COT 5405: Fall 2006

Lecture 26

Maximum Flow Problem

Flow Networks

Definition: A *flow network* $G=(V, E)$ is a directed graph where each edge $(u, v) \in E$ has a non-negative capacity $c(u, v) \geq 0$.

- If $(u, v) \notin E$, then we take $c(u, v) = 0$.
- Two special vertices, $s, t \in V$, $s \neq t$, are denoted as the *sink* and *source* respectively.
- For every $v \in V$, there is a path from s to t through v.
    - So, $|E| \geq |V| - 1$.

Definition: A *flow* is $f: V \times V \rightarrow \Re$ that satisfies the following constraints:

- *Capacity constraint:* $\forall u, v \in V, f(u\ v) \leq c(u, v)$.
- *Skew symmetry:* $\forall u, v \in V, f(u, v) = -f(v, u)$.
- *Flow conservation:* $\forall u \in V - \{s, t\}, \Sigma_{v \in V} f(u, v) = 0$.

Definition: The value of a *flow f* is defined as $|f| = \Sigma_{v \in V} f(s, v)$.

Maximum flow problem: Given a flow network, find a flow of maximum value.

- Multiple sources/multiple sinks maximum flow problem can be solved by creating a supersink (and a supersource) with edges to (and from) the sinks (and sources) having infinite capacities.

Implicit summation notation: $f(X, Y) = \Sigma_{x \in X} \Sigma_{y \in Y} f(x, y)$, when $X, Y \subseteq V$.

Lemma 26.1

1. $\forall X \subseteq V, f(X, X) = 0$.
2. $\forall X, Y \subseteq V, f(X, Y) = -f(Y, X)$.
3. $\forall X, Y, Z \subseteq V$ with $X \cap Y = \varnothing$, $f(X \cup Y, Z) = f(X, Z) + f(Y, Z)$ and $f(Z, X \cup Y) = f(Z, X) + f(Z, Y)$.

Proof for #1 (the rest are review questions):

Label the vertices in X as $x_1, x_2, \ldots, x_k$.

$f(X, X) = \Sigma_{x \in X} \Sigma_{y \in X} f(x, y) = \Sigma^k_{i=1} \Sigma_{j \neq i} f(x_i, x_j) = \Sigma^k_{i=1} \Sigma_{j>i} f(x_i, x_j) + \Sigma^k_{i=1} \Sigma_{j<i} f(x_i, x_j)$
$= \Sigma^k_{i=1} \Sigma_{j>i} f(x_i, x_j) - \Sigma^k_{i=1} \Sigma_{j<i} f(x_j, x_i) = \Sigma^k_{i=1} \Sigma_{j>i} f(x_i, x_j) - \Sigma^k_{j=1} \Sigma_{i>j} f(x_j, x_i)$
$= \Sigma^k_{i=1} \Sigma_{j>i} f(x_i, x_j) - \Sigma^k_{i=1} \Sigma_{j>i} f(x_i, x_j) = \Sigma^k_{i=1} \Sigma_{j>i} (f(x_i, x_j) - f(x_i, x_j)) = 0$.