COT 5405: Fall 2006

Lecture 27

Ford-Fulkerson Method

Residual Networks

Given a flow, the residual network consists of edges that can admit more flow.

Definition: The *residual capacity* of edge (u, v) is given by $c_f(u, v) = c(u, v) - f(u, v)$.

Definition: A *residual network* is defined by $G_f = (V, E_f)$, where $E_f = \{(u, v) \in V \times V \mid c_f(u, v) > 0\}$.

Lemma: If f is the flow for $G = (V, E)$ and $G_f = (V, E_f)$ is the induced residual network with flow f', then $f+f'$, defined by $(f+f')(u, v) = f(u, v) + f'(u, v)$, is a flow in G with value $|f+f'| = |f| + |f'|$.

Proof: We will prove that $f+f'$ has all the properties of a flow, and then determine that value of this flow.

Capacity constraints: $(f+f')(u, v) = f(u, v) + f'(u, v) \leq f(u, v) + c_f(u, v)$
$= f(u, v) + (c(u, v) - f(u, v)) = c(u, v)$.

Skew symmetry: $(f+f')(u, v) = f(u, v) + f'(u, v) = -f(v, u) - f'(v, u) = -(f(v, u) + f'(v, u))$
$= -(f+f')(v, u)$.

Flow conservation: Let $u \in V - \{s, t\}$. $\Sigma_{v \in V} (f+f')(u, v) = \Sigma_{v \in V} (f(u, v) + f'(u, v))$
$= \Sigma_{v \in V} f(u, v) + \Sigma_{v \in V} f'(u, v) = 0 + 0 = 0$.

Flow value: $|f+f'| = \Sigma_{v \in V} (f+f')(s, v) = \Sigma_{v \in V} (f(s, v) + f'(s, v)) = \Sigma_{v \in V} f(s, v) + \Sigma_{v \in V} f'(s, v)$
$= |f| + |f'|$.

Augmenting Path

Definition: Given $G = (V, E)$ and flow f, an *augmenting path p* is a simple path from s to t in G_f.

Definition: The *residual capacity* of an augmenting path p is given by $c_f(p) = \min\{c_f(u, v) \mid (u, v) \in p\}$.

Lemma: *Define* $f_p: V \times V \to \Re$ by $f_p(u, v) =$ (i) $c_f(p)$ if $(u, v) \in p$, *(ii)* $-c_f(p)$ if $(v, u) \in p$, *and (iii)* 0 otherwise. The f_p is a flow in G_f with value $|f_p| = c_f(p) > 0$.

Corrollary: $f' = f + f_p$ is a flow in G with value $|f'| = |f| + |f_p| > |f|$.

Cuts of Flow Networks

Definition: A *cut (S, T)* of $G = (V, E)$ is a partition of V into S and T such that $T = V - S$, $s \in S$ and $t \in T$.

Definition: The *net flow* across the cut *(S, T)* is defined as $f(S, T)$.

Definition: The *cut capacity* is $c(S, T)$.

Definition: A *minimum cut* is a cut with minimum capacity.

Lemma: Flow across any cut *(S, T)* is $f(S, T) = |f|$.

Proof: $f(S, T) = f(S, V) - f(S, S) = f(S, V) = f(s, V) + f(S\text{-}s, V) = f(s, V) = |f|$.
(Note that $f(S\text{-}s, V) = 0$ by flow conservation.)

Corrollary: The value of any flow is bounded above by the capacity of any cut.

Max-Flow Min-Cut Theorem: If f is a flow in a flow network $G = (V, E)$ with source s and sink t, then the following are equivalent:

1. f is a max flow in G.
2. G_f contains no augmenting path.
3. $|f| = c(S, T)$ for some cut *(S, T)* of G.

Proof:

$1 \Rightarrow 2$: Assume the contrary. The flow sum $f + f_p$ is a valid flow with $|f + f_p| > |f|$, leading to a contradiction.

$2 \Rightarrow 3$: Define $S = \{v \text{ in } V \mid \exists \text{ a path from } s \text{ to } V \text{ in } G_f\}$ and $T = V - S$. For each $u \in S$ and $v \in T$, $f(u, v) = c(u, v)$. So $|f| = f(S, T) = c(S, T)$.

$3 \Rightarrow 1$: $|f| \le c(S, T)$ for all cuts. So, if $|f| = c(S, T)$, then the flow is the maximum possible.