendiculars to the sides of the polygon thus obtaining N shaded sectors (where N is the number of sides or vertices of our polygon).

Now let ρ be a point in Ω. Stuff diffusing from $\vec{\rho}$ will either "see" the boundary as a straight line or, if $\vec{\rho}$ is near a vertex, as an infinite wedge.

We may as well say that the boundary will appear to the diffusing particle as the nearest wedge, and that consequently we may replace $P_\Omega(\vec{\rho}\,|\,\vec{\rho};\,t)$ by $P_{S(\theta_0)}(\vec{\rho}\,|\,\vec{\rho};\,t)$, where $S(\theta_0)$ is the wedge nearest to $\vec{\rho}$.

Now, each $P_S(\vec{\rho}\,|\,\vec{\rho};\,t)$ has $1/2\pi t$ as a term and after integration over Ω this gives the principal term $|\Omega|/2\pi t$. Next, each $P_S(\vec{\rho}\,|\,\vec{\rho};\,t)$ contains two complicated looking integrals which have to be integrated over the wedge.

Fortunately, the second of these integrates out to 0, while the first yields, upon integration over $S(\theta_0)$,

$$-\frac{1}{8\pi}\left(\sin\frac{\pi^2}{\theta_0}\right)\int_{-\infty}^{\infty}\frac{dy}{(1+\cosh y)\left(\cosh\frac{\pi}{\theta_0}y-\cos\frac{\pi^2}{\theta_0}\right)}.$$

This is only the contribution of one wedge; to get the total contribution one must sum over all wedges.

Thus the total contribution is

$$-\frac{1}{8\pi}\sum_{\theta_0}\left(\sin\frac{\pi^2}{\theta_0}\right)\int_{-\infty}^{\infty}\frac{dy}{(1+\cosh y)\left(\cosh\frac{\pi}{\theta_0}y-\cos\frac{\pi^2}{\theta_0}\right)}.$$

Finally, if $\vec{\rho}$ is in the shaded sector of the wedge $S(\theta_0)$ we get, on integrating over the sector,

$$-\int_{\theta_0-\pi/2}^{\pi/2}d\alpha\int_0^{\infty}\left\{\frac{\exp\left[-\frac{r^2}{t}(1-\cos 2\alpha)\right]}{2\pi t}+\frac{\exp\left[-\frac{r^2}{t}(1-\cos 2(\theta_0-\alpha))\right]}{2\pi t}\right\}rdr=-\tfrac{1}{2}\,\frac{1}{2\pi}\cot\left(\theta_0-\frac{\pi}{2}\right),$$

and the total contribution from the shaded sectors is $-\frac{1}{2}1/2\pi\sum_{\theta_0}\cot\,(\theta_0-\pi/2)$.

The remaining contribution is easily seen to be

$$-\frac{1}{2\pi t}\int_0^{\infty}\left(L-2\delta\sum_{\theta_0}\cot\left(\theta_0-\frac{\pi}{2}\right)\right)e^{-2\delta^2/t}\,d\delta$$
$$=-\frac{L}{4}\,\frac{1}{\sqrt{2\pi t}}+\tfrac{1}{2}\,\frac{1}{2\pi}\sum_{\theta_0}\cot\left(\theta_0-\frac{\pi}{2}\right).$$

Finally, for a polygonal drum

$$\sum_{n=1}^{\infty} e^{-\lambda_n t} \sim \frac{|\Omega|}{2\pi t} - \frac{L}{4}\frac{1}{\sqrt{2\pi t}}$$
$$- \frac{1}{8\pi}\sum_{\theta_0}\left(\sin\frac{\pi^2}{\theta_0}\right)\int_{-\infty}^{\infty}\frac{dy}{(1+\cosh y)(\cos \pi/\theta_0\, y - \cos \pi^2/\theta_0)},$$

with the understanding that each θ_0 satisfies the inequality $\pi/2 < \theta_0 < \pi$. If the polygon has N sides, and if we let $N \to \infty$ in such a way that each $\theta_0 \to \pi$, then the constant term approaches

$$+\frac{2\pi}{8\pi}\int_{-\infty}^{\infty}\frac{dy}{(1+\cosh y)^2} = \tfrac{1}{6}\,.$$

This should strengthen our belief that for simply connected smooth drums the constant is universal and equal to $\frac{1}{6}$.

15. What happens for multiply connected drums?

If the drum as well as the holes are polygonal the answer is easily obtained. One only needs $P_{S(\theta_0)}(\vec{\rho}\,|\,\vec{\rho};\, t)$ for θ_0 satisfying the inequality $\pi < \theta_0 < 2\pi$ and this is easily gotten from the general formula quoted above.

Near the holes the diffusing particles will "see" concave wedges but nothing will change in principle.

If we let all polygons approach smooth curves it turns out the constant approaches $(1-r)\frac{1}{6}$, where r is the number of holes. It is thus natural to conjecture that for a *smooth* drum with r *smooth* holes

$$\sum_{n=1}^{\infty} e^{-\lambda_n t} \sim \frac{|\Omega|}{2\pi t} - \frac{L}{4}\frac{1}{\sqrt{2\pi t}} + (1-r)\,\tfrac{1}{6}\,,$$

and that therefore one can "hear" the connectivity of the drum!

One can, of course, speculate on whether in general one can hear the Euler-Poincaré characteristic and raise all sorts of other interesting questions.

As our study of the polygonal drum shows, the structure of the constant term is quite complex since it combines metric and topological features. Whether these can be properly disentangled remains to be seen.

This is an expanded version of a lecture which was filmed under the auspices of the Committee on Educational Media of the Mathematical Association of America.

References

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