COT5405: Fall 2006

Lecture 3

Proof for Lemma 2.3

Let e_1, e_2, ..., e_n be the order in which the elements were covered. (If multiple elements were covered at the same time, then they can be numbered in any order amongst themselves.)

Lemma 2.3: *For each k in {1, ..., n}, price* $e_k \leq OPT/(n-k+1)$

Proof: Consider the iteration in which element e_k was covered. At that time, there must be some sets from an optimal solution that have not yet been picked. Let us call these sets as T_1, ..., T_m. The sum of the costs of these sets is at most *OPT*. Taken together, these sets can cover the remaining elements with cost-effectiveness of at most $OPT/(|U-C|)$. Consequently, at least one of those sets must have cost effectiveness $\leq OPT/(|U-C|)$, for the following reason.

> Assume that this is not the case, and the cost effectiveness of each of these sets is
>
> $c(T_i)/(|T_i-C|) > OPT/(|U-C|)$, $1 \leq i <= m$.
>
> So $c(T_i) > OPT\ (|T_i-C|)/(|U-C|)$.
>
> So $\Sigma\ c(T_i) > \Sigma\ OPT\ /(|T_i-C|)/(|U-C|) = OPT/(|U-C|)\ \Sigma\ (|T_i-C|) \geq OPT$ (because $\Sigma\ (|Ti-C|) \geq (|U-C|)$). This is not possible, because the optimal solution would then have cost > *OPT*. Therefore, our assumption was wrong and there was at least one set T_i with cost effectiveness $\leq OPT/(|U-C|)$.

The set picked in this iteration had the smallest cost effectiveness and so its cost effectiveness $\leq OPT/(|U-C|) \leq OPT/(n-k+1)$. So, $e_k \leq OPT/(n-k+1)$.