Shift Register Sequences

Based on S. W. Golomb, *Shift Register Sequences*,
Aegean Park Press (1982, Revised Edition)

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Overview

Introduction & Motivation

Linear Feedback Shift Registers (§2)

Properties of m-Sequences (§3–4)

Cycle Structure of Shift Registers (§2)

Cross-Correlation & Gold Sequences (§5–6)

Non-Linear Shift Registers & Applications (§7–8)

Summary & Further Reading

What Is a Shift Register?

Definition:

A **shift register of length** n is a clocked device consisting of n storage cells (stages), each holding one symbol from a finite alphabet $\mathcal{A}$. At each clock tick:

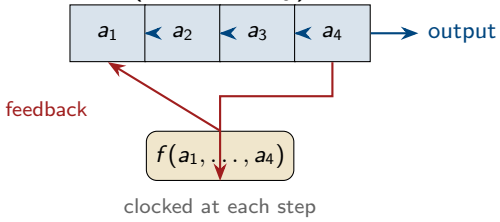
1. The content of each cell shifts *one position to the right*.
2. The new leftmost cell is filled by a **feedback function** $f(a_1, a_2, \ldots, a_n)$ applied to the current state.

Output: the sequence of symbols exiting from the rightmost cell.

Binary case (most common)

$\mathcal{A} = \{0, 1\} = \mathrm{GF}(2)$.
The feedback function $f : \mathrm{GF}(2)^n \to \mathrm{GF}(2)$ is a Boolean function.

Schematic ($n = 4$, binary):



State: the n-tuple $(a_1, a_2, \ldots, a_n) \in \mathrm{GF}(2)^n$.
There are 2^n possible states.

Why Shift Registers Matter for RNG

Shift registers are everywhere:

- **Pseudorandom number generation** — LFSRs produce long, statistically excellent sequences at very low hardware cost
- **Spread-spectrum communications** — PN sequences used for synchronization and channel coding
- **Cryptography** — stream cipher key-stream generators
- **Error-correcting codes** — BCH codes, Reed-Solomon codes
- **VLSI testing** — built-in self-test (BIST)
- **Radar ranging** — pulse compression via m-sequences

Golomb's central question

Under what conditions does a shift register of length n produce a **single long cycle** of period $2^n - 1$, and what are the statistical properties of the output sequence?

This course's focus

- **Linear** feedback shift registers (LFSRs)
- **Maximal-length** sequences (m-sequences)
- **Golomb's three randomness postulates**
- **Correlation** properties

Linear Feedback Shift Registers (LFSRs)

The linearity restriction:

Restrict the feedback function to be **linear over** $\mathrm{GF}(2)$:

$$a_{n+k} = c_1 a_{n+k-1} \oplus c_2 a_{n+k-2} \oplus \cdots \oplus c_n a_k, \quad c_i \in \mathrm{GF}(2)$$

where $\oplus$ denotes addition mod 2 (XOR).

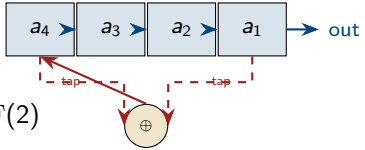
Characteristic polynomial:

$$\boxed{f(x) = 1 + c_1 x + c_2 x^2 + \cdots + c_n x^n \in \mathrm{GF}(2)[x]}$$

Example ($n = 4$):
$f(x) = 1 + x + x^4$ (taps at positions 1 and 4)

$$a_{k+4} = a_{k+3} \oplus a_k$$

LFSR with $f(x) = 1 + x + x^4$:



Key fact

The all-zero state **0** maps to itself and is **excluded** from useful sequences. At most $2^n - 1$ non-zero states exist.

Period, Primitive Polynomials, and m-Sequences

Period of an LFSR:
The *period* T of the output sequence divides $2^n - 1$.
T is determined by the characteristic polynomial $f(x)$.

Key definitions

- $f(x)$ is **irreducible** over $\mathrm{GF}(2)$ if it has no non-trivial factors in $\mathrm{GF}(2)[x]$
- $f(x)$ is **primitive** if it is irreducible and its roots have multiplicative order $2^n - 1$ in $\mathrm{GF}(2^n)$
- An LFSR with a primitive $f(x)$ produces a **maximal-length sequence (m-sequence)** of period $T = 2^n - 1$

Primitive polynomials for small n:

n	Primitive polynomial(s)	Period
2	$1 + x + x^2$	3
3	$1 + x + x^3$; $1 + x^2 + x^3$	7
4	$1 + x + x^4$; $1 + x^3 + x^4$	15
5	$1 + x^2 + x^5$; $1 + x^3 + x^5$; ...	31
6	$1 + x + x^6$; $1 + x^5 + x^6$; ...	63
7	18 primitive polys	127
8	16 primitive polys	255
10	60 primitive polys	1023

Practical consequence

For $n = 31$: period $= 2^{31} - 1 \approx 2.1 \times 10^9$ — a very long sequence from just 31 bits of state!

State Diagram of an LFSR

Example: $n = 3$, $f(x) = 1 + x + x^3$ **(primitive)**

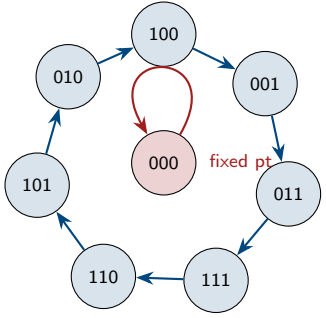
Starting from state $(1, 0, 0)$, the sequence of states is:

a_3	a_2	a_1	step	output
1	0	0	0	0
0	0	1	1	1
0	1	1	2	1
1	1	1	3	1
1	1	0	4	0
1	0	1	5	1
0	1	0	6	0
1	0	0	7	**(repeats)**

Output sequence: **0011101** | 0011101 | ...

Period $= 7 = 2^3 - 1$ ✓

State diagram (period-7 cycle + fixed point):



The all-zero state is always a fixed point (excluded from the m-sequence). A primitive polynomial gives a **single** *cycle of length* $2^n - 1$ *plus this fixed point.*

The Three Fundamental Properties of m-Sequences

Theorem (Golomb)

Every maximal-length sequence produced by an n-stage LFSR with a primitive characteristic polynomial satisfies the following three properties.

P1 — Balance

In each period of length $2^n - 1$, the number of 1s exceeds the number of 0s by exactly **one**:

$$\#\{1\text{s}\} = 2^{n-1}$$

$$\#\{0\text{s}\} = 2^{n-1} - 1$$

P2 — Run Property

Among all **runs** of consecutive identical symbols:

- $\frac{1}{2}$ of runs have length 1
- $\frac{1}{4}$ have length 2
- $\frac{1}{8}$ have length 3
- one run of 0s of length $n-1$
- one run of 1s of length n

P3 — Ideal Autocorrelation

The periodic autocorrelation function is **two-valued**:

$$C(\tau) = \begin{cases} 2^n - 1 & \tau \equiv 0 \\ -1 & \tau \not\equiv 0 \end{cases}$$

or equivalently (normalized):

$$\hat{C}(\tau) = \begin{cases} 1 & \tau = 0 \\ -\frac{1}{2^n - 1} & \tau \neq 0 \end{cases}$$

Golomb's Randomness Postulates in Detail

Why these three postulates?

A truly random infinite binary sequence has (with probability 1):

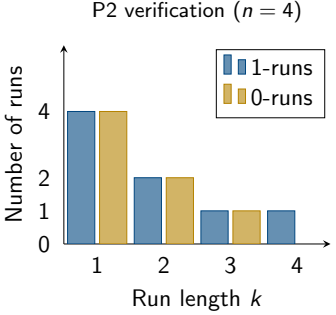
- Equal frequencies of 0s and 1s (by the Law of Large Numbers) $\rightarrow$ **P1**
- Runs of length k occurring with frequency $\approx 2^{-(k+1)} \rightarrow$ **P2**
- Autocorrelation $C(\tau) \approx 0$ for $\tau \neq 0 \rightarrow$ **P3**

A periodic sequence cannot achieve these *exactly*, but m-sequences achieve the **best possible approximation** for a given period length.

Golomb's definition (1955)

A binary sequence satisfying P1, P2, and P3 is called a **PN-sequence** (pseudonoise sequence).

Run distribution: $n = 4$, period=15



4 runs of length 1, 2 of length 2, 1 of length 3 (zeros) and 1 of length 4 (ones).

Autocorrelation of m-Sequences (P3)

Definition: For a ± 1 binary sequence $\{s_k\}$ with period $N = 2^n - 1$, the **periodic autocorrelation function (PACF)** is:

$$C(\tau) = \sum_{k=0}^{N-1} s_k \, s_{k+\tau}$$

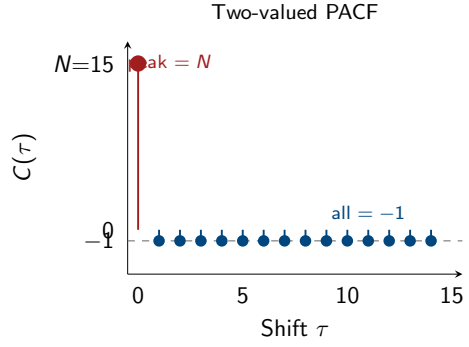
where $s_k = (-1)^{a_k}$ maps $\{0,1\} \to \{+1,-1\}$.
For an m-sequence of period $N = 2^n - 1$:

$$C(\tau) = \begin{cases} N & \text{if } \tau \equiv 0 \pmod{N} \\ -1 & \text{if } \tau \not\equiv 0 \pmod{N} \end{cases}$$

Why this matters for RNGs

The flat (nearly zero) off-peak autocorrelation means successive output values are **nearly uncorrelated** — key for a good PRNG.

PACF of an m-sequence ($n = 4$, $N = 15$):



Compare to a true random sequence where $C(\tau) \approx 0$ for $\tau \neq 0$.

The Cycle Structure Theorem

For a general (possibly non-primitive) characteristic polynomial:

If $f(x) = f_1(x)^{e_1} f_2(x)^{e_2} \cdots f_k(x)^{e_k}$ is the factorization over $\mathrm{GF}(2)$, where each $f_i(x)$ is irreducible of degree d_i and order $\mathrm{ord}(f_i) = r_i$, then the output sequences produced by the LFSR split into cycles whose lengths divide

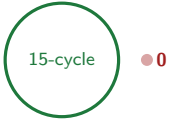
$$\mathrm{lcm}(r_1^{e_1},\ r_2^{e_2},\ \ldots,\ r_k^{e_k}).$$

The primitive case

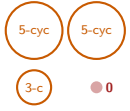
If $f(x)$ is primitive of degree n:
$\Rightarrow$ exactly **one** cycle of length $2^n - 1$
$\Rightarrow$ plus the fixed point **0**
$\Rightarrow$ total 2^n states accounted for

Comparison: primitive vs. non-primitive ($n = 4$):

Polynomial	Factorization	Cycle stru
$1 + x + x^4$	primitive	1 cycle of le
$1 + x^4$	$(1 + x)^4$	cycles of le
$1 + x + x^2 + x^3 + x^4$	reducible	mixed cy
$(1 + x + x^2)^2$	repeated	cycles div



primitive



non-primitive

Decimation of m-Sequences

Definition:
The d-**th decimation** of a sequence $\{a_k\}$ is the subsequence $\{a_{dk}\}$ (every d-th term).

Theorem (Golomb)

If $\{a_k\}$ is an m-sequence of period $N = 2^n - 1$ and $\gcd(d, N) = 1$, then the d-th decimation $\{a_{dk}\}$ is also an m-sequence of the same period.

Consequences:

- All $\phi(N)/n$ m-sequences of period N are decimations of one another — they form a single **equivalence class**
- Decimation by 2 gives a **cyclic shift**: $\{a_{2k}\}$ is just $\{a_k\}$ shifted
- This is the **shift-and-add property** of m-sequences

Shift-and-add property:

Theorem

If $\{a_k\}$ is an m-sequence and $\{b_k\}$ is any non-zero cyclic shift of $\{a_k\}$, then $a_k \oplus b_k = c_k$ where $\{c_k\}$ is *another* cyclic shift of $\{a_k\}$.

Implication for RNGs:
Two m-sequence generators running from different initial states produce sequences that XOR to yet another phase of the same sequence — a beautiful algebraic structure.

Gold sequences

Pairs of m-sequences with good cross-correlation (Gold, 1967) exploit this property for CDMA communications.

Cross-Correlation of Binary Sequences

Definition:
For two ± 1 binary sequences $\{s_k\}$ and $\{t_k\}$ of period N, the **periodic cross-correlation function (PCCF)** is:

$$C_{st}(\tau) = \sum_{k=0}^{N-1} s_k \, t_{k+\tau}$$

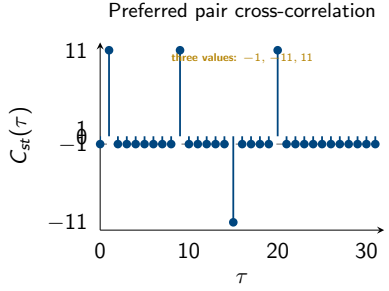
Why cross-correlation matters:

- In **CDMA**: multiple users share a channel; low cross-correlation between their codes prevents interference
- In **parallel RNG**: streams from different seeds should be uncorrelated
- In **radar/sonar**: cross-correlation used for ranging

Ideal cross-correlation

We want $|C_{st}(\tau)| \ll N$ for **all** τ when $\{s\} \neq \{t\}$.

Cross-correlation values for two m-sequences ($n = 5$):



A "preferred pair" of m-sequences achieves only **three** *cross-correlation values:* $\{-1, -(2^{(n+1)/2}+1), 2^{(n+1)/2}-1\}$ *for odd* n.

Gold Sequences and Kasami Sequences

Gold sequences (Gold, 1967):

Take a **preferred pair** of m-sequences $\{u_k\}$, $\{v_k\}$ of period $N = 2^n - 1$ (n odd or $n \equiv 2 \pmod 4$). Form:

$$g_\tau(k) = u_k \oplus v_{k+\tau}, \quad \tau = 0, 1, \ldots, N-1$$

Together with $\{u_k\}$ and $\{v_k\}$, this gives a **family of $N+2$ sequences** all of period N.

Gold's theorem

Every pair within the Gold family has cross-correlation taking only the three values $\{-1, -(2^{(n+1)/2}+1), 2^{(n+1)/2}-1\}$, achieving the **Welch lower bound** (nearly).

Summary of sequence families:

Family	Size	Max $\lvert C \rvert$	Period
m-sequences	$\phi(N)/n$	3-val	$2^n - 1$
Gold	$N+2$	$2^{(n+1)/2}+1$	$2^n - 1$
Kasami (small)	$2^{n/2}$	$2^{n/2}+1$	$2^n - 1$
Bent (even n)	large	$2^{n/2}$	varies

RNG relevance

Gold and Kasami families are used to generate **multiple independent streams** in parallel RNGs (e.g., SPRNG's LFSR generator uses Gold sequences to guarantee stream independence).

Non-Linear Feedback Shift Registers

Motivation:
Linear shift registers are **cryptographically weak**:

- Given $2n$ consecutive output bits, the Berlekamp-Massey algorithm recovers the characteristic polynomial in $O(n^2)$ time
- The sequence is *completely predictable* from short output

Non-linear alternatives:

1. **Non-linear feedback function** f — e.g., add a non-linear term to a linear recurrence
2. **Non-linear output filter** — pass LFSR output through a non-linear Boolean function (filter generator)
3. **Combination generator** — XOR outputs of multiple LFSRs through a non-linear combiner g
4. **Clock-controlled generators** — vary the clock rate of one LFSR based on the output of another (shrinking generator, alternating step generator)

Non-Linear Feedback Shift Registers (Cont.)

de Bruijn sequences

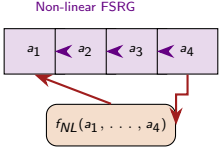
A **de Bruijn sequence of order** n is a binary sequence of period 2^n in which every n-bit pattern appears exactly once.

Produced by a **non-linear** shift register of length n.

The number of distinct de Bruijn sequences of order n is

$$2^{2^{n-1}-n}$$

For $n = 4$: $2^{8-4} = 16$ de Bruijn sequences of period 16.



LFSRs as Pseudorandom Number Generators

Practical LFSR-based PRNGs:

1. **Tausworthe generators** Combine the bits of an LFSR into w-bit words:

$$x_k = \sum_{i=0}^{w-1} a_{wk+i}\, 2^{-i-1}$$

The resulting floating-point values are uniform on $(0, 1)$.

2. **GFSR (Generalized FSR)** (Lewis & Payne, 1973) Use w-bit registers:

$$X_k = X_{k-p} \oplus X_{k-q}, \quad p > q$$

Period: $2^{wp} - 1$ if characteristic polynomial is primitive.

3. **Mersenne Twister** (Matsumoto & Nishimura, 1998)
GFSR variant with $n = 624$, period $2^{19937} - 1$ — the most widely used PRNG today.

LFSRs as Pseudorandom Number Generators (Cont.)

Comparison of LFSR-based generators:

Generator	Period	State bits
Basic LFSR	$2^n - 1$	n
Tausworthe	$2^n - 1$	n
GFSR	$2^{wp} - 1$	wp
Mersenne Twister	$2^{19937} - 1$	19937
Combined Tausworthe	$\approx 2^{88}$	88

Key advantage over LCGs

LFSR-based generators:

- No lattice structure in high dimensions
- Excellent autocorrelation (P3)
- Very fast in hardware (XOR gates only)
- Long provable periods

The Berlekamp-Massey Algorithm

Problem: Given a finite binary sequence $s_0, s_1, \ldots, s_{N-1}$, find the **shortest LFSR** that generates it.

The linear complexity $L(s)$: the length of the shortest LFSR generating $\{s_k\}$.

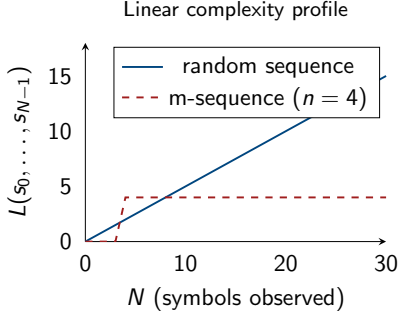
Berlekamp-Massey algorithm

Finds $f(x)$ (characteristic polynomial of length $L(s)$) from $\{s_k\}$ in $O(N^2)$ time — or $O(N \log N)$ with improvements.

Requires only $2L(s) \leq N$ output symbols to uniquely determine the minimal LFSR.

Cryptographic implication:
An m-sequence of length $2^n - 1$ has $L = n$ — catastrophically low! A stream cipher must have $L \gg n$ to resist this attack.

Linear complexity profile:



A random sequence has $L \approx N/2$ (unpredictable). An m-sequence has $L = n$ (predictable after $2n$ bits).

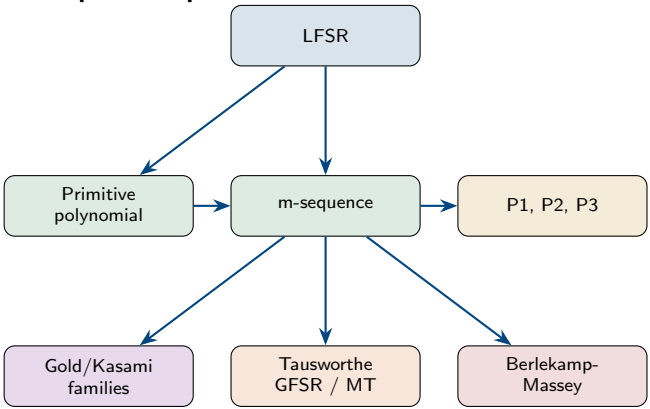
Key Results Summary

What we covered (Golomb, 1982):

Topic	Key Result
LFSR definition	$a_{n+k} = \bigoplus c_i a_{k+n-i}$
Primitive polynomial	Yields m-sequence of period $2^n - 1$
# Primitive polys	$\phi(2^n - 1)/n$
P1: Balance	2^{n-1} ones, $2^{n-1} - 1$ zeros
P2: Run property	2^{n-k-1} runs of length k
P3: Autocorrelation	$C(\tau) = N$ or -1
Shift-and-add	$a_k \oplus a_{k+\tau} = a_{k+\sigma}$
Decimation	$\gcd(d, N) = 1 \Rightarrow$ m-sequence
Gold sequences	$N + 2$ seqs, $\lvert C_{st}\rvert \leq 2^{(n+1)/2} + 1$
Linear complexity	$L = n$ (cryptographically weak)
Berlekamp-Massey	Recovers LFSR from $2n$ bits
Mersenne Twister	GFSR, period $2^{19937} - 1$

Key Results Summary (Cont.)

Conceptual map:



Discussion Questions

1. A 4-stage LFSR with characteristic polynomial $f(x) = 1 + x^2 + x^4$ is *not* primitive. Factor $f(x)$ over $\mathrm{GF}(2)$ and describe the resulting cycle structure. What is the maximum period achievable?

2. Verify Golomb's three randomness postulates P1, P2, P3 for the m-sequence generated by $f(x) = 1 + x + x^4$ with initial state $(1, 0, 0, 0)$. Write out the full period-15 sequence.

3. The Berlekamp-Massey algorithm can break any LFSR-based generator from $2n$ bits of output. Explain why the Mersenne Twister (despite using an LFSR structure) is still considered secure enough for simulation (though not for cryptography).

4. Explain the **shift-and-add property** of m-sequences geometrically. Why does this property make m-sequences attractive for generating *multiple independent streams* in parallel Monte Carlo simulations?

5. Compare and contrast LCGs (Chapter 3 of H&H) and LFSRs: period, statistical quality, lattice structure, hardware efficiency, and predictability. When would you prefer each?

Further Reading & References

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