3. Equivalence Relations

3.1. Definition of an Equivalence Relations.

DEFINITION 3.1.1. *A relation R on a set A is an* **equivalence relation** *if and only if R is*

- *reflexive,*
- *symmetric, and*
- *transitive.*

Discussion

Section 3.1 recalls the definition of an equivalence relation. In general an equivalence relation results when we wish to "identify" two elements of a set that share a common attribute. The definition is motivated by observing that any process of "identification" must behave somewhat like the equality relation, and the equality relation satisfies the reflexive ($x = x$ for all x), symmetric ($x = y$ implies $y = x$), and transitive ($x = y$ and $y = z$ implies $x = z$) properties.

3.2. Example.

EXAMPLE 3.2.1. *Let R be the relation on the set $\mathbb{R}$ real numbers defined by xRy iff $x - y$ is an integer. Prove that R is an equivalence relation on $\mathbb{R}$.*

PROOF.

I. Reflexive: Suppose $x \in \mathbb{R}$. Then $x - x = 0$, which is an integer. Thus, xRx.
II. Symmetric: Suppose $x, y \in \mathbb{R}$ and xRy. Then $x - y$ is an integer. Since $y - x = -(x - y)$, $y - x$ is also an integer. Thus, yRx.
III. Suppose $x, y \in \mathbb{R}$, xRy and yRz. Then $x - y$ and $y - z$ are integers. Thus, the sum $(x - y) + (y - z) = x - z$ is also an integer, and so xRz.

Thus, R is an equivalence relation on $\mathbb{R}$. □

Discussion

EXAMPLE 3.2.2. *Let R be the relation on the set of real numbers $\mathbb{R}$ in Example 1. Prove that if xRx' and yRy', then $(x + y)R(x' + y')$.*

PROOF. Suppose xRx' and yRy'. In order to show that $(x+y)R(x'+y')$, we must show that $(x + y) - (x' + y')$ is an integer. Since

$$(x + y) - (x' + y') = (x - x') + (y - y'),$$

and since each of $x - x'$ and $y - y'$ is an integer (by definition of R), $(x - x') + (y - y')$ is an integer. Thus, $(x + y)R(x' + y')$.

□

EXERCISE 3.2.1. *In the example above, show that it is possible to have xRx' and yRy', but $(xy)\not R(x'y')$.*

EXERCISE 3.2.2. *Let V be the set of vertices of a simple graph G. Define a relation R on V by vRw iff v is adjacent to w. Prove or disprove: R is an equivalence relation on V.*

3.3. Equivalence Classes.

DEFINITION 3.3.1.

(1) Let R be an equivalence relation on A and let $a \in A$. The set $[a] = \{x | aRx\}$ is called the **equivalence class** *of a.*
(2) The element in the bracket in the above notation is called the **Representative** *of the equivalence class.*

THEOREM 3.3.1. *Let R be an equivalence relation on a set A. Then the following are equivalent:*

(1) aRb
(2) $[a] = [b]$
(3) $[a] \cap [b] \neq \emptyset$

PROOF. $\underline{1 \to 2}$. Suppose $a, b \in A$ and aRb. We must show that $[a] = [b]$.

Suppose $x \in [a]$. Then, by definition of $[a]$, aRx. Since R is symmetric and aRb, bRa. Since R is transitive and we have both bRa and aRx, bRx. Thus, $x \in [b]$.

Suppose $x \in [b]$. Then bRx. Since aRb and R is transitive, aRx. Thus, $x \in [a]$.

We have now shown that $x \in [a]$ if and only if $x \in [b]$. Thus, $[a] = [b]$.

$\underline{2 \to 3}$. Suppose $a, b \in A$ and $[a] = [b]$. Then $[a] \cap [b] = [a]$. Since R is reflexive, aRa; that is $a \in [a]$. Thus $[a] = [a] \cap [b] \neq \emptyset$.

$\underline{3 \to 1}$. Suppose $[a] \cap [b] \neq \emptyset$. Then there is an $x \in [a] \cap [b]$. By definition, aRx and bRx. Since R is symmetric, xRb. Since R is transitive and both aRx and xRb, aRb. □

Discussion

The purpose of any identification process is to break a set up into subsets consisting of mutually identified elements. An equivalence relation on a set A does precisely this: it decomposes A into special subsets, called *equivalence classes*. Looking back at the example given in Section 3.2, we see the following equivalence classes:

- $[0] = \mathbb{Z}$, the set of integers.
- $[\frac{1}{2}] = \{\frac{m}{2} | m$ is an odd integer$\}$
- $[\pi] = \{\pi + n | n$ is an integer$\} = [\pi + n]$, for any integer n.

Notice that $[\frac{3}{4}] = [-\frac{37}{4}]$. The number $\frac{3}{4}$ is a representative of $[\frac{3}{4}]$, but $-\frac{37}{4}$ is also a representative of $[\frac{3}{4}]$. Indeed, any element of an equivalence class can be used to represent that equivalence class.

These ideas are summed up in Theorem 3.3.1 in Section 3.3. When we say several statements, such as P_1, P_2, and P_3 are equivalent, we mean $P_1 \leftrightarrow P_2 \leftrightarrow P_3$ is true. Notice that in order to prove that the statements are mutually equivalent, it is sufficient to prove a circle of implications, such as $P_1 \to P_2 \to P_3 \to P_1$. This is how we set up the proof of Theorem 3.3.1.

3.4. Partition.

Definition 3.4.1. *A collection $\mathcal{S}$ of nonempty subsets of a set A is a* **partition** *of A if*

(1) $S \cap S' = \emptyset$, if S and S' are in $\mathcal{S}$ and $S \neq S'$, and
(2) $A = \bigcup\{S | S \in \mathcal{S}\}$.

Theorem 3.4.1. *The equivalence classes of an equivalence relation on A form a partition of A. Conversely, given a partition on A, there is an equivalence relation with equivalence classes that are exactly the partition given.*

Discussion

The definition in Section 3.4 along with Theorem 3.4.1 describe formally the properties of an equivalence relation that motivates the definition. Such a decomposition is called a **partition**. For example, if we wish to identify two integers if they are either both even or both odd, then we end up with a partition of the integers into two sets, the set of even integers and the set of odd integers. The converse of Theorem 3.4.1 allows us to create or define an equivalence relation by merely partitioning a set into mutually exclusive subsets. The common "attribute" then might just be that elements belong to the same subset in the partition.

the notation used in the second part of Theorem 3.4.1 means that we take the union of all the sets that are members of the set to the far right and this union is defined to be set A.

DEFINITION 3.4.2. *If R is an equivalence relation on a set A, the set of equivalence classes of R is denoted A/R.*

Theorem 3.4.1 follows fairly easily from Theorem 3.3.1 in Section 3.3. Here is a proof of one part of Theorem 3.4.1.

PROOF. Suppose R is an equivalence relation on A and $\mathcal{S}$ is the set of equivalence classes of R. If S is an equivalence class, then $S = [a]$, for some $a \in A$; hence, S is nonempty, since aRa by the reflexive property of R.

By Theorem 3.3.1, if $S = [a]$ and $S' = [b]$ are in $\mathcal{S}$, then $[a] = [b]$ iff $[a] \cap [b] \neq \emptyset$. Since this is a biconditional, this statement is equivalent to $[a] \neq [b]$ iff $[a] \cap [b] = \emptyset$.

Since each equivalence class is contained in A, $\bigcup\{S|S \in \mathcal{S}\} \subseteq A$. But, as we just saw, every element in A is in the equivalence class it represents, so $A \subseteq \bigcup\{S|S \in \mathcal{S}\}$. This shows $\bigcup\{S|S \in \mathcal{S}\} = A$. □

EXERCISE 3.4.1. *Prove the converse statement in Theorem 3.4.1.*

3.5. Intersection of Equivalence Relations.

THEOREM 3.5.1. *If R_1 and R_2 are equivalence relations on a set A then $R_1 \cap R_2$ is also an equivalence relation on A.*

Discussion

To prove Theorem 3.5.1, it suffices to show the intersection of

- reflexive relations is reflexive,
- symmetric relations is symmetric, and
- transitive relations is transitive.

But these facts were established in the section on the Review of Relations.

3.6. Example.

EXAMPLE 3.6.1. *Let m be a positive integer. The relation $a \equiv b \,(mod\ m)$, is an equivalence relation on the set of integers.*

PROOF. Reflexive. If a is an arbitrary integer, then $a - a = 0 = 0 \cdot m$. Thus $a \equiv a \pmod{m}$.

Symmetric. If $a \equiv b \pmod{m}$, then $a - b = k \cdot m$ for some integer k. Thus, $b - a = (-k) \cdot m$ is also divisible by m, and so $b \equiv a \pmod{m}$.

Transitive. Suppose $a \equiv b \pmod{m}$ and $b \equiv c \pmod{m}$. Then $a - b = k \cdot m$ and $b - c = \ell \cdot m$ for some integers k and ℓ. Then

$$a - c = (a - b) + (b - c) = k \cdot m + \ell \cdot m = (k + \ell)m$$

is also divisible by m. That is, $a \equiv c \pmod{m}$. □

Discussion

Recall the "congruence" relations on the set $\mathbb{Z}$ of integers: Given an positive integer m and integers a and b, $a \equiv b \pmod{m}$ (read "a is congruent to b modulo m) iff $m|(a - b)$; that is, $a - b = k \cdot m$ for some integer k.

EXERCISE 3.6.1. *What are the equivalence classes for the congruence relation*

(1) $a \equiv b \pmod{2}$*?*
(2) $a \equiv b \pmod{3}$*?*
(3) $a \equiv b \pmod{5}$*?*

Given a positive integer m, the equivalence classes under the relation $a \equiv b \pmod{m}$ have *canonical* representatives. If we use the Division Algorithm to divide the integer a by the integer m, we get a quotient q and remainder r, $0 \leq r < m$, satisfying the equation $a = mq + r$. Recall that $r = a \,\mathbf{mod}\, m$ and that $a \equiv r \pmod{m}$. Thus $[a] = [r]$, and so there are exactly m equivalence classes

$$[0], [1], ...[m - 1].$$

If R is the congruence modulo m relation on the set $\mathbb{Z}$ of integers, the set of equivalence classes, $\mathbb{Z}/R$ is usually denoted by either $\mathbb{Z}/m$ or $\mathbb{Z}/m\mathbb{Z}$. That is,

$$\mathbb{Z}/m = \{[0], [1], ...[m - 1]\}.$$

REMARK 3.6.1. *If A is an infinite set and R is an equivalence relation on A, then A/R may be finite, as in the example above, or it may be infinite. As the following exercise shows, the set of equivalences classes may be very large indeed.*

EXERCISE 3.6.2. *Let R be the equivalence relation defined on the set of real numbers $\mathbb{R}$ in Example 3.2.1 (Section 3.2). That is, xRy iff $x - y$ is an integer. Prove that every equivalence class $[x]$ has a* unique *canonical representative r such that $0 \leq r < 1$. That is, for every x there is a unique r such that $[x] = [r]$ and $0 \leq r < 1$. [Hint: You might recall the "floor" function $f(x) = \lfloor x \rfloor$.]*

3.7. Example.

EXAMPLE 3.7.1. *Let R be the relation on the set of ordered pairs of positive integers such that $(a, b)R(c, d)$ if and only if $ad = bc$.*

- *R is an equivalence relation.*
- *The equivalence class of $(2, 3)$:*
$$[(2, 3)] = \{(2k, 3k) | k \in \mathbb{Z}^+\}.$$
- *There is a natural bijection between the equivalence classes of this relation and the set of positive rational numbers.*

Discussion

Notice that the relation R in Example 3.7.1 is a relation on the set $\mathbb{Z}^+ \times \mathbb{Z}^+$, and so $R \subseteq (\mathbb{Z}^+ \times \mathbb{Z}^+) \times (\mathbb{Z}^+ \times \mathbb{Z}^+)$.

PROOF R IN EXAMPLE 3.7.1 IS AN EQUIVALENCE RELATION. We must show that R is reflexive, symmetric, and transitive.

I. Reflexive: Let (a, b) be an ordered pair of positive integers. To show R is reflexive we must show $((a, b), (a, b)) \in R$. Multiplication of integers is commutative, so $ab = ba$. Thus $((a, b), (a, b)) \in R$.
II. Symmetric: Let (a, b) and (c, d) be ordered pairs of positive integers such that $(a, b)R(c, d)$ (recall this notation is equivalent to $((a, b), (c, d)) \in R$). Then $ad = bc$. This equation is equivalent to $cb = da$, so $(c, d)R(a, b)$. This shows R is symmetric.
III. Transitive: Let (a, b), (c, d), and (e, f) be ordered pairs of positive integers such that $(a, b)R(c, d)$ and $(c, d)R(e, f)$. Then $ad = bc$ and $cf = de$. Thus, $adf = bcf$ and $bcf = bde$, which implies $adf = bde$. Since $d \neq 0$, we can cancel it from both sides of this equation to get $af = be$. This shows $(a, b)R(e, f)$, and so R is transitive.

□

One of the points of this example is that there is a bijection between the equivalence classes of this relation and the set of positive rational numbers. In other words, the function

$$f : (\mathbb{Z}^+ \times \mathbb{Z}^+)/R = \{[(a, b)] | [(a, b)] \text{ is an equivalence class of } R\} \to \mathbb{Q}^+$$

defined by $f([(a, b)]) = a/b$ is well-defined and is a bijection. This follows from the fact that

$$[(a, b)] = [(c, d)] \Leftrightarrow (a, b)R(c, d) \Leftrightarrow ad = bc \Leftrightarrow \frac{a}{b} = \frac{c}{d}.$$

EXERCISE 3.7.1. *Let R be the relation defined on the set of ordered pairs $\mathbb{Z}^+ \times \mathbb{Z}^+$ of positive integers defined by*

$$(a, b)R(c, d) \Leftrightarrow a + d = b + c.$$

1. *Prove that R is an equivalence relation on $\mathbb{Z}^+ \times \mathbb{Z}^+$.*
2. *List 5 different members of the equivalence class $[(1, 4)]$.*

EXERCISE 3.7.2. *Let R be the relation defined on the set of ordered pairs $\mathbb{Z}^+ \times \mathbb{Z}^+$ of positive integers defined by*

$$(a, b)R(c, d) \Leftrightarrow a + d = b + c.$$

Prove that the function $f\colon \mathbb{Z}^+ \times \mathbb{Z}^+/R \to \mathbb{Z}$, defined by $f([a, b]) = a - b$, is well-defined and a bijection.

3.8. Isomorphism is an Equivalence Relation.

THEOREM 3.8.1. *Let S be a set of simple graphs. Define a relation R on S as follows:*

If $G, H \in S$, then $(G, H) \in R$ if and only if $G \simeq H$.

(This is equivalent to $(G, H) \in R$ if and only if there exists an isomorphism $f\colon V(G) \to V(H)$ that preserves adjacencies.)

Then R is an equivalence relation on S.

PROOF. Reflexive. Suppose $G \in S$. We need to show $(G, G) \in R$.

Define the function $f\colon V(G) \to V(G)$ by $f(v) = v$ for every vertex $v \in V(G)$. Then f is the identity function on $V(G)$; hence, f is a bijection. ($f^{-1} = f$!)

Clearly, v and u are adjacent in G if and only if $f(v) = v$ and $f(u) = u$ are adjacent.

Thus, $(G, G) \in R$.

Symmetric. Suppose $G, H \in S$ and $(G, H) \in R$. Then there exists an isomorphism $f\colon V(G) \to V(H)$. We need to find an isomorphism $g\colon V(H) \to V(G)$.

Since f is a bijection, f is invertible. Thus the map $f^{-1}\colon V(H) \to V(G)$ is defined, and we shall show it is an isomorphism. We know the inverse of a bijection is itself a bijection, so all we need to show is that f^{-1} preserves adjacency.

Suppose $u, v \in V(H)$. Then $f^{-1}(u) = x$ and $f^{-1}(v) = y$ are vertices of G.

Now, we know f preserves adjacency, so x and y are adjacent in G if and only if $f(x) = u$ and $f(y) = v$ are adjacent in H. Use the previous equations to rewrite this statement in terms of u and v: $f^{-1}(u)(= x)$ and $f^{-1}(v)(= y)$ are adjacent in G if and only if $u(= f(x))$ and $v(= f(y))$ are adjacent in H.

Thus f^{-1} preserves adjacency, and so $(H, G) \in R$.

Transitive. Suppose $G, H, K \in S$ are graphs such that $(G, H), (H, K) \in R$. We need to prove $(G, K) \in R$.

Since (G, H) and (H, K) are in R, there are isomorphisms $f: V(G) \to V(H)$ and $g: V(H) \to V(K)$. We need to find an isomorphism $h: V(G) \to V(K)$. *Notice that we have used different letters for the functions here. The function g is not necessarily the same as the function f, so we cannot call it f as well.*

Let $h = g \circ f$. We will show h is an isomorphism.

Since the composition of bijections is again a bijection, $g \circ f: V(G) \to V(K)$ is a bijection.

What we still need to show is that the composition preserves adjacency. Let u and v be vertices in G. Recall that f must preserve adjacency. Therefore, u and v are adjacent in G if and only if $f(u)$ and $f(v)$ are adjacent in H. But since g preserves adjacency, $f(u)$ and $f(v)$ are adjacent in H if and only if $g(f(u))$ and $g(f(v))$ are adjacent in K. Using the fact that "if and only if" is transitive, we see that u and v are adjacent in G if and only if $(g \circ f)(u)$ and $(g \circ f)(v)$ are adjacent in K. This implies that $g \circ f$ preserves adjacency, and so $g \circ f: V(G) \to V(K)$ is an isomorphism.

□

Discussion

Section 3.8 recalls the notion of graph isomorphism. Here we prove that graph isomorphism is an equivalence relation on any set of graphs. It is tempting to say that graph isomorphism is an equivalence relation on the "set of all graphs," but logic precludes the existence of such a set.

3.9. Equivalence Relation Generated by a Relation R.

Definition 3.9.1. *Suppose R is a relation on a set A. The* **equivalence relation on A generated by a** R, *denoted R_e, is the smallest equivalence relation on A that contains R.*

Discussion

There are occasions in which we would like to define an equivalence relation on a set by starting with a primitive notion of "equivalence", which, in itself, may not satisfy one or more of the three required properties. For example, consider the set of vertices V of a simple graph G and the adjacency relation R on V: uRv iff u is adjacent to v. You would have discovered while working through Exercise 3.2.2 that, for most graphs, R is neither reflexive nor transitive.

EXERCISE 3.9.1. *Suppose V is the set of vertices of a simple graph G and R is the adjacency relation on V: uRv iff u is adjacent to v. Prove that R_e is the relation*

$$uR_ev \text{ iff either } u = v \text{ or there is a path in } G \text{ from } u \text{ to } v.$$

3.10. Using Closures to find an Equivalence Relation.

THEOREM 3.10.1. *Suppose R is a relation on a set A. Then R_e, the equivalence relation on A generated by R, is the relation $t(s(r(R)))$. That is, R_e may be obtained from R by taking*

(1) the reflexive closure $r(R)$ of R, then
(2) the symmetric closure $s(r(R))$ of $r(R)$, and then
(3) the transitive closure $t(s(r(R)))$ of $s(r(R))$.

PROOF. Suppose R is a relation on a set A. We must show
(1) $t(s(r(R)))$ is an equivalence relation containing R, and
(2) if S is an equivalence relation containing R, then $t(s(r(R))) \subseteq S$.

Proof of (1).

I. Reflexive: If $a \in A$, then $(a, a) \in r(R)$; hence, $(a, a) \in t(s(r(R)))$, since $r(R) \subseteq t(s(r(R)))$.
II. Symmetric: Suppose $(a, b) \in t(s(r(R)))$. Then there is a chain $(a, x_1), (x_1, x_2), ..., (x_n, b)$ in $s(r(R))$. Since $s(r(R))$ is symmetric, $(b, x_n), ..., (x_2, x_1), (x_1, a)$ are in $s(r(R))$. Hence, $(b, a) \in t(s(r(R)))$, since $t(s(r(R)))$ is transitive.
III. Transitive: $t(s(r(R)))$, being the transitive closure of $s(r(R))$, is transitive, by definition.

Proof of (2). Suppose S is an equivalence relation containing R.

I. Since S is reflexive, S contains the reflexive closure of R. That is, $r(R) \subseteq S$.
II. Since S is symmetric and $r(R) \subseteq S$, S contains the symmetric closure of $r(R)$. That is, $s(r(R)) \subseteq S$.
III. Since S is transitive and $s(r(R)) \subseteq S$, S contains the transitive closure of $s(r(R))$. That is, $t(s(r(R))) \subseteq S$.

□

Discussion

Theorem 3.10.1 in Section 3.10 describes the process by which the equivalence relation generated by a relation R can be constructed using the closure operations discussed in the notes on Closure. As it turns out, it doesn't matter whether you take the reflexive closure before you take the symmetric and transitive closures, but it is important that the symmetric closure be taken *before* the transitive closure.

EXERCISE 3.10.1. *Given a relation R on a set A, prove that $R_e = (R \cup \Delta \cup R^{-1})^*$. [See the lecture notes on Closure for definitions of the terminology.]*

EXERCISE 3.10.2. *Suppose A is the set of all people (alive or dead) and R is the relation "is a parent of". Describe the relation R_e in words. What equivalence class do you represent?*

EXERCISE 3.10.3. *Give an example of a relation R on a set A such that $R_e \neq s(t(r(R)))$.*