COP 5405: Advanced Algorithms

Fall 2006

Lecture 17

Analysis of *Find*

Let $T(n)$ be the expected number of comparisons for selecting the k th smallest element from a sequence of n elements. We will prove, using induction, that $T(n) \leq 4n$.

Induction hypothesis: $T(m) \leq 4m$, $m \geq 1$.

Base case: $T(1) = 0$, satisfying the induction hypothesis.

Induction step: Assume that the induction hypothesis is true for all $m < n$. We shall show that it is true for n too.

$T(n) = n-1 + 1/n\ \Sigma^{k-1}_{i=1} T(n-i) + 1/n\ \Sigma^{n-1}_{i=k} T(i)$
$\leq n + 4/n\ \Sigma^{k-1}_{i=1} (n-i) + 4/n\ \Sigma^{n-1}_{i=k}\ i$ (from the induction hypothesis)
$= n + 4/n\ [n(n-1)/2 – (n-k-1)(n-k-2)/2 + n(n-1)/2 – k(k-1)/2]$
$= n + 2/n\ [n(n-1) – (n-k-1)(n-k-2) + n(n-1) – k(k-1)]$
$= n + 2/n\ [2n(n-1) – (n-k-1)(n-k-2) – k(k-1)]$. (1)

Let $Q(n) = n + 2/n\ [2n(n-1) – (n-k-1)(n-k-2) – k(k-1)]$. We wish to show that $Q(n) \leq 4n$. We will accomplish this by showing that the maximum value of $Q(n)$ is $4n$, when $n \geq 1$.

$d\ Q(n)/d\ k = -2/n\ [2k-1 – (n-k-2) – (n-k-1)] = -2/n\ [4k-2n+2]$.
Therefore, $d\ Q(n)/d\ k = 0$ when $k = (n-1)/2$.
Furthermore, $d^2\ Q(n)/d\ k^2 = -8/n < 0$, implying that $k =(n-1)/2$ yields the maximum.

Substituting $k = (n-1)/2$ into (1) yields
$T(n) \leq n + 2/n[2n(n-1) – (n-1)(n-3)/4 – (n-1)(n-3)/4]$
$= n + 2/n[2n^2 - 2n – (n-1)(n-3)/2]$
$= n + 2/n[3/2n^2 - 2n + 2\ n - 3/2]$
$\leq n + 2[3n/2]$
$= 4n$.