COT 5405: Fall 2006

Lecture 28

Edmonds-Karp Algorithms

Ford-Fulkerson Method

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Ford-Fulkerson(G, s, t)

o  For each edge (u, v) ∈ E
      o  f(u, v) ← 0,  f(v, u) ← 0
o  While there is an augmenting path p from s to t in G_f
      o  c_f(p) ← min{c_f(u, v) | (u, v) ∈ p}
      o  for each (u, v) ∈ p
            ▪  f(u, v) ← f(u, v) + c_f(p)
            ▪  f(v, u) ← -f(u, v)
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An arbitrary search for an augmenting path leads to $O(|E|\ |f^*|)$ algorithm with *integer* capacities, where $|f^*|$ is the maximum flow.

Edmonds-Karp Algorithm Method

In the Ford-Fulkerson method, when finding p, perform breadth first search and choose the shortest path from s to t as p. Each BFS takes $O(|E|)$ time (since $|E| \geq |V|-1$, from our definition of a flow network). We will show below that the number of augmentations is $O(|V|\ |E|)$, and so the total time complexity is $O(|V|\ |E|^2)$.

Lemma: The shortest path distance from the source, in the residual, $\delta_f(s, v)$, increases monotonically with each augmentation, for each v in $V - \{s, t\}$.

Proof: Assume the contrary. Then there exists $v \in V - \{s, t\}$ such that some augmentation causes the shortest path to decrease. Let v be the vertex with smallest $\delta_{f'}(s, v)$ for which this happens. Let f' be the flow when this happens, and f be the previous flow. $\delta_{f'}(s, v) < \delta_f(s, v)$ from our assumption.

Let $p = s \rightarrow \ldots \rightarrow u \rightarrow v$ be a shortest path from s to v in $G_{f'}$. So, $\delta_{f'}(s, u) = \delta_{f'}(s, v) - 1$. Also, $\delta_{f'}(s, u) \geq \delta_f(s, u)$.

Note that $(u, v) \notin E_f$. Otherwise, we would have $\delta_f(s, v) \leq \delta_f(s, u) + 1 \leq \delta_{f'}(s, u) + 1 = \delta_{f'}(s, v)$, contradicting our assumption.

Since $(u, v) \notin E_f$ but is in $E_{f'}$, (v, u) must have been on the augmenting path in G_f. So, $\delta_f(s, v) = \delta_f(s, u) - 1 \leq \delta_{f'}(s, u) - 1 = \delta_{f'}(s, v) - 2$, contradicting our assumption. Q.E.D.

Definition: Edge $(u, v) \in E_f$ is critical if $|f_p| = c(u, v)$.

Theorem 26.9: The number of augmentations in the Edmonds-Karp algorithm is O(|V| |E|).

Proof:

We will show that an edge can become critical at most $|V|/2$ times. Since each augmentation requires at least one critical edge, this will prove that the number of augmentations is $O(|V|\ |E|)$.

Let the flow be f the first time some edge (u, v) becomes critical. It can become critical again (in fact, appear in the residual again) only after (v, u) appears in an augmenting path. Let f' be the flow when (v, u) appears on an augmenting path. Then, $\delta_{f'}(s, u) = \delta_{f'}(s, v) + 1 \geq \delta_f(s, v) + 1 = \delta_f(s, u) + 2$.

Since $\delta_f(s, u) \leq |V|$, (u, v) can become critical at most $|V|/2$ times. Q.E.D.