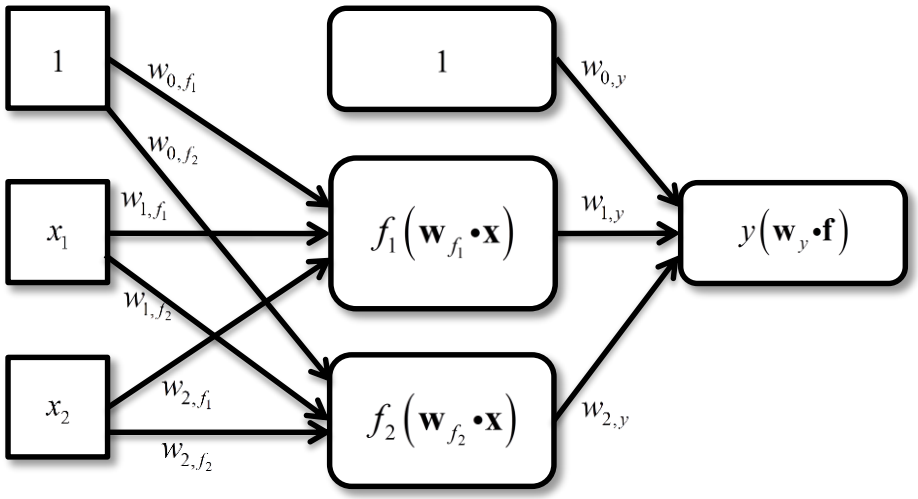
3. Neural Networks (100 Points).

Given the following Neural Network:



where $\mathbf{w}_{f_1}\bullet\mathbf{x}=w_{0,f_1}\cdot 1+w_{1,f_1}\cdot x_1+w_{2,f_1}\cdot x_2$, $\mathbf{w}_{f_2}\bullet\mathbf{x}=w_{0,f_2}\cdot 1+w_{1,f_2}\cdot x_1+w_{2,f_2}\cdot x_2$, $\mathbf{w}_y\bullet\mathbf{f}=w_{0,y}\cdot 1+w_{1,y}\cdot f_1+w_{2,y}\cdot f_2$, and activation functions f_1, f_2, and y are step functions defined as follows:

$$f_1\left(\mathbf{w}_{f_1}\bullet\mathbf{x}\right)=\begin{cases}\mathbf{w}_{f_1}\bullet\mathbf{x}\geq 0, & 1\\ \mathbf{w}_{f_1}\bullet\mathbf{x}<0, & 0\end{cases}\qquad \mathbf{w}_{f_1}\bullet\mathbf{x}=\left(\cos\left(135°\right),\sin\left(135°\right)\right)\bullet\left(\left(x_1,x_2\right)-\left(0,\frac{1}{2}\right)\right)$$

$$f_2\left(\mathbf{w}_{f_2}\bullet\mathbf{x}\right)=\begin{cases}\mathbf{w}_{f_2}\bullet\mathbf{x}\geq 0, & 1\\ \mathbf{w}_{f_2}\bullet\mathbf{x}<0, & 0\end{cases}\quad\text{where}\quad \mathbf{w}_{f_2}\bullet\mathbf{x}=\left(\cos\left(-45°\right),\sin\left(-45°\right)\right)\bullet\left(\left(x_1,x_2\right)-\left(\frac{1}{2},0\right)\right)$$

$$y\left(\mathbf{w}_y\bullet\mathbf{f}\right)=\begin{cases}\mathbf{w}_y\bullet\mathbf{f}\geq 0, & 1\\ \mathbf{w}_y\bullet\mathbf{f}<0, & 0\end{cases}\qquad \mathbf{w}_y\bullet\mathbf{f}=\left(\cos\left(45°\right),\sin\left(45°\right)\right)\bullet\left(\left(f_1,f_2\right)-\left(0,\frac{1}{2}\right)\right)$$

used to create this function:

x_1	x_2	y
0	0	**0**
0	1	**1**
1	0	**1**
1	1	**0**

HINT: It will be very helpful to plot the $\left(x_1,x_2\right)$ points, draw the lines $\mathbf{w}_{f_1}\bullet\mathbf{x}=0$ and $\mathbf{w}_{f_2}\bullet\mathbf{x}=0$ on that plot, and shade each positive region (i.e. where $f_1=1$ and $f_2=1$).

HINT: It will be very helpful to plot the (f_1, f_2) points, draw the line $\mathbf{w}_y \bullet \mathbf{f} = 0$ on that plot, and shade the positive region (i.e. where $y = 1$).

Example: w_{0,f_1} may be written as w0f1.

Example: w_{1,f_2} may be written as w1f2.

Example: $w_{2,y}$ may be written as w2y.

Answer the following questions:

NOTE: To receive partial credit, show your work.

$$\begin{aligned}
\mathbf{w}_{f_1} \bullet \mathbf{x} &= \left(\cos(135°), \sin(135°)\right) \bullet \left((x_1, x_2) - \left(0, \frac{1}{2}\right)\right) \\
&= \cos(135°)(x_1 - 0) + \sin(135°)\left(x_2 - \frac{1}{2}\right) \\
&= \left(-0\cos(135°) - \frac{1}{2}\sin(135°)\right) + \left(\cos(135°)\right)x_1 + \left(\sin(135°)\right)x_2 \\
&= \underbrace{\left(-\frac{\sqrt{2}}{4}\right)}_{w_{0,f1}} + \underbrace{\left(-\frac{\sqrt{2}}{2}\right)}_{w_{1,f1}} x_1 + \underbrace{\left(\frac{\sqrt{2}}{2}\right)}_{w_{2,f1}} x_2 \\
&\approx \underbrace{(-0.353553)}_{w_{0,f1}} + \underbrace{(-0.707107)}_{w_{1,f1}} x_1 + \underbrace{(0.707107)}_{w_{2,f1}} x_2
\end{aligned}$$

1) What is the value of parameter w0f1?

$$w_{0,f_1} = -\frac{\sqrt{2}}{4} \approx -0.353553$$

2) What is the value of parameter w1f1?

$$w_{1,f_1} = -\frac{\sqrt{2}}{2} \approx -0.707107$$

3) What is the value of parameter w2f1?

$$w_{2,f_1} = \frac{\sqrt{2}}{2} \approx 0.707107$$

$$\mathbf{w}_{f_2} \bullet \mathbf{x} = \left(\cos(-45°), \sin(-45°)\right) \bullet \left((x_1, x_2) - \left(\frac{1}{2}, 0\right)\right)$$

$$= \cos(-45°)\left(x_1 - \frac{1}{2}\right) + \sin(-45°)(x_2 - 0)$$

$$= \left(-\frac{1}{2}\cos(-45°) - 0\sin(-45°)\right) + \left(\cos(-45°)\right)x_1 + \left(\sin(-45°)\right)x_2$$

$$= \underbrace{\left(-\frac{\sqrt{2}}{4}\right)}_{w_{0,f_2}} + \underbrace{\left(\frac{\sqrt{2}}{2}\right)}_{w_{1,f_2}} x_1 + \underbrace{\left(-\frac{\sqrt{2}}{2}\right)}_{w_{2,f_2}} x_2$$

$$\approx \underbrace{(-0.353553)}_{w_{0,f_2}} + \underbrace{(0.707107)}_{w_{1,f_2}} x_1 + \underbrace{(-0.707107)}_{w_{2,f_2}} x_2$$

4) What is the value of parameter w0f2?

$$w_{0,f_2} = -\frac{\sqrt{2}}{4} \approx -0.353553$$

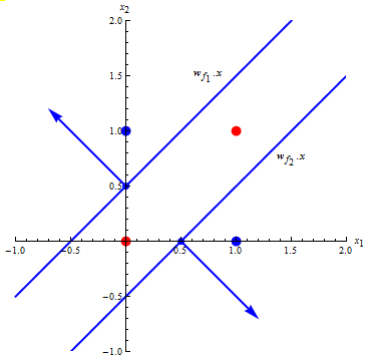
5) What is the value of parameter w1f2?

$$w_{1,f_2} = \frac{\sqrt{2}}{2} \approx 0.707107$$

6) What is the value of parameter w2f2?

$$w_{2,f_2} = -\frac{\sqrt{2}}{2} \approx -0.707107$$

f_1 and f_2 produce the following plot:



$$\mathbf{w}_y \bullet \mathbf{f} = \left(\cos(45°), \sin(45°)\right) \bullet \left((f_1, f_2) - \left(0, \frac{1}{2}\right)\right)$$

$$= \cos(45°)(f_1 - 0) + \sin(45°)\left(f_2 - \frac{1}{2}\right)$$

$$= \left(-0\cos(45°) - \frac{1}{2}\sin(45°)\right) + \left(\cos(45°)\right) f_1 + \left(\sin(45°)\right) f_2$$

$$= \underbrace{\left(-\frac{\sqrt{2}}{4}\right)}_{w_{0,y}} + \underbrace{\left(\frac{\sqrt{2}}{2}\right)}_{w_{1,y}} f_1 + \underbrace{\left(\frac{\sqrt{2}}{2}\right)}_{w_{2,y}} f_2$$

$$\approx \underbrace{(-0.353553)}_{w_{0,y}} + \underbrace{(0.707107)}_{w_{1,y}} f_1 + \underbrace{(0.707107)}_{w_{2,y}} f_2$$

7) What is the value of parameter w0y?

$$w_{0,y} = -\frac{\sqrt{2}}{4} \approx -0.353553$$

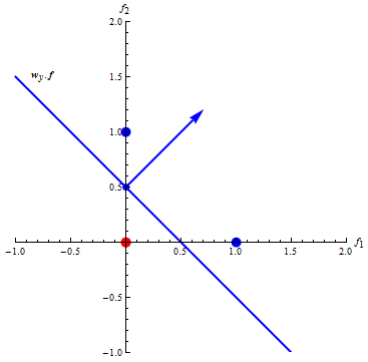
8) What is the value of parameter w1y?

$$w_{1,y} = \frac{\sqrt{2}}{2} \approx 0.707107$$

9) What is the value of parameter w2y?

$$w_{2,y} = \frac{\sqrt{2}}{2} \approx 0.707107$$

y produces the following plot:



10) For functions f1, f2, and y above, calculate the following tables:

x_1	x_2	f_1
0	0	**0**
0	1	**1**
1	0	**0**
1	1	**0**

x_1	x_2	f_2
0	0	**0**
0	1	**0**
1	0	**1**
1	1	**0**

f_1	f_2	y
0	**0**	**0**
1	**0**	**1**
0	**1**	**1**

Since f_1, f_2, and y produce the following plots, respectively: