ϵ-Identifiability of Causal Quantities

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Abstract

Identifying the effects of causes and causes of effects is vital in virtually every scientific field. Often, however, the needed probabilities may not be fully identifiable from the available data sources. This paper shows how approximate identifiability is still possible for several probabilities of causation. We term this ϵ -identifiability and demonstrate its usefulness in cases where the behavior of certain subpopulations can be restricted within sufficiently narrow bounds. In particular, we show how unidentifiable causal effects and counterfactual probabilities can be ϵ -identified when such allowances are made. Often, these allowances are easily measured and reasonably assumed. Finally, ϵ -identifiability is applied to the unit selection problem.

Introduction

Both Effects of Causes (EoC) and Causes of Effects (CoE) play an important role in many fields, such as health science, social science, and business. This is due, in part, to several features of EoC and CoE analysis. First, causal effects identified by the adjustment formula (Pearl 1993) help decision-makers avoid randomized controlled trials and use purely observational data. Second, probabilities of causation have been proven critical in policy-making and personalized decision-making (Mueller and Pearl 2022). Third, a linear combination of probabilities of causation has been used to solve the unit selection problem defined by Li and Pearl (Li and Pearl 2019, 2022b, 2024b). Additionally, causal quantities can increase the accuracy of machine learning models by combining causal quantities with the model's label (Li et al. 2020).

The identification of causal quantities has been studied for decades. Pearl first defined causal quantities such as causal effects (Pearl 1993), probability of necessity and sufficiency (PNS), probability of sufficiency (PS), and probability of necessity (PN) (Pearl 1999) and their identifiability (Pearl 2009) using the structural causal model (SCM) (Galles and Pearl 1998; Halpern 2000). Pearl also developed identification conditions of causal effects, such as back-door and front-door criteria (Pearl 1993), along with the *do*-calculus mathematically complete axiomatic identification system

(Pearl 1994). Pearl and Bareinboim have studied more conditions for identifying causal effects (Shpitser and Pearl 2009a; Bareinboim and Pearl 2012). If causal effects are not identifiable, informative bounds can be obtained. Balke and Pearl bounded causal effects from a randomized controlled trial (RCT) with imperfect compliance (Balke and Pearl 1997a). Li and Pearl bounded causal effects when adjustment variables are partially observed (Li and Pearl 2022a). Tian and Pearl proposed monotonicity as an identification condition of the binary probabilities of causation (Tian and Pearl 2000). Tian and Pearl (Tian and Pearl 2000) additionally derived tight bounds, using linear programming, when probabilities of causation are not identifiable (Balke and Pearl 1997b). Mueller, Li, and Pearl (Mueller, Li, and Pearl 2022), as well as Dawid (Dawid, Musio, and Murtas 2017), narrowed those bounds using additional covariate information and the corresponding causal structure. Recently, Li and Pearl proposed theoretical work for non-binary probabilities of causation (Li and Pearl 2024a). Zhang, Tian, and Bareinboim proposed numerical bounds for non-binary probabilities of causation (Zhang, Tian, and Bareinboim 2022) and named it "Partial identification" (the difference between partial identification and the proposed ϵ -Identifiability will be discussed in the discussion section). Furthermore, Shpitser and Pearl introduced another essential causal measure. known as the effect of treatment on the treated (ETT), within the SCM framework and extensively investigated its identification (Shpitser and Pearl 2009b).

In real-world applications, decision-makers are more likely to have identifiable cases (i.e., the causal quantities have point estimations) because the bounds under unidentifiable cases may be less informative (e.g., $0.1 \leq \text{PNS} \leq 0.9$). Besides, estimating the bounds often requires a combination of experimental and observational data. However, the identifiability is often hard to achieve, so we wonder if something is sitting between the identifiable and the bounds. Inspired by the idea of the confidence interval, in this paper, we proposed the definition of ϵ -identifiability, in which more conditions of ϵ -identifiability can be found while the estimations of the causal quantities are still near point estimations.

Preliminaries

Here, we review the definition of ETT, PNS, PS, and PN defined by Pearl (Pearl 1999; Shpitser and Pearl 2009a), as

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well as the definition of identifiable and the conditions for identifying ETT, PNS, PS, and PN (Tian and Pearl 2000; Shpitser and Pearl 2009b). Besides, we review the tight bounds of PNS, PS, and PN when they are unidentifiable (Tian and Pearl 2000). Readers who are familiar with the above knowledge may skip this section.

Similarly to any works mentioned above, we used the causal language of the SCM (Galles and Pearl 1998; Halpern 2000). The introductory counterfactual sentence "Variable Y would have the value y, had X been x" in this language is denoted by $Y_x = y$, and shorted as y_x . We have two types of data: experimental data, which is in the form of causal effects (denoted as $P(y_x)$), and observational data, which is in the form of a joint probability function (denoted as P(x, y)). Besides, in the absence of further specification, let X and Y be two binary variables in a causal model M, let x and y stand for the propositions X = true and Y = true, respectively, and x' and y' for their complements. For notational simplicity, we limit the discussion to binary variables; extensions to multi-valued variables are discussed by Pearl (see (Pearl 2009) p. 286, footnote 5).

First, the definition of identifiable for any causal quantities defined using SCM is as follows:

Definition 1 (Identifiability). Let Q(M) be any computable quantity of a class of SCM M that is compatible with graph G. We say that Q is identifiable in M if, for any pairs of models M_1 and M_2 from M, $Q(M_1) = Q(M_2)$ whenever $P_{M_1}(v) = P_{M_2}(v)$, where P(v) is the statistical distribution over the set V of observed variables. If our observations are limited and permit only a partial set F_M of features (of $P_M(v)$) to be estimated, we define Q to be identifiable from F_M if $Q(M_1) = Q(M_2)$ whenever $F_{M_1} = F_{M_2}$. (Pearl 2009)

Second, the definitions of four binary probabilities of causation defined using SCM are as follow (Pearl 1999; Shpitser and Pearl 2009a):

Definition 2 (Effect of treatment on the treated (ETT)). Let X and Y be two binary variables in a causal model M, let x and y stand for the propositions X = true and Y = true, respectively, and x' and y' for their complements. The effect of treatment on the treated is defined as the expression

$$\begin{array}{rcl} \mathsf{ETT} & \triangleq & P(Y_{X=true} = true | X = false) \\ & \triangleq & P(y_x | x') \end{array}$$

Definition 3 (Probability of necessity (PN)). Let X and Y be two binary variables in a causal model M, let x and y stand for the propositions X = true and Y = true, respectively, and x' and y' for their complements. The probability of necessity is defined as the expression

$$PN \triangleq P(Y_{X=false} = false | X = true, Y = true) \\ \triangleq P(y'_{x'} | x, y)$$

Definition 4 (Probability of sufficiency (PS)). Let X and Y be two binary variables in a causal model M, let x and y stand for the propositions X = true and Y = true, respectively, and x' and y' for their complements. The probability

of sufficiency is defined as the expression

$$PS \triangleq P(Y_{X=true} = true | X = false, Y = false) \\ \triangleq P(y_x | x', y')$$

Definition 5 (Probability of necessity and sufficiency (PNS)). Let X and Y be two binary variables in a causal model M, let x and y stand for the propositions X = true and Y = true, respectively, and x' and y' for their complements. The probability of necessity and sufficiency is defined as the expression

PNS
$$\triangleq P(Y_{X=true} = true, Y_{X=false} = false)$$

 $\triangleq P(y_x, y'_{x'})$

Third, we review the identification conditions for causal effects (Pearl 1993, 1995).

Definition 6 (Back-door criterion). Given an ordered pair of variables (X, Y) in a directed acyclic graph G, a set of variables Z satisfies the back-door criterion relative to (X, Y), if no node in Z is a descendant of X, and Z blocks every path between X and Y that contains an arrow into X.

If a set of variables Z satisfies the back-door criterion for X and Y, the causal effects of X on Y are identifiable and given by the adjustment formula:

$$P(y_x) = \sum_{z} P(y|x, z) P(z).$$
(1)

Definition 7 (Front-door criterion). A set of variables Z is said to satisfy the front-door criterion relative to an ordered pair of variables (X, Y) if:

- Z intercepts all directed paths from X to Y;
- there is no back-door path from X to Z; and
- all back-door paths from Z to Y are blocked by X.

If a set of variables Z satisfies the front-door criterion for X and Y, and P(x, Z) > 0, then the causal effects of X on Y are identifiable and given by the adjustment formula:

$$P(y_x) = \sum_z P(z|x) \sum_{x'} P(y|x',z) P(x').$$

If causal effects are not identifiable, Tian and Pearl (Tian and Pearl 2000) provided the following bounds for the causal effects.

$$P(x,y) \le P(y_x) \le 1 - P(x,y').$$
 (2)

Fourth, we review the identification conditions for ETT (Shpitser and Pearl 2009b).

Theorem 8. If a set of variables Z satisfies the back-door criterion for X and Y, the ETT of X on Y are identifiable and given by the following formula:

$$P(y_{x}|x') = \sum_{z} P(y|x, z) P(z|x').$$
 (3)

Theorem 9. The ETT of X on Y are identifiable if $P(Y_X)$ are identifiable and given by the following formula:

$$P(y_x|x') = \frac{P(y_x) - P(x,y)}{P(x')}.$$
(4)

Finally, we review the identification conditions for PNS, PS, and PN (Tian and Pearl 2000).

Definition 10. (Monotonicity) A Variable Y is said to be monotonic relative to variable X in a causal model M iff

$$y'_x \wedge y_{x'} =$$
false.

Theorem 11. If Y is monotonic relative to X, then PNS, PN, and PS are all identifiable, and

$$PNS = P(y_x) - P(y_{x'}),$$
$$PN = \frac{P(y) - P(y_{x'})}{P(x, y)},$$
$$PS = \frac{P(y_x) - P(y)}{P(x', y')}.$$

If PNS, PN, and PS are not identifiable, informative bounds are given by Tian and Pearl (Tian and Pearl 2000).

$$\max \left\{ \begin{array}{c} 0, \\ P(y_{x}) - P(y_{x'}), \\ P(y) - P(y_{x'}), \\ P(y_{x}) - P(y) \end{array} \right\} \le \text{PNS}$$
(5)

$$\min \left\{ \begin{array}{c} P(y_x), \\ P(y'_x), \\ P(x,y) + P(x',y'), \\ P(y_x) - P(y_{x'}) + \\ P(x,y') + P(x',y) \end{array} \right\} \ge \text{PNS}$$
(6)

$$\max\left\{\begin{array}{c}0,\\\frac{P(y)-P(y_{x'})}{P(x,y)}\end{array}\right\} \le \mathsf{PN}$$
(7)

$$\min\left\{\begin{array}{c}1,\\\frac{P(y'_{x'})-P(x',y')}{P(x,y)}\end{array}\right\} \ge \mathsf{PN}$$
(8)

$$\max\left\{\begin{array}{c}0,\\\frac{P(y')-P(y'_x)}{P(x',y')}\end{array}\right\} \le \mathbf{PS}$$
(9)

$$\min\left\{\begin{array}{c}1,\\\frac{P(y_x)-P(x,y)}{P(x',y')}\end{array}\right\} \ge \mathsf{PS} \tag{10}$$

The identification conditions mentioned above (i.e., backdoor and front-door criteria and monotonicity) are robust. However, it may still be hard to achieve in real-world applications. In this work, we extend the definition of identifiability, in which a sufficiently small interval is allowed. By the new definition, the estimates of causal quantities are still near point estimations, and more conditions for identifiability could be discovered. Again, if nothing is specified, the discussion in this paper will be restricted to binary treatment and effect (i.e., X and Y are binary).

Main Results

First, we extend the definition of identifiability, which we call ϵ -identifiability.

Definition 12 (ϵ -Identifiability). Let Q(M) be any computable quantity of a class of SCM M that is compatible with graph G. We say that Q is ϵ -identifiable in M (and ϵ -identified to q) if for every P(v), there exists q s.t. for any model M from M, $Q(M) \in [q - \epsilon, q + \epsilon]$ whenever the statistical data $P_M(v) = P(v)$, where P(v) is the statistical distribution over the set V of observed variables. If our observations are limited and permit only a partial set F_M of features (of $P_M(v)$) to be estimated, we define Q to be ϵ -identifiable from F_M if $Q(m) \in [q - \epsilon, q + \epsilon]$ with statistical data F_M .

With the above definition, the causal quantity is at a maximum distance of ϵ from its true value. We will use the infix operator symbol \approx_{ϵ} to represent its left-hand side being within ϵ of its right-hand side:

$$r \approx_{\epsilon} q \iff r \in [q - \epsilon, q + \epsilon].$$
 (11)

The following sections explicate conditions for ϵ -identifiability of causal effects, ETT, PNS, PS, and PN.

ϵ -Identifiability of Causal Effects

The causal effect $P(Y_X)$ can be ϵ -identified with information about the observational joint distribution P(X, Y). This can be seen by rewriting Equation (2) as:

$$P(x,y) \leqslant P(y_x) \leqslant P(x,y) + P(x'). \tag{12}$$

Here, $P(y_x)$ is ϵ -identified to $P(x, y) + \epsilon$ when $P(x') \leq 2\epsilon$. This ϵ -identification indicates a lower bound of P(x, y) and an upper bound of $P(x, y) + 2\epsilon$. Since $P(x') \leq 2\epsilon$, these bounds are equivalent to (12). Notably, only P(x, y) and an upper bound on P(x') are necessary to ϵ -identify $P(y_x)$. This is generalized in Theorem 13, without any assumptions of the causal structure.

Theorem 13. The causal effect $P(Y_X)$ is ϵ -identified as follows:

$$P(y_x) \approx_{\epsilon} P(x, y) + \epsilon$$
 if $P(x') \leq 2\epsilon$, (13)

$$P(y'_x) \approx_{\epsilon} P(x, y') + \epsilon \quad \text{if } P(x') \leqslant 2\epsilon, \quad (14)$$

$$P(y_{x'}) \approx_{\epsilon} P(x', y) + \epsilon \qquad \text{if } P(x) \leq 2\epsilon, \qquad (15)$$

$$P(y'_{x'}) \approx_{\epsilon} P(x', y') + \epsilon \quad \text{if } P(x) \leq 2\epsilon.$$
 (16)

When the complete distribution P(X, Y) is known, Theorem 13 provides no extra precision over Equation (12). Its power comes from when only part of the distribution is known and only an upper bound on P(X) is available or

able to be assumed. Knowledge of a causal structure can aid ϵ -identification. In Figure 1, there is a binary confounder U. If the full joint distribution P(X, Y, U) was available, the causal effect $P(Y_X)$ could be computed simply through the backdoor adjustment formula. In the absence of the full joint distribution, Theorem 14 allows ϵ -identification of $P(y_x)$ with only knowledge of P(x) and the conditional probability P(y|x)as well as an upper bound on P(u).



Figure 1: The causal graph, where X is a binary treatment, Y is a binary effect, and U is a binary confounder.

Theorem 14. Given the causal graph in Figure 1 and $P(u) \leq P(x) - c$ for some constant c, where $0 < c \leq P(x)$,

$$P(y_x) \approx_{\epsilon} P(y|x) + \frac{P(x) - c}{2cP(x) + P(x) + c} \cdot \epsilon,$$

if $P(u) \leq \frac{2cP(x)}{2cP(x) + P(x) + c} \cdot \epsilon.$ (17)

Specifically, if $P(x) \ge 0.5$ and $P(u) \le \min\{0.1, \frac{4}{13}\epsilon\}$, then the causal effect $P(y_x)$ is ϵ -identified to $P(y|x) + \frac{\epsilon}{13}$.

Proof. See Appendix.

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Note that $x \in \{x, x'\}$, $y \in \{y, y'\}$, and $u \in \{u, u'\}$ in Theorem 14 (e.g., you can substitute x with x' and x' with x). The constant c should be maximized satisfying both $c \leq P(x) - P(u)$ and the condition in Equation (17) for a given ϵ . The larger c is, the closer $P(y_x)$ is ϵ -identified to P(y|x). This needs to be balanced with minimizing ϵ .

As an example, if $P(x) \ge 0.5$ and $P(u) \le 0.1$, then the causal effect $P(y_x)$ is ϵ -identified to $P(y|x) + \frac{\epsilon}{13}$ if $P(u) \le \frac{4}{13}\epsilon$.

Essentially, $P(y_x)$ is ϵ -identified to P(y|x) plus some fraction of ϵ when P(u) is sufficiently small. Therefore, the causal effect $P(y_x)$ is near P(y|x) if P(U) is specific (i.e., P(u) or P(u') is minimal). In this case, Theorem 14 can be advantageous over the backdoor adjustment formula to compute $P(y_x)$, even when data on X, Y, and U are available, because P(Y|X, U), required for the adjustment formula, is impractical to estimate with P(U) close to 0.

e-Identifiability of ETT

The effect of treatment on the treated (ETT) is a crucial quantity in the field of causal inference, as defined in Definition 2. ETT has been shown to be valuable for detecting latent heterogeneity (Pearl 2015). Shpitser and Pearl proposed the identification of ETT through backdoor criteria, as outlined in Theorem 8. Furthermore, Shpitser and Pearl demonstrated the relationship between the ETT and both experimental and observational data as shown in Theorem 9. Therefore, Theorem 14 can be extended to incorporate ETT accordingly.

Theorem 15. Given the causal graph in Figure 1 and $P(u) \leq P(x) - c$ for some constant c, where $0 < c \leq c$

P(x) < 1,

$$P(y_x|x')$$

$$\approx_{\epsilon} \frac{P(y|x) - P(x,y)}{P(x')} + \frac{P(x) - c}{2cP(x) + P(x) + c} \cdot \epsilon$$

$$if P(u) \leqslant \frac{2cP(x)P(x')}{2cP(x) + P(x) + c} \cdot \epsilon.$$
(18)

Specifically, if P(x) = 0.5 and $P(u) \leq \min\{0.1, \frac{2}{13}\epsilon\}$, then the $P(y_x|x')$ is ϵ -identified to $P(y|x) + \frac{\epsilon}{13}$.

Proof. See Appendix.

Note that P(x) cannot equal 1 because doing so would result in the conditional space x' in $P(y_x|x')$ being undefined. Additionally, it should be noted that $x \in \{x, x'\}$, $y \in \{y, y'\}$, and $u \in \{u, u'\}$ in Theorem 15.

e-Identifiability of PNS

Even though Tian and Pearl derived tight bounds on PNS (Tian and Pearl 2000), the PNS can potentially be sufficiently narrow when taking into account particular upperbound assumptions on causal effects or observational probabilities. This can be seen by analyzing the bounds of PNS in Equations (5) and (6). Picking any of the arguments to the max function of the lower bound and any of the arguments to the min function of the upper bound, we can make a condition that the range of those two values is less than 2ϵ . For example, let us pick the second argument of the max function, $P(y_x) - P(y_{x'})$, and the first argument of the min function, $P(y_x)$:

$$P(y_x) - [P(y_x) - P(y_{x'})] \leq 2\epsilon,$$

$$P(y_{x'}) \leq 2\epsilon.$$
(19)

Equation (19) is the assumption and the PNS is the ϵ -identified to ϵ above the lower bound or ϵ below the upper bound:

$$PNS \approx_{\epsilon} P(y_x) - P(y_{x'}) + \epsilon, \text{ or}$$
 (20)

$$PNS \approx_{\epsilon} P(y_x) - \epsilon.$$
(21)

Since it is assumed that $P(y_{x'}) \leq 2\epsilon$, Equation (20) is equivalent to Equation (21). The complete set of ϵ -identifications and associated conditions are stated in Theorem 16.

Theorem 16. *The PNS is* ϵ *-identified as follows:*

$$\begin{array}{ll} PNS \approx_{\epsilon} \epsilon & if P(y_x) \leqslant 2\epsilon, \qquad (22) \\ PNS \approx_{\epsilon} \epsilon & if P(y'_{x'}) \leqslant 2\epsilon, \qquad (23) \\ PNS \approx_{\epsilon} \epsilon & if P(x,y) + P(x',y') \leqslant 2\epsilon, \qquad (24) \\ PNS \approx_{\epsilon} \epsilon & if P(y_x) - P(y_{x'}) + \\ P(x,y') + P(x',y) \leqslant 2\epsilon, \qquad (25) \\ PNS \approx_{\epsilon} P(y_x) - \epsilon & if P(y_{x'}) \leqslant 2\epsilon, \qquad (26) \\ PNS \approx_{\epsilon} P(y_{x'}) - \epsilon & if P(y'_x) \leqslant 2\epsilon, \qquad (27) \\ PNS \approx_{\epsilon} P(y_x) - \\ P(y_{x'}) + \epsilon & if P(x,y') + P(x',y) \leqslant 2\epsilon, \qquad (28) \\ PNS \approx_{\epsilon} P(y_x) - & if P(y_{x'}) - P(y_x) + \\ P(y_{x'}) + \epsilon & P(x,y) + P(x',y') \leqslant 2\epsilon, \qquad (29) \\ PNS \approx_{\epsilon} P(x,y) - & if P(y_{x'}) - P(y_x) + \\ P(x',y') - \epsilon & P(x,y) + P(x',y') \leqslant 2\epsilon, \qquad (30) \end{array}$$

$$PNS \approx_{\epsilon} P(y'_{x'}) - \epsilon \quad \text{if } P(y') \leqslant 2\epsilon, \tag{31}$$

$$PNS \approx_{\epsilon} P(y_x) - \epsilon \quad \text{if } P(y_x) + P(y_{x'}) - P(y) \leqslant 2\epsilon,$$
(32)

$$PNS \approx_{\epsilon} P(y) - \qquad if P(y_x) + P(y_{x'}) - P(y_{x'}) + \epsilon \qquad P(y) \leq 2\epsilon,$$
(33)

$$PNS \approx_{\epsilon} P(x, y) + \qquad if P(x', y') + P(y_{x'}) - P(x', y') - \epsilon \qquad P(x', y) \leq 2\epsilon, \qquad (34)$$

$$PNS \approx_{\epsilon} P(y) - \qquad if P(x', y') + P(y_{x'}) - P(y_{x'}) - P(y_{x'}) + \epsilon \qquad P(x', y) \leq 2\epsilon \qquad (35)$$

$$PNS \approx_{\epsilon} P(y) - if P(x', y) + P(y'_{x'}) - P(y'_{x'}) + \epsilon$$

$$P(y) = P(y'_{x'}, y) + P(y'_{x'}) - P(y'_{x'}) = P(y'_{x'}, y') \leq 2\epsilon$$

$$P(y) = P(y'_{x'}, y') \leq 2\epsilon$$

$$P(y_{x'}) + \epsilon \qquad P(x', y') \leq 2\epsilon,$$
(36)

$$PNS \approx_{\epsilon} P(y_x) - \epsilon \qquad if P(y) \leq 2\epsilon,$$
(37)

$$PNS \approx_{\epsilon} P(y'_{x'}) - \epsilon \qquad if P(y'_{x'}) - P(y_x) +$$

$$P(y) \leq 2\epsilon, \qquad (38)$$

$$PNS \approx_{\epsilon} P(y) - \qquad if P(y'_{x'}) - P(y_x) + \qquad P(y) \leq 2\epsilon, \qquad (20)$$

$$P(y_{x'}) + \epsilon \qquad P(y) \leq 2\epsilon, \tag{39}$$

$$PNS \approx_{\epsilon} P(x, y) + \qquad if P(x, y) + P(y'_{x}) - P(x, y') \leq 2\epsilon \qquad (40)$$

$$P(x', y') - \epsilon \qquad P(x, y') \leq 2\epsilon, \qquad (40)$$

$$PNS \approx_{\epsilon} P(y_x) - \qquad if P(x, y) + P(y'_x) -$$

$$P(y) + \epsilon \qquad P(x, y') \leq 2\epsilon, \qquad (41)$$

$$PNS \approx_{\epsilon} P(y_x) - \qquad if P(x', y) + P(y'_{x'}) -$$

$$P(y) + \epsilon$$
 $P(x', y') \leq 2\epsilon$. (42)
Proof. See Appendix.

Note that in the above theorem, eight conditions consist solely of experimental probabilities or solely of observational probabilities. This potentially eliminates the need for some types of studies, at least partially, even when estimating a counterfactual quantity such as PNS. For example, if a decision-maker knows that P(y) is large $(P(y) \ge 0.95)$, they can immediately conclude PNS $\approx_{0.025} P(y'_{x'}) - 0.025$

(Equation (31)), without knowing the specific value of P(y). Thus, only a control group study would be sufficient to estimate PNS.

$\epsilon\text{-Identifiability of PN and PS}$

Tian and Pearl derived tight bounds on PN and PS, in addition to PNS. Similarly to the derivation of Theorem 16, we can potentially achieve sufficiently narrow bounds by considering upper bound assumptions on causal effects or observational probabilities. The set of ϵ -identifications and the associated conditions are stated in Theorems 17 and 18.

Theorem 17. *The PN is* ϵ *-identified as follows:*

$$PN \approx_{\epsilon} \epsilon \qquad \qquad if P(y'_{x'}) - P(x', y') \\ \leqslant 2\epsilon P(x, y), \qquad (43) \\ PN \approx_{\epsilon} 1 - \epsilon \qquad \qquad if P(y_{x'}) - P(x', y)$$

$$PN \approx_{\epsilon} \frac{P(y) - P(y_{x'})}{P(x, y)} + \epsilon$$
 if $P(y_{x'}) - P(x', y)$

$$\leqslant 2\epsilon P(x,y), \qquad (45)$$

 $\leq 2\epsilon P(x,y),$

(44)

$$PN \approx_{\epsilon} \frac{P(y'_{x'}) - P(x', y')}{P(x, y)} - \epsilon \quad if P(x, y') \leq 2\epsilon P(x, y), \qquad (46)$$

$$PN \approx_{\epsilon} \frac{P(y) - P(y_{x'})}{P(x, y)} + \epsilon \qquad if P(x, y') \\ \leqslant 2\epsilon P(x, y). \tag{47}$$

Proof. See Appendix.

Theorem 18. *The PS is* ϵ *-identified as follows:*

$$PS \approx_{\epsilon} \epsilon \qquad \qquad if P(y_x) - P(x, y) \\ \leqslant 2\epsilon P(x', y'), \qquad (48) \\ PS \approx_{\epsilon} 1 - \epsilon \qquad \qquad if P(y'_x) - P(x, y') \\ \leqslant 2\epsilon P(x', y'), \qquad (49) \end{cases}$$

$$PS \approx_{\epsilon} \frac{P(y') - P(y'_x)}{P(x', y')} + \epsilon \quad if P(y'_x) - P(x, y')$$
$$\leq 2\epsilon P(x', y'), \quad (50)$$

$$PS \approx_{\epsilon} \frac{P(y_x) - P(x, y)}{P(x', y')} - \epsilon \quad if P(x', y)$$
$$\leq 2\epsilon P(x', y'), \quad (51)$$

$$PS \approx_{\epsilon} \frac{P(y') - P(y'_x)}{P(x', y')} + \epsilon \qquad \text{if } P(x', y) \\ \leqslant 2\epsilon P(x', y'). \tag{52}$$

Proof. See Appendix.

Examples

Here, we illustrate how to apply ϵ -Identifiability in real applications by two simulated examples.

Table 1: Results of an observational study with 1500 individuals who have access to the medicine, where 1260 individuals chose to receive the medicine and 240 individuals chose not to.

| | Medicine | No medicine |
|---------------|----------|-------------|
| Recovered | 780 | 210 |
| Not recovered | 480 | 30 |

Causal Effects of Medicine

Consider a medicine manufacturer who wants to know the causal effect of a new medicine on a disease. They conducted an observational study where 1500 patients were given access to the medicine; the results of the study are summarized in Table 1. In addition, the expert from the medicine manufacturer acknowledged that family history is the only confounder of taking medicine and recovery, and the family history of the disease is extremely rare; only 1% of the people have the family history.

Let X = x denote that a patient chose to take the medicine, and X = x' denote that a patient chose not to take the medicine. Let Y = y denote that a patient recovered, and Y = y' denote that a patient did not recover. Let U = u denote that a patient has the family history, and U = u' denote that a patient has no family history.

The simulated data in Table 1 provides the estimates

$$P(x) = \frac{1260}{1500} = 0.84,$$

$$P(x') = \frac{240}{1500} = 0.16,$$

$$P(y|x) = \frac{780}{1260} \approx 0.62,$$

$$P(y|x') = \frac{210}{240} = 0.875.$$

To obtain the causal effect of the medicine (i.e., using adjustment formula (1)), we have to know the observational data associated with family history, which is difficult to obtain.

Fortunately, we have the prior that P(u) = 0.01. Since $0.01 = P(u) \leq P(x) - 0.8$ (let c = 0.8) and $0.01 = P(u) < \frac{2c*0.025P(x)}{2cP(x)+P(x)+c} \approx 0.0113$, we can apply Theorem 14 to obtain that $P(y_x)$ is 0.025-identified to $P(y|x) + \frac{P(x)-c}{2cP(x)+P(x)+c} 0.025 \approx 0.62$. This means the causal effect of the medicine is very close to 0.62 (i.e., 0.025 close), which can not be 0.025 far from 0.62.

Or even simpler, note that P(x) = 0.84 > 0.5 and P(u) = 0.01 < 0.1, $P(u) = 0.01 < \frac{4}{13} * 0.035 \approx 0.0108$. We obtain that $P(y_x)$ is 0.035-identified to $P(y|x) + \frac{0.035}{13} \approx 0.62$.

Similarly, since $0.01 = P(u) \le P(x') - 0.15$ (let c = 0.15) and $0.01 = P(u) < \frac{2c*0.1P(x')}{2cP(x')+P(x')+c} \approx 0.0134$, we can apply Theorem 14 again to obtain that $P(y_{x'})$ is 0.1-identified to $P(y|x') + \frac{P(x')-c}{2cP(x')+P(x')+c} 0.1 \approx 0.878$.

In contrast, without the concept of " ϵ -Identifiability," we would have to apply the general bounds in equation (2),

because the joint distribution P(X, Y, U) is not available; therefore, the adjustment formula (1) cannot be applied. We have, by equation (2), $0.52 \leq P(y_x) \leq 0.68$ and $0.14 \leq P(y_{x'}) \leq 0.98$, making it impossible to reach a decision. However, by Theorem 14, we find $P(y_x) \approx_{0.025} 0.62$ and $P(y_{x'}) \approx_{0.1} 0.878$, leading to the conclusion that the medicine is ineffective, without knowing the observational data associated with the family history.

PNS of Flu Shot

Consider a newly invented flu shot. After a vaccination company introduced a new flu shot, the number of people infected by flu reached the lowest point in 20 years (i.e., less than 5% of people infected by flu). The government concluded that the new flu shot is the key to success. However, some anti-vaccination associations believe it is because people's physical quality increases yearly. Therefore, they all want to know how many percentages of people are uninfected because of the flu shot. The PNS of the flu shot (i.e., the percentage of individuals who would not infect by the flu if they had taken the flu shot and would infect otherwise) is indeed what they want.

Let X = x denote that an individual has taken the flu shot and X = x' denote that an individual has not taken the flu shot. Let Y = y denote an individual infected by the flu and Y = y' denote an individual not infected by the flu.

If they want to apply the bounds of PNS in Equations (5) and (6), they must conduct both experimental and observational studies. However, note that P(y) < 0.05, one could apply Equation (37) in Theorem 16, which PNS is 0.025-identified to $P(y_x)-0.025$ (i.e., PNS is very close to $P(y_x)$). Thus, according to the analysis of sample size in (Li, Mao, and Pearl 2022), only an experimental study for the treated group with a sample size of 385 is adequate for estimating PNS.

ϵ-Identifiability in Unit Selection Problem

One utility of the causal quantities is the unit selection problem (Li and Pearl 2019, 2024b), in which Li and Pearl defined an objective causal function to select a set of individuals that have the desired mode of behavior.

Let X denote the binary treatment and Y denote the binary effect. According to Li and Pearl, individuals were divided into four response types: Complier (i.e., $P(y_x, y'_{x'}))$, always-taker (i.e., $P(y_x, y_{x'}))$, never-taker (i.e., $P(y'_x, y'_{x'}))$, and defier (i.e., $P(y'_x, y_{x'}))$. Suppose the payoff of selecting a complier, always-taker, never-taker, and defier is β , γ , θ , δ , respectively (i.e., benefit vector). The objective function (i.e., benefit function) that optimizes the composition of the four types over the selected set of individuals c is as follows:

$$f(c) = \beta P(y_x, y'_{x'}|c) + \gamma P(y_x, y_{x'}|c) + \theta P(y'_x, y'_{x'}|c) + \delta P(y'_x, y_{x'}|c).$$

Li and Pearl provided two types of identifiability conditions for the benefit function. One is about the response type such that there is no defier in the population (i.e., monotonicity). Another is about the benefits vector's relations, such that

Table 2: Results of an experimental study with 1500 randomly selected customers were forced to apply the discount, and 1500 randomly selected customers were forced not to.

| | Discount | No discount |
|---------------------|----------|-------------|
| Bought the purchase | 900 | 750 |
| No purchase | 600 | 750 |

 $\beta + \delta = \gamma + \theta$ (i.e., gain equality). These two conditions are helpful but still too specific and challenging to satisfy in real-world applications. If the benefit function is not identifiable, it can be bounded using experimental and observational data. Here in this paper, we extend the gain equality to the ϵ -identifiability as stated in the following theorem.

Theorem 19. Given a causal diagram G and distribution compatible with G, let C be a set of variables that does not contain any descendant of X in G, then the benefit function $f(c) = \beta P(y_x, y'_{x'}|c) + \gamma P(y_x, y_{x'}|c) + \theta P(y'_x, y'_{x'}|c) + \delta P(y_{x'}, y'_{x}|c)$ is $\frac{|\beta - \gamma - \theta + \delta|}{2}$ -identified to $(\gamma - \delta)P(y_x|c) + \delta P(y_{x'}|c) + \theta P(y'_{x'}|c) + \frac{\beta - \gamma - \theta + \delta}{2}$.

One critical use case of the above theorem is that decisionmakers usually only care about the sign (gain or lose) of the benefit function. Decision-makers can apply the above theorem before conducting any observational study to see if the sign of the benefit function can be determined, as we will illustrate in the next section.

Example: Non-immediate Profit

Consider the most common example in (Li and Pearl 2019). A sale company proposed a discount on a purchase in order to increase the total non-immediate profit. The company assessed that the profit of offering the discount to complier, always-taker, never-taker, and defier is \$100, -\$60, \$0, -\$140, respectively. Let X = x denote that a customer applied the discount, and X = x denote that a customer did not apply the discount. Let Y = y denote that a customer did not. The benefit function is then (here *c* denote all customers)

$$f(c) = 100P(y_x, y'_{x'}|c) - 60P(y_x, y_{x'}|c) + 0P(y'_x, y'_{x'}|c) - 140P(y'_x, y_{x'}|c).$$

The company conducted an experimental study where 1500 randomly selected customers were forced to apply the discount, and 1500 randomly selected customers were forced not to. The results are summarized in Table 2. The experimental data reads $P(y_x|c) = 0.6$ and $P(y_{x'}|c) = 0.5$.

Before conducting any observational study, one can conclude that the benefit function is 10-identified to -12 using Theorem 19. This result indicates that the benefit function is at most 10 away from -12; thus, the benefit function is negative regardless of the observational data. The decisionmaker then can easily conclude that the discount should not offer to the customers.

Discussion

We have defined the ϵ -identifiability of causal quantities and provided a list of ϵ -identifiable conditions for causal effects, ETT, PNS, PN, and PS. We still have some further discussions about the topic.

First, all conditions except Theorems 14 and 15 are conditions from observational or experimental data. In other words, if some of the observational or experimental distributions satisfied a particular condition, then the causal quantities are ϵ -identifiable. These conditions are advantageous in real-world applications as no specific causal graph is needed. However, we still love to discover more graphical conditions (similar to back-door or front-door criterion) of ϵ -identifiability. In addition, as illustrated by the example in Section , Theorems 14 and 15 were derived and stated to give an example and a starting point for future research where specific causal graphs, combined with our other work in this paper, can provide causal effect estimates that were otherwise unobtainable.

Second, the bounds of PNS, PS, PN, and the benefit function can be narrowed by covariates information with their causal structure (Dawid, Musio, and Murtas 2017; Li and Pearl 2022b; Mueller, Li, and Pearl 2022). The ϵ -identifiability can also be extended if covariates information and their causal structure are available, which should be an exciting direction in the future.

Third, monotonicity is defined using a causal quantity, and in the meantime, monotonicity is also an identifiable condition for other causal quantities (e.g., PNS). Thus, another charming direction is how the ϵ -identifiability of monotonicity affects the ϵ -identifiability of other causal quantities.

Fourthly, it can be argued that the concept of ϵ -Identifiability shares similarities with the notion of partial identification, as discussed in the works of Tian & Pearl (Tian and Pearl 2000), Zhang et al. (Zhang, Tian, and Bareinboim 2022), and Li & Pearl (Li and Pearl 2024a). Partial identification provides bounds for causal quantities when full identification is not achievable. In contrast, ϵ identifiability specifies conditions under which these causal quantities fall within an ϵ range of their true values. Simply put, while partial identification defines the bounds, ϵ identifiability identifies the conditions that ensure these bounds are within an acceptably small ϵ range. In other words, ϵ -identifiability seeks to establish conditions where the causal quantities are satisfied to a predefined ϵ -wide interval, determined by the decision-maker as an acceptable width, thus making them nearly identifiable.

Fifth, for each of the causal quantities, we have outlined multiple conditions using observational and/or experimental data. Significantly, the concept of ϵ -identifiability allows for the computation of informative probabilities in scenarios where this would otherwise be impossible. Our theorems do more than just assert that the permissible range is narrow; they also define what this permissible range is.

Sixth, the primary contribution of this paper is the introduction of the ϵ -identifiability concept, suggesting that further conditions could be unearthed in subsequent research. The main goal is to pave the way for exploring additional conditions that adhere to ϵ -identifiability in future studies.

Lastly, this paper does not address tightness because its focus is on identifying conditions that meet a predetermined ϵ requirement. This involves specifying the criteria necessary for ensuring that the causal quantities align within the ϵ range of the true value. With a given ϵ , we aim to minimize the required conditions. However, defining tightness for multiple overlapping conditions remains a task for future work, where tightness can be properly defined and proven.

Conclusion

In this paper, we defined the ϵ -identifiability of causal quantities, which is easier to satisfy in real-world applications. We provided the ϵ -identifiability conditions for causal effects, ETT, PNS, PS, PN, and the unit selection problem. We further illustrated the use cases of the proposed conditions by simulated examples.

We believe this paper introduces a concept of practical significance and, although the core concept is algebraically straightforward, useful derivations (e.g., Theorems 14 and 15) can grow quite complex. The aim is to introduce the concept of ϵ -identifiability and encourage the development of similar theorems and derivations in the future.

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Reproducibility Checklist

This paper:

- Includes a conceptual outline and/or pseudocode description of AI methods introduced (**yes**)
- Clearly delineates statements that are opinions, hypothesis, and speculation from objective facts and results (**yes**)
- Provides well marked pedagogical references for lessfamiliare readers to gain background necessary to replicate the paper (**yes**)

Does this paper make theoretical contributions? (**yes**) If yes, please complete the list below.

- All assumptions and restrictions are stated clearly and formally. (**yes**)
- All novel claims are stated formally (e.g., in theorem statements). (**yes**)
- Proofs of all novel claims are included. (yes)
- Proof sketches or intuitions are given for complex and/or novel results. (yes)
- Appropriate citations to theoretical tools used are given. (yes)
- All theoretical claims are demonstrated empirically to hold. (**partial**)
- All experimental code used to eliminate or disprove claims is included. (NA)

Does this paper rely on one or more datasets? (**no**) Does this paper include computational experiments? (**no**)

Appendix

Proof of Theorem 13

Proof. From Equation (2) we have,

$$P(x,y) \le P(y_x) \le 1 - P(x,y').$$

Let $1 - P(x, y') - P(x, y) \le 2\epsilon$, we obtain $P(x') \le 2\epsilon$. Therefore, $P(y_x)$ is ϵ -identified to $P(x, y) + \epsilon$ if $P(x') \le 2\epsilon$, Equation (13) holds. Similarly, we can substitute x, y with x', y', respectively. Equations (14) to (16) hold. \Box

Proof of Theorem 14

Proof. First, by adjustment formula in Equation (1), we have,

$$P(y_x) = P(y|x, u)P(u) + P(y|x, u')P(u').$$

Thus,

$$P(y_{x}) \\ \geq P(y|x, u')P(u') \\ = P(y|x, u')(1 - P(u)) \\ = \frac{P(x, y, u')}{P(x, u')}(1 - P(u)) \\ \geq \frac{P(x, y) - P(u)}{P(x)}(1 - P(u)) \\ = P(y|x) - P(y|x)P(u) - \frac{P(u)}{P(x)} + \frac{P^{2}(u)}{P(x)} \\ \geq P(y|x) - P(u) - \frac{P(u)}{P(x)} \\ = P(y|x) - (1 + \frac{1}{P(x)})P(u).$$

Also if $P(x) \ge P(u) + c$ for some constant c > 0, we have,

$$P(y_x)$$

$$\leq P(u) + P(y|x, u')(1 - P(u))$$

$$\leq P(u) + \frac{P(x, y, u')}{P(x, u')}(1 - P(u))$$

$$\leq P(u) + \frac{P(x, y)}{P(x) - P(u)}(1 - P(u))$$

$$\leq P(u) + \frac{P(x, y)}{P(x) - P(u)}$$

$$= P(u) + \frac{P(x, y)}{P(x)(1 - \frac{P(u)}{P(x)})}$$

$$= P(u) + \frac{P(x, y)(1 - \frac{P(u)}{P(x)}) + P(y|x)P(u)}{P(x)(1 - \frac{P(u)}{P(x)})}$$

$$= P(u) + P(y|x) + \frac{P(y|x)P(u)}{P(x) - P(u)}$$

$$\leq P(y|x) + P(u) + \frac{P(u)}{P(x) - P(u)}$$

$$\leq P(y|x) + P(u) + \frac{P(u)}{c}$$

$$= P(y|x) + P(u)(1 + \frac{1}{c})$$

Therefore, we have,

$$P(y|x) - (1 + \frac{1}{P(x)})P(u) \le P(y_x),$$

 $P(y|x) + (1 + \frac{1}{c})P(u) \ge P(y_x).$

Let

$$(1+\frac{1}{c})P(u) + (1+\frac{1}{P(x)})P(u) \le 2\epsilon.$$

We have,

$$P(u)$$

$$\leq \frac{2}{2 + \frac{1}{c} + \frac{1}{P(x)}}\epsilon$$

$$= \frac{2cP(x)}{2cP(x) + P(x) + c}\epsilon$$

Then we know that if $P(u) \leq \frac{2cP(x)}{2cP(x)+P(x)+c}\epsilon$,

$$P(y|x) - (1 + \frac{1}{P(x)}) \frac{2cP(x)}{2cP(x) + P(x) + c} \epsilon \le P(y_x),$$

$$P(y|x) + (1 + \frac{1}{c})\frac{2cP(x)}{2cP(x) + P(x) + c}\epsilon \ge P(y_x),$$

$$P(y|x) - \frac{2cP(x) + 2c}{2cP(x) + P(x) + c}\epsilon \le P(y_x),$$

$$P(y|x) + \frac{2cP(x) + 2P(x)}{2cP(x) + P(x) + c}\epsilon \ge P(y_x).$$

Therefore, $P(y_x)$ is ϵ -identified to $P(y|x) - \frac{2cP(x)+2c}{2cP(x)+P(x)+c}\epsilon + \epsilon = P(y|x) + \frac{P(x)-c}{2cP(x)+P(x)+c}\epsilon$. Besides, if $P(x) \ge 0.5$ and $P(u) \le 0.1$, let c = 0.4, we have

$$\begin{split} P(y|x) - (1 + \frac{1}{P(x)})P(u) &\leq P(y_x), \\ P(y|x) + (1 + \frac{1}{c})P(u) &\geq P(y_x). \\ P(y|x) - (1 + \frac{1}{0.5})P(u) &\leq P(y_x), \\ P(y|x) + (1 + \frac{1}{0.4})P(u) &\geq P(y_x). \\ P(y|x) - 3P(u) &\leq P(y_x) \leq P(y|x) + 3.5P(u). \\ \text{Let } 3.5P(u) + 3P(u) &\leq 2\epsilon, \text{ we have } P(u) &\leq \frac{4}{13}\epsilon, \text{ and} \end{split}$$

 $P(y|x) - \frac{12}{13}\epsilon \le P(y_x) \le P(y|x) + \frac{14}{13}\epsilon.$

Therefore, $P(y_x)$ is ϵ -identified to $P(y|x) - \frac{12}{13}\epsilon + \epsilon = P(y|x) + \frac{\epsilon}{13}$.

Proof of Theorem 15

Proof. Similarly to the proof of Theorem 14, we have,

$$P(y|x) - (1 + \frac{1}{P(x)})P(u) \le P(y_x),$$
$$P(y|x) + (1 + \frac{1}{c})P(u) \ge P(y_x).$$

then we obtain,

$$\frac{P(y|x)}{P(x')} - \left(1 + \frac{1}{P(x)}\right)\frac{P(u)}{P(x')} \le \frac{P(y_x)}{P(x')}$$
$$\frac{P(y|x)}{P(x')} + \left(1 + \frac{1}{c}\right)\frac{P(u)}{P(x')} \ge \frac{P(y_x)}{P(x')}$$

Let

$$(1+\frac{1}{c})\frac{P(u)}{P(x')} + (1+\frac{1}{P(x)})\frac{P(u)}{P(x')} \le 2\epsilon.$$

We have,

$$P(u)$$

$$\leq \frac{2P(x')}{2 + \frac{1}{c} + \frac{1}{P(x)}} \epsilon$$

$$= \frac{2cP(x)P(x')}{2cP(x) + P(x) + c} \epsilon.$$

Then we know that if $P(u) \leq \frac{2cP(x)P(x')}{2cP(x)+P(x)+c}\epsilon$,

$$\frac{P(y|x)}{P(x')} - (1 + \frac{1}{P(x)})\frac{2cP(x)}{2cP(x) + P(x) + c}\epsilon \le \frac{P(y_x)}{P(x')}$$

$$\frac{P(y|x)}{P(y|x)} \leftarrow 1, \quad 2cP(x) \quad c \in P(y_x)$$

$$\frac{\overline{P(x')}}{P(x')} + (1+\frac{1}{c})\frac{\overline{P(x)}}{2cP(x) + P(x) + c}\epsilon \ge \frac{\overline{P(yx)}}{P(x')},$$
$$\frac{P(y|x)}{\overline{P(x')}} - \frac{2cP(x) + 2c}{2cP(x) + 2c}\epsilon \le \frac{P(yx)}{P(x')},$$

$$\frac{P(x')}{P(x')} = \frac{2cP(x) + P(x) + c}{2cP(x) + 2P(x)}$$
$$\frac{P(y|x)}{P(x')} + \frac{2cP(x) + 2P(x)}{2cP(x) + P(x) + c} \epsilon \ge -\frac{P(y_x)}{P(x')}.$$

By Theorem 9, we have,

$$P(y_x|x') = \frac{P(y_x)}{P(x')} - \frac{P(x,y)}{P(x')}.$$

Therefore, $P(y_x|x')$ is ϵ -identified to $\frac{P(y|x)}{P(x')} - \frac{P(x,y)}{P(x')} - \frac{2cP(x)+2c}{2cP(x)+P(x)+c}\epsilon + \epsilon = \frac{P(y|x)}{P(x')} - \frac{P(x,y)}{P(x')} + \frac{P(x)-c}{2cP(x)+P(x)+c}\epsilon$. Besides, if P(x) = 0.5 and $P(u) \leq 0.1$, let c = 0.4, similarly to the proof of Theorem 14, we have,

$$\begin{split} P(y|x) - (1 + \frac{1}{0.5})P(u) &\leq P(y_x), \\ P(y|x) + (1 + \frac{1}{0.4})P(u) &\geq P(y_x). \\ 2P(y|x) - 2(1 + \frac{1}{0.5})P(u) &\leq 2P(y_x), \\ 2P(y|x) + 2(1 + \frac{1}{0.4})P(u) &\geq 2P(y_x). \\ 2P(y|x) - 6P(u) &\leq 2P(y_x) &\leq 2P(y|x) + 7P(u). \end{split}$$

By Theorem 9 and P(x) = P(x') = 0.5, we have,

$$P(y_x|x') = \frac{P(y_x)}{P(x')} - \frac{P(x,y)}{P(x')}$$
$$= \frac{P(y_x)}{0.5} - \frac{P(x,y)}{P(x)}$$
$$= 2P(y_x) - P(y|x).$$

Let $7P(u) + 6P(u) \le 2\epsilon$, we have $P(u) \le \frac{2}{13}\epsilon$, and

$$P(y|x) - \frac{12}{13}\epsilon \le P(y_x|x') \le P(y|x) + \frac{14}{13}\epsilon.$$

Therefore, $P(y_x|x')$ is ϵ -identified to $P(y|x) - \frac{12}{13}\epsilon + \epsilon = P(y|x) + \frac{\epsilon}{13}$.

Proof of Theorem 16

Proof. From the bounds of PNS in Equations (5) and (6) is as follows:

$$\max \begin{cases} 0, \\ P(y_x) - P(y_{x'}), \\ P(y) - P(y_{x'}), \\ P(y_x) - P(y) \end{cases} \leq \text{PNS}$$
$$\min \begin{cases} P(y_x), \\ P(y_{x'}), \\ P(x, y) + P(x', y'), \\ P(y_x) - P(y_{x'}) + \\ + P(x, y') + P(x', y) \end{cases} \geq \text{PNS}.$$

Let $P(y_x) - 0 \le 2\epsilon$, we obtain that PNS is ϵ -identified to ϵ if $P(y_x) \le 2\epsilon$, Equation (22) holds.

Similarly, the rest of $20\ {\rm equations}$ can be obtained by letting

$$\begin{array}{rclcrcl} P(y'_{x'}) - 0 &\leq & 2\epsilon, \\ P(x,y) + P(x',y') - 0 &\leq & 2\epsilon, \\ P(y_x) - P(y_{x'}) + P(x,y') + P(x',y) - 0 &\leq & 2\epsilon, \\ P(y_x) - (P(y_x) - P(y_{x'})) &\leq & 2\epsilon, \\ P(y_{x'}) - (P(y_x) - P(y_{x'})) &\leq & 2\epsilon, \\ P(x,y) + P(x',y') - (P(y_x) - P(y_{x'})) &\leq & 2\epsilon, \\ P(y_x) - P(y_{x'}) + P(x,y') + P(x',y) - & \\ & & (P(y_x) - P(y_{x'})) &\leq & 2\epsilon, \\ P(y_x) - (P(y) - P(y_{x'})) &\leq & 2\epsilon, \\ P(y_x) - (P(y) - P(y_{x'})) &\leq & 2\epsilon, \\ P(y_x) - P(y_{x'}) + P(x,y') + P(x',y) - & \\ & & (P(y_x) - P(y_{x'})) &\leq & 2\epsilon, \\ P(y_x) - P(y_{x'}) + P(x,y') + P(x',y) - & \\ & & (P(y_x) - (P(y_x) - P(y))) &\leq & 2\epsilon, \\ P(x,y) + P(x',y') - (P(y_x) - P(y)) &\leq & 2\epsilon, \\ P(y_x) - (P(y_x) - P(y)) &\leq & 2\epsilon, \\ P(y_x) - P(y_{x'}) + P(x,y') + P(x',y) - & \\ & (P(y_x) - P(y_x) - P(y)) &\leq & 2\epsilon, \\ P(y_x) - P(y_{x'}) + P(x,y') + P(x',y) - & \\ & (P(y_x) - P(y_x) - P(y)) &\leq & 2\epsilon. \\ \end{array}$$

Proof of Theorem 17

Proof. From the bounds of PN in Equations (7) and (8) is as follows:

 $\max\left\{\begin{array}{c}0,\\\frac{P(y)-P(y_{x'})}{P(x,y)}\end{array}\right\} \le \mathsf{PN} \le \min\left\{\begin{array}{c}1,\\\frac{P(y'_{x'})-P(x',y')}{P(x,y)}\end{array}\right\}$

Let $\frac{P(y'_{x'}) - P(x',y')}{P(x,y)} - 0 \leq 2\epsilon$, we obtain that PN is ϵ -identified to ϵ if $P(y'_{x'}) - P(x',y') \leq 2P(x,y)\epsilon$, Equation (43) holds.

Similarly, the rest of 4 equations can be obtained by letting

$$\frac{1 - \frac{P(y) - P(y_{x'})}{P(x, y)} \leq 2\epsilon, \\
\frac{P(y'_{x'}) - P(x', y')}{P(x, y)} - \frac{P(y) - P(y_{x'})}{P(x, y)} \leq 2\epsilon.$$

Proof of Theorem 18

Proof. From the bounds of PS in Equations (9) and (10) is as follows:

$$\max\left\{\begin{array}{c}0,\\\frac{P(y')-P(y'_x)}{P(x',y')}\end{array}\right\} \le \mathbf{PS} \le \min\left\{\begin{array}{c}1,\\\frac{P(y_x)-P(x,y)}{P(x',y')}\end{array}\right\}$$

Let $\frac{P(y_x) - P(x,y)}{P(x',y')} - 0 \le 2\epsilon$, we obtain that PS is ϵ -identified to ϵ if $P(y_x) - P(x,y) \le 2P(x',y')\epsilon$, Equation (48). Similarly, the rest of 4 conditions can be obtained by letting

$$1 - \frac{P(y') - P(y'_x)}{P(x', y')} \le 2\epsilon,$$

$$\frac{P(y_x) - P(x, y)}{P(x', y')} - \frac{P(y') - P(y'_x)}{P(x', y')} \le 2\epsilon.$$

Proof of Theorem 19

Proof. $f(c) = \beta P(y_x, y'_{x'}|c) + \gamma P(y_x, y_{x'}|c) + \theta P(y'_x, y'_{x'}|c) + \delta P(y'_x, y_{x'}|c) = \beta P(y_x, y'_{x'}|c) + \gamma [P(y_x|c) - P(y_x, y'_{x'}|c)] + \theta [P(y'_{x'}) - P(y_x, y'_{x'}|c)] + \delta P(y'_x, y_{x'}|c) = \gamma P(y_x|c) + \theta P(y'_{x'}|c) + (\beta - \gamma - \theta) P(y_x, y'_{x'}|c)$

$$= \gamma P(y_{x}|c) + \theta P(y'_{x'}|c) + (\beta - \gamma - \theta) P(y_{x}, y'_{x'}|c) + \delta P(y'_{x}, y_{x'}|c).$$
(53)

Note that, we have,

 $P(y'_{x}, y_{x'}|c) = P(y_{x}, y'_{x'}|c) - P(y_{x}|c) + P(y_{x'}|c).$ (54) Substituting Equation (54) into Equation (53), we have,

$$f(c) = \gamma P(y_{x}|c) + \theta P(y'_{x'}|c) + (\beta - \gamma - \theta)P(y_{x}, y'_{x'}|c) + \delta P(y'_{x}, y_{x'}|c) = \gamma P(y_{x}|c) + \theta P(y'_{x'}|c) + (\beta - \gamma - \theta)P(y_{x}, y'_{x'}|c) + \delta [P(y_{x}, y'_{x'}|c) - P(y_{x}|c) + P(y_{x'}|c)] = (\gamma - \delta)P(y_{x}|c) + \delta P(y_{x'}|c) + \theta P(y'_{x'}|c) + \theta P(y'_{x'}|c) + \delta P(y_{x'}|c) + \delta P(y'_{x'}|c) + \delta P(y''_{x'}|c) + \delta P(y'$$

 $(\beta - \gamma - \theta + \delta)P(y_x, y'_{x'}|c).$ Case 1: If $\beta - \gamma - \theta + \delta \ge 0$,

case 1. If p = y = 0 + f(a)

$$\leq (\gamma - \delta)P(y_x|c) + \delta P(y_{x'}|c) + \theta P(y'_{x'}|c) + \frac{\beta - \gamma - \theta + \delta}{2} + \frac{|\beta - \gamma - \theta + \delta|}{2}$$
$$= (\gamma - \delta)P(y_x|c) + \delta P(y_{x'}|c) + \theta P(y'_{x'}|c) + \frac{\beta - \gamma - \theta + \delta}{2}.$$

and,

$$\begin{aligned} &f(c)\\ \geq & (\gamma-\delta)P(y_x|c) + \delta P(y_{x'}|c) + \theta P(y'_{x'}|c) + \\ & \frac{\beta-\gamma-\theta+\delta}{2} - \frac{|\beta-\gamma-\theta+\delta|}{2} \\ = & (\gamma-\delta)P(y_x|c) + \delta P(y_{x'}|c) + \theta P(y'_{x'}|c). \end{aligned}$$

 $\begin{array}{l} \text{Therefore, } f(c) \text{ is } \frac{|\beta - \gamma - \theta + \delta|}{2} \text{-identified to } (\gamma - \delta) P(y_x | c) + \\ \delta P(y_{x'} | c) + \theta P(y'_{x'} | c) + \frac{\beta - \gamma - \theta + \delta}{2}. \\ \text{Case 2: If } \beta - \gamma - \theta + \delta < 0, \end{array}$

$$f(c) \leq (\gamma - \delta)P(y_x|c) + \delta P(y_{x'}|c) + \theta P(y'_{x'}|c) + \frac{\beta - \gamma - \theta + \delta}{2} + \frac{|\beta - \gamma - \theta + \delta|}{2} = (\gamma - \delta)P(y_x|c) + \delta P(y_{x'}|c) + \theta P(y'_{x'}|c).$$

and,

$$f(c) \\ \geq (\gamma - \delta)P(y_x|c) + \delta P(y_{x'}|c) + \theta P(y'_{x'}|c) + \frac{\beta - \gamma - \theta + \delta}{2} - \frac{|\beta - \gamma - \theta + \delta|}{2} \\ = (\gamma - \delta)P(y_x|c) + \delta P(y_{x'}|c) + \theta P(y'_{x'}|c) + \frac{\beta - \gamma - \theta + \delta}{2}.$$

Therefore, f(c) is $\frac{|\beta - \gamma - \theta + \delta|}{2}$ -identified to $(\gamma - \delta)P(y_x|c) + \delta P(y_{x'}|c) + \theta P(y'_{x'}|c) + \frac{\beta - \gamma - \theta + \delta}{2}$.