Estimating Causal Quantities via Linear and Nonlinear Programming: Current Status, Challenges, and Future Directions

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Abstract

Identifying causal quantities, such as causal effects and probabilities of causation, is crucial across various scientific disciplines. However, these causal quantities often cannot be estimated using the closed-form solutions or available data. Fortunately, approximate numerical solutions can still be derived for several causal quantities through linear and nonlinear programming techniques. This paper reviews one existing formalization of such problems, discusses their applications in critical decision-making domains such as healthcare and economics, and outlines the challenges associated with these approximations.

1 Introduction

Causal quantities are crucial for advanced decision-making across disciplines such as business, health science, and social science. For example, the linear combinations of the probabilities of causation effectively solve the unit selection problem (Li and Pearl 2019, 2022b, 2024). The health impacts of artificial sweetener drinks have been investigated through an exploration of one of these probabilities (Qi and Li 2024). Mueller and Pearl demonstrated that the probabilities of causation can also be leveraged for advanced personalized decision-making in health science (Mueller and Pearl 2022). Furthermore, causal quantities can be integrated into the loss functions of machine learning models to improve their accuracy (Li et al. 2020).

The exploration of causal quantities has a rich history. Pearl first defined causal effects (Pearl 1993), as well as the probabilities of necessity and sufficiency (PNS), sufficiency (PS), and necessity (PN) (Pearl 1999), using the structural causal model (SCM) (Galles and Pearl 1998; Halpern 2000). Subsequent research has focused on estimating these quantities, with results categorized as either identifiable or partially identifiable. Identifiable conditions allow for the point estimation of causal quantities when satisfied, while partially identifiable cases require techniques such as linear and non-linear programming to obtain informative bounds on the causal quantities.

Pearl introduced the widely used identification criteria known as the back-door and front-door rules (Pearl 1993) for causal effects. However, there are situations where these criteria cannot be met, or the necessary variables remain unobservable. In such scenarios, informative bounds can still be obtained. For instance, Balke and Pearl established bounds on causal effects derived under conditions of imperfect compliance (Balke and Pearl 1997a). Li and Pearl also derived bounds on causal effects when adjustment variables are only partially observed, solving the problem through non-linear programming (Li and Pearl 2022a).

Tian and Pearl identified monotonicity as a key identification condition for PNS, PS, and PN (Tian and Pearl 2000), noting that point estimation of probabilities of causation is generally not possible without additional assumptions. To address this, they derived tight bounds for these probabilities using linear programming (Tian and Pearl 2000) and even provided a closed-form solution by applying Balke and Pearl's method of considering the corresponding dual linear programming problem (Balke and Pearl 1997b).

The next section reviews one such formalization of non-linear programming problems.

2 Estimation of Causal Effects via Nonlinear Programming

Causal effects are typically estimated using the back-door or front-door criteria. However, these back-door or front-door variables are not always observable. Li and Pearl (Li and Pearl 2022a) then formalized the following theorem, providing bounds on causal effects that can be obtained by solving two nonlinear programming problems.

Theorem 1 *Given a causal diagram G and a distribution compatible with G, let $W \cup U$ be a set of variables satisfying the back-door criterion in G relative to an ordered pair (X, Y), where $W \cup U$ is partially observable, i.e., only probabilities $P(X, Y, W)$ and $P(U)$ are given. The causal effects of X on Y are then bounded as follows:*

$$\mathit{LB} \leq P(y|do(x)) \leq \mathit{UB}$$

where LB is the solution to the nonlinear optimization problem in Equation 1 and UB is the solution to the nonlinear optimization problem in Equation 2.

$$LB = \min \sum_{w,u} \frac{a_{w,u} b_{w,u}}{c_{w,u}}, \tag{1}$$

$$UB = \max \sum_{w,u} \frac{a_{w,u} b_{w,u}}{c_{w,u}}, \tag{2}$$

$$\textit{where, } \sum_{u} a_{w,u} = P(x, y, w), \sum_{u} b_{w,u} = P(w), \sum_{u} c_{w,u} = P(x, w) \textit{ for all } w \in W,$$

$$\textit{and for all } w \in W \textit{ and } u \in U,$$

$$b_{w,u} \geq c_{w,u} \geq a_{w,u},$$

$$\max\{0, p(x, y, w) + p(u) - 1\} \leq a_{w,u}, \min\{P(x, y, w), p(u)\} \geq a_{w,u},$$

$$\max\{0, p(w) + p(u) - 1\} \leq b_{w,u}, \min\{P(w), p(u)\} \geq b_{w,u},$$

$$\max\{0, p(x, w) + p(u) - 1\} \leq c_{w,u}, \min\{P(x, w), p(u)\} \geq c_{w,u}.$$

3 Challenges

In the example above, we used the "SLSQP" solver from the SciPy package to address the nonlinear programming problems. However, we observed that each run of the solver does not consistently produce identical solutions, leading to variations in the resulting bounds for the causal effects. This necessitates running the solver multiple times and manually selecting some of the results. Additionally, the true causal effects are not uniformly distributed within these bounds. Therefore, even slight shifts in the bounds, due to the solver's accuracy, could cause us to miss the true causal effects.

As mentioned earlier, Tian and Pearl (Tian and Pearl 2000) derived tight bounds for PNS, PS, and PN using linear programming. In those cases, the PNS, PS, and PN are binary, resulting in a linear programming problem with 8 variables and 6 constraints. However, the number of variables and constraints increases exponentially for non-binary cases. With just one additional dimension, the problem's size expands to 81 variables and 15 constraints, rendering high-dimensional cases impractical.

Future research could explore more robust solvers and approximation techniques to enhance accuracy in the linear and nonlinear programming formalization of causal quantities. Advances in computational efficiency may also help address the complexity challenges in high-dimensional cases. Additionally, efforts could be directed toward finding closed-form solutions for nonlinear programming and high-dimensional linear programming problems, as demonstrated by Balke and Pearl in (Balke and Pearl 1997b) for simpler linear cases.

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