COP 5405: Advanced Algorithms

Fall 2006

Lecture 16

Analysis of Time Complexity for Randomized Quicksort

Let $S_{(i)}$ be the ith smallest element of S, and let X_{ij} be the number of comparisons between $S_{(i)}$ and $S_{(j)}$.

The expected total number of comparisons is:

$E(\Sigma^n_{i=1}\ \Sigma_{j>i} X_{ij}) = \Sigma^n_{i=1}\ \Sigma_{j>i} E(X_{ij})$.

If p_{ij} denotes the probability of $S_{(i)}$ and $S_{(j)}$ being compared, then $E(X_{ij}) = p_{ij} \times 1 + (1-p_{ij}) \times 0 = p_{ij}$.
(Note that X_{ij} is 0 or 1, because and element can be the pivot at most once, and after being a pivot, it is never compared again.)

We can draw a binary tree with y as root, and its children being the pivots of S_1 and S_2, recursively. A level order traversal of this tree yields a permutation π.

We make the following two observations.

1. There is a comparison between $S_{(i)}$ and $S_{(j)}$ iff $S_{(i)}$ or $S_{(j)}$ appears in π before any $S_{(l)}$, $i < l < j$. Otherewise, $S_{(l)}$ would be a pivot that separates $S_{(i)}$ and $S_{(j)}$ into different subtrees, and so they would never be compared.
2. Any of $S_{(i)}, S_{(i+1)}, \ldots, S_{(j)}$ is equally likely to be the first of these elements to be chosen as a pivot. So, the probability that $S_{(i)}$ or $S_{(j)}$ is the first of these to be chosen is $p_{ij} = 2/(j-i+1)$.

So, the expected number of comparisons is $\Sigma^n_{i=1}\ \Sigma_{j>i}\ p_{ij} = \Sigma^n_{i=1}\ \Sigma_{j>i}\ 2/(j-i+1) = 2\ \Sigma^n_{i=1}\ \Sigma_2^{n-i+1}\ 1/k \leq 2\ \Sigma^n_{i=1}\ \Sigma_1^n\ 1/k = 2nH_n$, and $H_n = ln\ n + \theta(1)$.