**Integer Functions**

\[ x = -e \]

\[ f(x) = x \]

\[ [x] = \]

\[ \lfloor -e \rfloor = -3 \]

\[ \lceil -e \rceil = -2 \]

**Floor:** \( \lfloor \frac{x}{1} \rfloor \)

**Ceiling:** \( \lceil x \rceil \)

Many calculators and languages have \( \text{Int}(x) \) function (rounds towards zero), and has the property \( \text{Int}(-x) = -\text{Int}(x) \).

**Facts:** \( \lfloor x \rfloor \leq x \leq \lceil x \rceil \)

Equality \( \iff \) \( x \) is an integer

\( \lceil x \rceil - \lfloor x \rfloor = [x \text{ is not an integer}] \)

By shifting the graphs we see that

\[ x - 1 \leq \lfloor x \rfloor \leq x \leq \lceil x \rceil < x + 1 \]
And \( \lfloor -x \rfloor = -\lceil x \rceil \) \( \text{ and } \lceil -x \rceil = -\lfloor x \rfloor \), thus one can express one in terms of the other. Some useful identities (for proofs, etc.)

\[
\begin{align*}
\lfloor x \rfloor &= n & \iff & n \leq x < n+1 \\
\lceil x \rceil &= n & \iff & x-1 < n \leq x \\
\lfloor x \rfloor &= n & \iff & n-1 < x \leq n \\
\lceil x \rceil &= n & \iff & x \leq n < x+1
\end{align*}
\]

we assume \( n \in \mathbb{Z}, \ x \in \mathbb{R} \)

Clearly we may subtract out integers:

\[
\lfloor x + n \rfloor = \lfloor x \rfloor + n
\]

\[
\lfloor 2 \cdot \frac{1}{2} \rfloor = 1 \neq \lfloor 2 \rfloor \cdot \lfloor \frac{1}{2} \rfloor = 0
\]

Often floor/ceiling are redundant

\[
\begin{align*}
x < n & \iff \lfloor x \rfloor < n \\
n < x & \iff n \leq \lceil x \rceil \\
x \leq n & \iff \lceil x \rceil \leq n \\
n \leq x & \iff \lfloor x \rfloor \leq n
\end{align*}
\]

A related concept: fractional part of \( x \):

\[
\{ x \} = x - \lfloor x \rfloor \quad \text{“integer part of } x \text{”}
\]

If \( x \) can be written as \( x = n + \theta \)

\[
\theta \in \mathbb{Z} \quad 0 \leq \theta < 1,
\]

then
then \( n = L \times J \) and \( \Theta = \Sigma \times 3 \).

Consider \( L \times y \beta = L \times J + \Sigma \times 3 + J \times \beta + \Sigma \times \beta \)  

integer  

\( = L \Sigma \times 3 + \Sigma \times \beta + L \times J + J \times \beta \)  

\( 0 \leq \Sigma \times 3 + \Sigma \times \beta < 2 \)  

is either 0

Floor and Ceiling Applications

What is \( \beta \log 357 \):  
\( 2^5 \leq 35 \leq 2^6 \)  
\( 5 \leq \beta \log 35 \leq 6 \)  

so \( \beta \log 357 = 6 \).

Note: \( 35 = (100011)_2 \) is 6-bits long  
so is \( \beta \log 7 \) the length of \( n \) in bits?  

No: \( 32 = (100000)_2 \) and \( \beta \log 327 = 5 \).  

each number \( n \) has \( m \) bits when  

\( 2^{m-1} \leq n < 2^m \)  

so \( m-1 = \beta \log n \)  

\( \Leftrightarrow m = \beta \log n + 1 \)  

also \( m = \beta \log(n+1) \) and this holds for \( n = 0 \) as well.
What is \( \lfloor \sqrt{x} \rfloor \)? Since \( \lfloor x \rfloor \in \mathbb{Z}, \) \( = \lfloor x \rfloor. \)

Clearly, nesting floor/ceiling expressions has only the inner one being important.

Prove or disprove: \( \lfloor \sqrt{\lfloor x \rfloor} \rfloor = \lfloor \sqrt{x} \rfloor, \) \( x \in \mathbb{R}. \)

Clearly holds when \( x = \lfloor x \rfloor. \)

Trying examples \( \pi, e, \phi, \) does not give counterexample.
Let try to prove it.

Don't try \( x = \lfloor x \rfloor + \theta! \)

Define \( m = \lfloor \sqrt{\lfloor x \rfloor} \rfloor \iff m \leq \sqrt{\lfloor x \rfloor} < m + 1 \) (square,

\( m^2 \leq \lfloor x \rfloor < (m+1)^2 \iff m^2 \leq x < (m+1)^2 \)

\( (n \leq x \iff n^2 \leq \lfloor x \rfloor) \iff m \leq \sqrt{x} < m + 1 \) (square root

\( (x < n \iff \lfloor x \rfloor < n) \iff m = \lfloor \sqrt{x} \rfloor = \lfloor \sqrt{\lfloor x \rfloor} \rfloor \)

Can also prove \( \lfloor \sqrt{x} \rfloor = \lfloor \sqrt{x} \rfloor, \) \( x \geq 0. \)

Doesn't really depend on square root function much. Let \( f(x) \) be continuous, monotonically increasing and

\[ f(x) \in \mathbb{Z} \implies x \in \mathbb{Z}, \text{ then} \]

\[ \lfloor f(x) \rfloor = \lfloor f(\lfloor x \rfloor) \rfloor \text{ and } \lceil f(x) \rceil = \lceil f(\lceil x \rceil) \rceil \]
Proof of $[f(x)] = [f([x])]$. If $x = [x]$ there is nothing to prove. So $x > [x] \Rightarrow f(x) > f([x])$ as $f$ is increasing. Since ceiling is a non-decreasing function $[f([x])] \leq [f(x)]$. If $< \text{ holds, } \exists y \\
\text{ s.t. } x < y < [x] \text{ and } f(y) = f([x])$ since $f$ is continuous. But $[f(x)] \in \mathbb{Z}$ so $f(y) \in \mathbb{Z} = y \in \mathbb{Z}$, which is a contradiction so $x \neq [x]$ and $[f([x])] = [f(x)]$.

Important special case:
\[
\left\lfloor \frac{x+m}{n} \right\rfloor = \left\lfloor \frac{[x]+m}{n} \right\rfloor = \left\lfloor \frac{[x]+m}{n} \right\rfloor
\]
\[
m, n \in \mathbb{Z}, n > 0 \quad \text{L.L.L. } x/101/10/10 = Lx/1000\]

What about mixed-mode?
\[
[x] = \sqrt{x}, \quad x > 0
\]

Fails with $\phi = x$, so false. This begs the general question different math problem levels.

1: Given $x$ explicitly, prove $P(x)$: "prove $[f(x)] = f(x)$".
2: Given $X$ prove $P(x)$ $\forall x \in X$: "prove $[f(x)] = f(x) \forall x \in X$".
3: Given $X$ and $P(x)$, prove or disprove $P(x)$: "Prove or disprove $[\sqrt{[x]}] = \sqrt{x} \forall x \in \mathbb{R} > 0$".

Must first try to find a counterexample, or lacking that, try to prove it. Very similar to

-5- "real math."
4. Given $X$ and $P(x)$, find a necessary and sufficient condition $Q(x)$ that makes $P(x)$ true. “Find a necessary and sufficient condition so that $\lfloor x \rfloor \neq \lceil x \rceil$.” Must find $Q(x) \Rightarrow P(x)$

$x \in \mathbb{Z} = Q(x)$

After you find $Q(x)$, must prove $\Leftarrow$.

5. Given $X$ find an interesting property $P(x)$

This is pure research.

What are necessary and sufficient conditions so that $\lfloor \sqrt{x} \rfloor = \lceil \sqrt{x} \rceil$. We saw this failed for $\phi = 1.618$ + can see it fails for $\pi \times e < 10$. The bad cases occur when $m^2 < x < m^2 + 1$. Thus the condition is: $x \in \mathbb{Z}$ or $\lfloor \sqrt{x} \rfloor \not\in \mathbb{Z}$.

Intervals

- $\alpha \leq x \leq \beta$: $[\alpha \ldots \beta]$ closed
- $\alpha < x \leq \beta$: $(\alpha \ldots \beta]$ half-open
- $\alpha \leq x < \beta$: $[\alpha \ldots \beta)$ open
- $\alpha < x < \beta$: $(\alpha \ldots \beta)$ open

How many integers are in each of these intervals?

Start with half-open, which are additive:

$[\alpha \ldots \beta) \cup [\beta \ldots \gamma) = [\alpha \ldots \gamma)$

If $\alpha, \beta \in \mathbb{Z}$ then $[\alpha, \beta)$ contains $\alpha, \alpha + 1, \ldots, \beta - 1$ when $\alpha < \beta$: $\beta - \alpha$
Similarly \((a \leq \alpha < \beta \leq \beta)\) contains \(\alpha - \alpha\) integers. What if \(a, \beta \in IR\):
\[
\begin{align*}
   \alpha \leq n < \beta & \iff [\alpha] \leq n < [\beta] \\
   a < n \leq \beta & \iff [a] < n \leq [\beta]
\end{align*}
\]

Thus we convert to integer intervals:
\[
\begin{align*}
   [a \leq \beta) & \text{ contains } [\beta] - [\alpha] \text{ integers} \\
   (a \leq \beta) & \text{ contains } [\beta] - [\alpha] \text{ integers}
\end{align*}
\]

Also, can show that \([a \leq \beta) \text{ contains } [\beta] - [\alpha] + 1\) integers, and \((a \leq \beta) \text{ contains } [\beta] - [\alpha] - 1\) integers. Here we must also impose \(a \neq \beta\).

Summary
\[
\begin{align*}
   [a \leq \beta] & \quad [\beta] - [\alpha] + 1 \quad a \leq \beta \\
   [a \leq \beta) & \quad [\beta] - [\alpha] \quad a \leq \beta \\
   (a \leq \beta] & \quad [\beta] - [\alpha] \quad a \leq \beta \\
   (a \leq \beta) & \quad [\beta] - [\alpha] - 1 \quad a < \beta
\end{align*}
\]

Consider a roulette wheel with 1000 slots, that pays off when the result, \(n\), is divisible by the floor of its cube root:
\[
[\sqrt[3]{n}] \mid n \quad \text{“divides”}
\]

In this case the house pay \$5 on a \$1 bet, otherwise we loose our \$1.

What are the “odds” of this game?
Let's compute the average winnings: in 1 to 1000 count # of winners = \( W \); \( L = 1000 - W \) is the # of losers. So the average is:

\[
\frac{5W - L}{1000} = \frac{5W - (1000 - W)}{1000} = \frac{6W - 1000}{1000}
\]

If \( 6W - 1000 > 0 \) we "beat the house" = 7

\( W > \frac{100}{6} = 166 \frac{2}{3} \), so \( W \geq 167 \) "beats the house"

\[ \left\lfloor \sqrt[3]{n} \right\rfloor \quad \text{Range of n's} \quad \# \]

1 \quad 1 \text{ to } 7 = 2^3 - 1 \quad 7

2 \quad 8 \text{ to } 26 = 3^3 - 1 \quad 10

3 \quad 3^3 \text{ to } 4^3 - 1 \quad 13

\vdots

\[ j^3 \text{ to } (j+1)^3 - 1 \quad ? \]

so the winners are those n's in each group divisibly by \( \left\lfloor \sqrt[3]{n} \right\rfloor \).

\[
W = \sum_{n=1}^{1000} [n \text{ is a winner}]
= \sum \left\lfloor \sqrt[3]{n} \right\rfloor \setminus n = \sum [k = \left\lfloor \sqrt[3]{n} \right\rfloor][k \setminus n]
\quad \text{for } 1 \leq n \leq 1000
\]
\[
\begin{align*}
&= \sum_{h,m,n} [h^3 \leq n < (h+1)^3][n = hm] [1 \leq h \leq 1000] \\
&= \sum_{h,m} [h^3 \leq hm < (h+1)^3][1 \leq h < \sqrt[3]{1000}] \\
&= 1 + \sum_{h,m} [m \in [h^2 \cdots (h+1)^3/h)] [1 \leq h < 10] \\
&= 1 + \sum_{13 \leq h < 10} \left( \left\lceil \frac{h^2 + 3h + 3 + \sqrt[3]{h}}{h} \right\rceil - \left\lceil \frac{h^2}{h} \right\rceil \right) \\
&= 1 + \sum_{14 \leq h < 10} (3h + 4) = 1 + 3 \cdot \left( \frac{9.10}{2} \right) + 4.9 \\
&= 1 + 3 \cdot 45 + 36 \\
&= 1 + 135 + 36 = 172 \\
\frac{6 \cdot 172 - 1000}{1000} &= 0.032, \text{ so this favors us.}
\end{align*}
\]

We generally: how many integers, \( n \), \( 1 \leq n \leq N \), satisfy \( \left\lfloor \sqrt[3]{n} \right\rfloor \leq n \)? Must take this sum to \( K = \sqrt[3]{N} \).

\[
N = \sum_{1 \leq k < K} (3h + 4) + \sum_{m} [K^3 \leq Km \leq N] \\
= \frac{3}{2} K(K-1) + 4(K-1) + \sum_{m} [m \in [K^2 \cdots \sqrt[3]{N}]]
\]
\[
= \frac{3}{2} K^2 + \frac{5}{2} K - 4 + LN/K - \Gamma K^2 + 1 \\
= LN/K + \frac{1}{2} K^2 + \frac{5}{2} K - 3, \quad K = \sqrt[3]{N}
\]

So \( w = \frac{3}{2} N^{2/3} + \mathcal{O}(N^{1/3}) \) as \( LN/K \approx K^3 \)

e get smaller with \( N \)

<table>
<thead>
<tr>
<th>( N )</th>
<th>( \frac{3}{2} N^{2/3} )</th>
<th>( \omega )</th>
<th>( \text{in relative terms} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 10^2 )</td>
<td>150.0</td>
<td>172</td>
<td>12.791</td>
</tr>
<tr>
<td>( 10^4 )</td>
<td>696.2</td>
<td>746</td>
<td>6.520 ( \sqrt{2} )</td>
</tr>
<tr>
<td>( 10^5 )</td>
<td>3231.7</td>
<td>3348</td>
<td>3.331 ( \sqrt{2} )</td>
</tr>
<tr>
<td>( 10^6 )</td>
<td>15000.0</td>
<td>15297</td>
<td>1.620 ( \sqrt{2} )</td>
</tr>
<tr>
<td>( 10^7 )</td>
<td>69623.8</td>
<td>70158</td>
<td>0.761 ( \sqrt{2} )</td>
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<tr>
<td>( 10^8 )</td>
<td>323165.2</td>
<td>324322</td>
<td>0.357 ( \sqrt{2} )</td>
</tr>
<tr>
<td>( 10^9 )</td>
<td>1500000.0</td>
<td>1502497</td>
<td>0.166 ( \sqrt{2} )</td>
</tr>
</tbody>
</table>

Let \( \alpha \in \mathbb{R} \) and define its "spectrum" as

\[
\text{Spec} (\alpha) = \{L\alpha, L2\alpha, L3\alpha, \ldots\}
\]

This is a multiset, as it can have repeated values. Can prove that \( \alpha \neq \beta \Rightarrow \text{Spec}(\alpha) \neq \text{Spec}(\beta) \).

Assume \( \alpha < \beta \), then \( \exists m \in \mathbb{Z}^+ \)

so that \( m(\beta - \alpha) > 1 \) \( \Rightarrow m\beta - m\alpha > 1 \), \( \Rightarrow m\beta > m\alpha \).

Thus \( \text{Spec}(\beta) \) has less than \( m \) elements \( \leq m\alpha \), while \( \text{Spec}(\alpha) \) has at least \( m \) elements.

Consider

\[
\text{Spec}(\sqrt{2}) = \{1, 2, 4, 5, 7, 8, 9, 11, 12, 14, 15, 16, 18, 19, \ldots\}
\]

\[
\text{Spec}(2\sqrt{2}) = \{3, 6, 10, 13, 17, 20, 23, 27, 30, 34, 37, 40, 44, \ldots\}
\]
\[ \text{Spec } (F_2)_n + 2n = \text{Spec } (2 + \sqrt{2})_n \]

Also it seems that \( j \in \mathbb{Z}^+ \) is either in \( \text{Spec } (F_2) \) or \( \text{Spec } (2 + \sqrt{2}) \), but not both. This is true and we say that \( \text{Spec } (F_2) \) and \( \text{Spec } (2 + \sqrt{2}) \) form a partition of \( \mathbb{Z}^+ \).

**Proof:** We will count \( \# \text{Spec } (F_2) \leq n \) and \( \# \text{Spec } (2 + \sqrt{2}) \leq n \)

If these two always sum up to \( n \), then they are a partition.

Let \( a \in \mathbb{R} \) be positive

\[ N(a, n) = \# \text{Spec } (a) \leq n = \sum_{0 < k} \left[ k \leq n \right] \]

\[ = \sum_{0 < k} \left[ k \leq n + 1 \right] \]

\[ = \sum_{0 < k} \left[ k > n + 1 \right] \]

\[ = \sum_{k} \left[ k \in \left( 0, \frac{(n+1)}{a} \right) \right] \]

\[ = \left\lceil \frac{n+1}{a} \right\rceil - \left\lfloor \frac{n+1}{a} \right\rfloor - 1 = \left\lfloor \frac{n+1}{a} \right\rfloor - 1 \]

**What is** \( N(\sqrt{2}, n) + N(2 + \sqrt{2}, n) \)?

\[ \left\lfloor \frac{n+1}{\sqrt{2}} \right\rfloor - 1 + \left\lfloor \frac{n+1}{2 + \sqrt{2}} \right\rfloor - 1 = \left\lfloor \frac{n+1}{\sqrt{2}} \right\rfloor + \left\lfloor \frac{n+1}{2 + \sqrt{2}} \right\rfloor \]

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\[
= \frac{n+1}{\sqrt{2}} - \left\{ \frac{n+1}{\sqrt{2}} \right\} + \frac{n+1}{2+\sqrt{2}} - \left\{ \frac{n+1}{2+\sqrt{2}} \right\}
\]

Note $\frac{1}{\sqrt{2}} + \frac{1}{2+\sqrt{2}} = \frac{Z + \sqrt{2} + \sqrt{2}}{\sqrt{2}(2+\sqrt{2})} = \frac{2\sqrt{2}+2}{\sqrt{2}2} = 1$

\[
= (n+1) \left( \frac{1}{\sqrt{2}} + \frac{1}{2+\sqrt{2}} \right) - \left[ \left\{ \frac{n+1}{\sqrt{2}} \right\} + \left\{ \frac{n+1}{2+\sqrt{2}} \right\} \right]
\]

but since $\frac{n+1}{\sqrt{2}} + \frac{n+1}{2+\sqrt{2}} = n+1$, and these are never integers for any $n$, the fractional parts must add up to 1, and so

\[
= n+1 - 1 = n.
\]

**Floor/Ceiling Recurrences**

Consider $K_0 = 1$

\[
K_{n+1} = 1 + \min(2K_{n^{/2}}, 3K_{n^{/3}}), \quad n \geq 2
\]

$K_1 = 1 + \min(2K_0, 3K_0) = 1 + 2 = 3$. And this begins 1, 3, 3, 4, 7, 7, 9, 9, 10, 13, ...

these are called the Knuth numbers, and it seems that $K_n \geq \frac{n+2}{2}$. 

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Attempted proof by induction:
\( n = 0 \) (basis) \( K_0 = 1 \geq 0 \)

Assume \( K_j \geq j \neq j \leq n \) and consider

\[
K_{n+1} = 1 + \min \left( 2K_{\lfloor n/2 \rfloor}, 3K_{\lfloor n/3 \rfloor} \right)
\]

\[
2K_{\lfloor n/2 \rfloor} \geq 2\left\lfloor \frac{n}{2} \right\rfloor \quad \text{and} \quad 3K_{\lfloor n/3 \rfloor} \geq 3\left\lfloor \frac{n}{3} \right\rfloor
\]

but \( 2\left\lfloor \frac{n}{2} \right\rfloor \) can be \( n-1 \) and \( 3\left\lfloor \frac{n}{3} \right\rfloor \) can be \( n/2 \), so we have only \( K_{n+1} \geq 1 + (n-2) = n-1 \)

not \( n+1 \)!

Recurrences involving floor/ceiling often arise in CS because of "divide and conquer" algorithmic analysis. In sorting we often recursively divide a list into 2 pieces of \( \left\lfloor \frac{n}{2} \right\rfloor \) and \( \left\lceil \frac{n}{2} \right\rceil \)

note: \( \left\lceil \frac{n}{2} \right\rceil + \left\lfloor \frac{n}{2} \right\rfloor = n \). With merging, and \( n-1 \) comparisons, we perform \( \text{few} \) comparisons where

\[
f(1) = 0
\]

\[
f(n) = f\left( \left\lfloor \frac{n}{2} \right\rfloor \right) + f\left( \left\lceil \frac{n}{2} \right\rceil \right) + n-1 \quad \text{for} \quad n > 1
\]

Can also rewrite Josephus as

\[
J(1) = 1
\]

\[
J(n) = 2J(\left\lfloor \frac{n}{2} \right\rfloor) - (-1)^n \quad \text{for} \quad n > 1
\]
Consider a variant on Josephus: every 3rd person is eliminated:

\[ J_3(m) = \left[ \frac{3}{2} J_3 \left( \left\lfloor \frac{3}{2} m \right\rfloor \right) + a_n \right] \mod n + a_n \]

\[ a_n = \begin{cases} 
-2 & n \equiv 0 \pmod{3} \\
+1 & 1 \\
-\frac{1}{2} & 2 
\end{cases} \]

This seems pretty hard. Why not re-number as follows (example with 10):

\[
\begin{array}{ccccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
11 & 12 & 13 & 14 & 15 & 16 & 17 \\
18 & 19 & 20 & 21 & 22 & 23 & 24 \\
25 & 26 & 27 & 28 & 29 & 30 \\
\end{array}
\]

Points: in this procedure the survivor is renumbered \(3k\). So if we can derive the original number of \(3k\), we solve the problem.

\[ 1 \rightarrow n+1, \ 2 \rightarrow n+2, \ \text{\cancel{3}}, \ 4 \rightarrow n+3, \ 5 \rightarrow n+4, \ \text{\cancel{6}}, \ \ldots \ 3k+1 \rightarrow n+2k+1, \ 3k+2 \rightarrow n+2k+2, \ \text{\cancel{3k+3}}, \ \ldots \]
If $N > n$, $N$ must have had a previous number, let's compute it:

\[ N = \binom{n + 2k + 1}{n + 2k + 2} \Rightarrow k = \left\lfloor \frac{N - n - 1}{2} \right\rfloor \]

and the previous number was $3k+1$ or $3k+2$, or

\[ 3k + (N - n - 2k) = k + N - n. \]

To compute $J_3(n)$ we can do so as follows:

(A)

\[ N := 3n; \]
while $N > n$ do $N := \left\lfloor \frac{N - n - 1}{2} \right\rfloor + N - n$;

$J_3(n) := N$, this counts down.

Let $D = 3n + 1 - N$ replace $N$, so

\[ D := 3n + 1 - (\left\lfloor \frac{2n + 1 - D - n - 1}{2} \right\rfloor + (3n + 1 - D) - n) \]
\[ = n + D - \left\lfloor \frac{2n + 0}{2} \right\rfloor = D - \left\lfloor \frac{-D}{2} \right\rfloor \]
\[ = D + \left\lceil \frac{D}{2} \right\rceil = \left\lceil \frac{D}{2} \right\rceil \]

Algorithm becomes

\[ D := 1; \]
while $D \leq 2$ do $D := \left\lceil \frac{D}{2} \right\rceil$;

$J_3(n) := 3n + 1 - D$.

This counts up, not down.
Can we generalize this to $J_q(n)$?

\[ D := 1; \]
\[ \text{while } D \leq (q-1)n \text{ do } D := \left\lfloor \frac{q}{q-1} D \right\rfloor; \]
\[ J_q(n) := q n + 1 - D \]

$q = 2$ is our “old friend” : \( \frac{q}{q-1} = 2 \), so we grow $D$ to $2^{m+1}$ with $n = 2^m + l$

\[ J_2(n) = 2n + 1 - D \]
\[ = 2(2^m + l) + 1 - 2^{m+1} \]
\[ = 2^{m+1} + 2l + 1 - 2^{m+1} = 2l + 1 \checkmark \]

Define \( D_0^{(q)} = 1 \); \( D_n^{(q)} = \left\lfloor \frac{q}{q-1} D_{n-1}^{(q)} \right\rfloor \) $n \geq 0$.

Then \( J_q(n) = q n + 1 - D_k^{(q)} \) where $k$ is the smallest index such that $D_k^{(q)} \geq (q-1)n$.

'MOD': The Binary Operation

Quotient/Remainder (given $n \in \mathbb{Z}$, $m \in \mathbb{Z}^+$)

\[ n = m \left\lfloor \frac{n}{m} \right\rfloor + n \mod m \]

quotient  remainder
\[ x \mod y = x - y \lfloor \frac{x}{y} \rfloor, \; y \neq 0 \]
for real \( x, y \) too.

Interpretation:

\[ \sim \text{radius is } \frac{y}{2\pi}, \text{ so} \]

\[ \text{circumference is } y \]

\[ x \mod y \text{ is final location} \]
\[ \lfloor \frac{x}{y} \rfloor \text{ is number of full circles} \]

To see impact of negative numbers consider:

\[ 5 \mod 3 = 5 - 3 \lfloor \frac{5}{3} \rfloor = 2; \; 5 \mod -3 = 5 - (-3) \lfloor \frac{5}{-3} \rfloor = - \]
\[ -5 \mod 3 = -5 - 3 \lfloor \frac{-5}{3} \rfloor = 1; \; -5 \mod -3 = -5 - (-3) \lfloor \frac{-5}{-3} \rfloor = -2 \]

these are called the modulus

Mostly modulus is positive, but

\[ 0 \leq x \mod y \leq y \text{ for } y > 0 \]

\[ 0 \geq x \mod y \geq y \text{ for } y < 0 \]

With \( y = 0 \) we can define \( x \mod 0 = x \)

Recall \[ x = \lfloor x \rfloor + \{ x \} \]
\[ \xrightarrow{\text{mod } 1} x \mod 1 \]
Note: mod is defined via “floor” what about

\[ x \text{ “mumble” } y = y \lfloor x/y \rfloor - x \]

this is the “distance left to zero.”

Some properties:

Distributive \[ c(x \mod y) = (cx) \mod (cy) \]

\[ \forall c, x, y \in \mathbb{R} \]

\[ c(x \mod y) = c(x - y\lfloor x/y \rfloor) \]

\[ = cx - cy \lfloor cx/ cy \rfloor = (cx) \mod (cy) \]

An application of mod: partition n things into m groups as “equally” as possible.

Example: n lines of text and we want to arrange them in m columns. Should be in decreasing order of length, and no two columns should differ in length by more than one line. We wish to distribute column-wise. We decide how many in the 1st column, then the second, etc.
As an example, consider $n = 37$ and $m = 5$

Two possible arrangements are below. The one on the right is preferred, as column length differ only by one.

\[
\begin{array}{ccccccc}
8 & 8 & 8 & 8 & 5 & 8 & 8 & 7 & 7 & 7 \\
\text{line 1} & \text{line 9} & \text{line 17} & \text{line 25} & \text{line 33} & \text{line 1} & \text{line 9} & \text{line 17} & \text{line 24} & \text{line 31} \\
\text{line 2} & \text{line 10} & \text{line 18} & \text{line 26} & \text{line 34} & \text{line 2} & \text{line 10} & \text{line 18} & \text{line 25} & \text{line 32} \\
\text{line 3} & \text{line 11} & \text{line 19} & \text{line 27} & \text{line 35} & \text{line 3} & \text{line 11} & \text{line 19} & \text{line 26} & \text{line 33} \\
\text{line 4} & \text{line 12} & \text{line 20} & \text{line 28} & \text{line 36} & \text{line 4} & \text{line 12} & \text{line 20} & \text{line 27} & \text{line 34} \\
\text{line 5} & \text{line 13} & \text{line 21} & \text{line 29} & \text{line 37} & \text{line 5} & \text{line 13} & \text{line 21} & \text{line 28} & \text{line 35} \\
\text{line 6} & \text{line 14} & \text{line 22} & \text{line 30} & & \text{line 6} & \text{line 14} & \text{line 22} & \text{line 29} & \text{line 30} \\
\text{line 7} & \text{line 15} & \text{line 23} & \text{line 31} & & \text{line 7} & \text{line 15} & \text{line 23} & \text{line 30} & \text{line 37} \\
\text{line 8} & \text{line 16} & \text{line 24} & \text{line 32} & & \text{line 8} & \text{line 16} & & & \\
\end{array}
\]

Clearly the long columns contain $\lceil \frac{n}{m} \rceil$ lines and the short columns $\lfloor \frac{n}{m} \rfloor$ lines. There will be $n \mod m$ long columns and $n \"mumble\" m$ short columns.

Generalize to "things" and "groups." The first group will contain $\lceil \frac{n}{m} \rceil$ things, so to distribute $n$ things in $m$ groups we iterate

\[
\begin{align*}
& n' = n \\
& m' = m \\
& \text{for } i = 1 \text{ to } m \quad (\text{until } m' = 1 / n = 0) \\
& \text{put } \lfloor \frac{n'}{m'} \rfloor \text{ things in a group} \\
& n' = n' - \lfloor \frac{n'}{m'} \rfloor \\
& m' = m' - 1 \\
& \text{do} \\
\end{align*}
\]

(This is iterative, but can be defined recursively.)
Example: \( n = 316, m = 6 \)

\[
314, 6: \quad \left\lceil \frac{314}{6} \right\rceil = 53 \rightarrow 261, 5: \quad \left\lceil \frac{261}{5} \right\rceil = 53 \rightarrow 208, 4: \quad \left\lceil \frac{208}{4} \right\rceil = 52 \rightarrow 156, 3: \quad \left\lceil \frac{156}{3} \right\rceil = 52 \rightarrow 104, 2: \quad \left\lceil \frac{104}{2} \right\rceil = 52 \rightarrow 52, 1: \quad \left\lceil \frac{52}{1} \right\rceil = 52 \quad 53, 53, 52, 52, 52, 52
\]

How many things are in the \( k \)th group?

\[
? = \begin{cases} 
\left\lfloor \frac{n}{m} \right\rfloor & \text{when } k \leq n \mod m \\
\left\lceil \frac{n}{m} \right\rceil & \text{when } k > n \mod m
\end{cases}
\]

\[
\left\lceil \frac{n - k + 1}{m} \right\rceil = \left\lceil \frac{8m + r - k + 1}{m} \right\rceil = 8 + \left\lceil \frac{r - k + 1}{m} \right\rceil
\]

and \( \left\lceil \frac{r - k + 1}{m} \right\rceil = \left\lceil \frac{k}{r} \right\rceil \) with \( 1 \leq k \leq m, \quad 0 \leq r \leq m \).

Since the sum of each group must equal the total we also have

\[
n = \sum_{k=1}^{m} \left\lceil \frac{n - k + 1}{m} \right\rceil = \left\lfloor \frac{n}{m} \right\rfloor + \left\lfloor \frac{n+1}{m} \right\rfloor + \cdots + \left\lfloor \frac{n+m}{m} \right\rfloor
\]

with \( m = 2 \Rightarrow \quad n = \left\lfloor \frac{n}{2} \right\rfloor + \left\lceil \frac{n}{2} \right\rceil = \left\lfloor \frac{n}{2} \right\rfloor \times \left\lceil \frac{1}{2} \right\rceil \).

What if we want the groups to be presented in non-decreasing order (smaller groups first):

\[
n = \sum_{k=1}^{m} \left\lfloor \frac{n+k-1}{m} \right\rfloor = \left\lfloor \frac{n}{m} \right\rfloor + \left\lfloor \frac{n+1}{m} \right\rfloor + \cdots + \left\lfloor \frac{n+m}{m} \right\rfloor
\]
If we replace $n$ by $\lfloor mx \rfloor$ in the previous, we get:

$$\lfloor mx \rfloor = \sum_{k=1}^{m} \left\lfloor x + \frac{k-1}{m} \right\rfloor = \left\lfloor x \right\rfloor + \left\lfloor x + \frac{1}{m} \right\rfloor + \cdots + \left\lfloor x + \frac{m-1}{m} \right\rfloor \text{ (m terms)}$$

This is remarkable. To see why assume $\lfloor x \rfloor \approx x - \frac{1}{2}$ "on average"

$$mx - \frac{1}{2} \approx \text{lhs} \left( x - \frac{1}{2} \right) + \left( x - \frac{1}{2} + \frac{1}{m} \right) + \left( x - \frac{1}{2} + \frac{2}{m} \right) + \cdots + \left( x - \frac{1}{2} + \frac{m-1}{m} \right) \approx \text{rhs}$$

$$= mx - \frac{m}{2} + \frac{1}{m} \frac{(m-1)}{2}$$

$$= mx - \frac{m}{2} + \frac{m}{2} - \frac{1}{2} = mx - \frac{1}{2}$$

Floor / Ceiling Sums

Consider $\sum \left\lfloor \sqrt{n} \right\rfloor$, let $m = \lfloor \sqrt{n} \rfloor$

$$= \sum_{h, m \geq 0} m \left[ h < n \right] \left[ m = \lfloor \sqrt{n} \rfloor \right]$$

$$= \sum_{h, m \geq 0} m \left[ h < n \right] \left[ m \leq \sqrt{n} < m+1 \right] = \sum_{h, m \geq 0} m \left[ h < n \right] \left[ m^2 \leq h < (m+1)^2 \right]$$

$$= \sum_{h, m \geq 0} m \left[ m^2 \leq h < (m+1)^2 \leq n \right] + \sum_{h, m \geq 0} m \left[ m^2 \leq h < n < (m+1)^2 \right]$$

The boundary conditions are tricky, first assume $n = a^2$ is a perfect square.
the second sum is zero:

\[ \sum_{m \geq 0} m \left[ m^2 \leq k < (m+1)^2 \leq a^2 \right] = \sum_{m \geq 0} (m(m+1)^2 - m^3) \left[ m \leq a^2 \right] \]

\[ = \sum_{m \geq 0} m (2m+1) \left[ m < a \right] = \sum_{m \geq 0} \left( 2m^3 + 3m^2 \right) \left[ m < a \right] \]

\[ = \sum_{m \geq 0} \left( 2m^3 + 3m^2 \right) \delta_m = \frac{2}{3} a(a-1)(a-2) + \frac{3}{2} a(a-1) \]

\[ = \frac{1}{6} (4a+1)a(a-1) \]

When \( a = \lfloor \sqrt{n} \rfloor \), we must consider the second sum, which are for \( a^2 \leq k < n \). Each term equals \( a \), and there are \( (n-a^2) \) of them, so

\[ \sum_{a^2 \leq k < n} \lfloor \sqrt{k} \rfloor = na - \frac{1}{3} a^3 - \frac{1}{2} a^2 - \frac{1}{6} a \text{, } a = \lfloor \sqrt{n} \rfloor \]

(adding in \( (n-a^2) \cdot a \))

Consider replacing \( L(x) \) by \( \sum_{j} \left[ 1 \leq j \leq x \right] \), \( n = a^2 \):

\[ \sum_{a^2 \leq k < n} \lfloor \sqrt{k} \rfloor = \sum_{j,k} \left[ 1 \leq j \leq \sqrt{k} \right] \left[ 0 \leq k < a^2 \right] \]

\[ = \sum_{j,a^2} \sum_{k} \left[ j^2 \leq k \leq a^2 \right] \]

\[ = \sum_{j,a^2} (a^2 - j^2) = a^3 - \frac{1}{3} a(a+\frac{1}{2})(a+1) \]

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Theorem: (Bohl, Sierpinski, Wely) If \( \alpha \) is irrational, then

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k \text{ oaken}} f(Ek < \beta) = \int_0^1 f(x) \, dx
\]

with all bounded and continuous (almost everywhere) \( f(x) \). If \( f(x) = x \) then \( \int_0^1 x \, dx = \frac{x^2}{2} \bigg|_0^1 = \frac{1}{2} \).

\[
= \lim_{n \to \infty} \frac{1}{n} \sum_{k \text{ oaken}} \{k < \beta\}
\]

Consider \( f_\beta(x) = [0 \leq x < \beta] \), then we wish to see how close

\[
\sum_{\text{oaken}} \{k \alpha < \beta\} \text{ is to } n \beta.
\]

Define \( s(x,n,\beta) = \sum_{\text{oaken}} \{k \alpha < \beta \} \) - \( \beta \) and

\[
D(\alpha,n) = \sup_{0 \leq \beta \leq 1} |s(x,n,\beta)|
\]

we wish to show \( D(\alpha,n) \) is not too large compared to \( n \), by showing \( |s(x,n,\beta)| \) is reasonably small when \( \alpha \) is irrational.

\[
\sum_{\text{oaken}} \{k \alpha < \beta \} - \beta = \sum_{\text{oaken}} (|k\alpha| - k\alpha - \beta - \beta) = 0.
\]
\[
= -n\nu + \sum \sum [ k\alpha - \nu < j \leq k\alpha ] \\
= -n\nu + \sum \sum [ j\alpha^{-1} < k < (j+\nu)\alpha^{-1} ] \\
\text{or}\ j < \lfloor n\alpha \rfloor \text{ then}
\]

Can assume \(0 < \alpha < 1\), and define:

\[
\alpha' = \lfloor \alpha^{-1} \rfloor, \quad \alpha^{-1} = \lfloor \alpha^{-1} \rfloor + \alpha' \\
b = \lfloor n\alpha^{-1} \rfloor, \quad n\alpha^{-1} = \lfloor n\alpha^{-1} \rfloor - \nu' \\
so \quad \alpha' = \lfloor \alpha^{-1} \rfloor \quad \text{and} \quad \nu' \text{is the "mumble" fractional part of } n\alpha^{-1}.
\]

Remove the boundary condition \(k<n\):

\[
\sum \sum [ k \in [j\alpha^{-1}, (j+\nu)\alpha^{-1}) ] = \sum \left[ k \in [j\alpha + j\alpha' + \nu\alpha^{-1}] - [j\alpha + j\alpha'] \right] = \Gamma (j\alpha + j\alpha' + \nu\alpha^{-1}) - \Gamma j\alpha + j\alpha' ] = \\
\text{integers cancel} \\
= \Gamma j\alpha' + b - \nu' = \lfloor j\alpha' \rfloor = b + \lfloor j\alpha' - \nu' \rfloor - \lfloor j\alpha' \rfloor \\
\text{So} \quad \frac{sc(\alpha, n, \nu)}{\alpha} = -n\nu + \lfloor n\alpha \rfloor b + \\
\sum_{0 < j < \lfloor n\alpha \rfloor} (\Gamma j\alpha' - \nu' - \lfloor j\alpha' \rfloor) - \sum_{0 < j < \lfloor n\alpha \rfloor} \left( \Gamma j\alpha' - \nu' - \lfloor j\alpha' \rfloor \right) - S \quad \text{for } k<n\text{ not excluded}
\]
\( s(\alpha', n, r) = -n \nu + \Gamma n \alpha' b - \sum_{0 \leq j < \Gamma n \alpha'} (Lj \alpha' - Lj \alpha' - r') \)

Note \( s(\alpha', \Gamma n \alpha', r') = \sum_{0 \leq j < \Gamma n \alpha'} (Lj \alpha' - Lj \alpha' - r') \)

can be related as:

\[
s(\alpha, n, r) = -n \nu + \Gamma n \alpha' b - \Gamma n \alpha' \nu
- s(\alpha', \Gamma n \alpha', r') - S + \varepsilon + E_0 \text{ or } 13
\]

Here \( 0 < \varepsilon < n \alpha' \), and \( 0 < S < \Gamma n \alpha' - 1 \). We also remove the term for \( j = \Gamma n \alpha' - 1 = \text{Lmax} \) since it contributes either \( r' \) or \( r' - 1 \). Max over absolute values and \( \nu \) to get

\[
D(\alpha, n) \leq D(\alpha', \text{Lmax}) + \alpha' + 2
\]

This implies (by more advanced methods) that

\[
D(\alpha, n) = o(n).
\]

Last sum:

\[
\sum_{0 \leq k < m} \left| \frac{n \nu + k}{m} \right|, \quad m \in \mathbb{Z}^+, \quad n \in \mathbb{Z}^-
\]
With \( n = 1 \) we get an old *fixed*
\[
L x 1 = \left[ \frac{x}{m} \right] + \left[ \frac{x+1}{m} \right] + \ldots + \left[ \frac{x+m-1}{m} \right]
\]
which is what we saw before but with \( \frac{x}{m} \) instead of \( x \)

With \( n = 0 \)
\[
\sum \left[ \frac{x}{m} \right] = m \left[ \frac{x}{m} \right]. \quad \text{What about for small } m?
\]

\( m = 2 \)
\[
\left[ \frac{x}{2} \right] + \left[ \frac{x+1}{2} \right] = 2 \left\lfloor \frac{x}{2} \right\rfloor + \frac{n}{2} \quad \text{if even}
\]
\[
= L x 1 + \frac{n-1}{2} \quad \text{if odd}
\]

\( m = 3 \)
- Cases \( n \mod 3 = 0, 1, 2 \).
  - \( 0: \frac{n}{3}, \frac{2n}{3} \) are integers
    \[
    \left[ \frac{x}{3} \right] + \left( \left[ \frac{x+1}{3} \right] + \frac{n}{3} \right) + \left( \left[ \frac{x+2}{3} \right] + \frac{2n}{3} \right) = 2 \left\lfloor \frac{x}{3} \right\rfloor + n
    \]
  - \( 1: \frac{n-1}{3}, \frac{2n-2}{3} \) are integers
    \[
    \left[ \frac{x}{3} \right] + \left( \left[ \frac{x+1}{3} \right] + \frac{n-1}{3} \right) + \left( \left[ \frac{x+2}{3} \right] + \frac{2n-2}{3} \right) = L x 1 + n-1
    \]
  - \( 2: \frac{n-2}{3}, \frac{2n-1}{3} \) are integers
    \[
    \left[ \frac{x}{3} \right] + \left( \left[ \frac{x+1}{3} \right] + \frac{n-2}{3} \right) + \left( \left[ \frac{x+2}{3} \right] + \frac{2n-1}{3} \right) = L x 1 + n-1
    \]

\( m = 4 \)
\[
\left[ \frac{x}{4} \right] + \left[ \frac{x+1}{4} \right] + \left[ \frac{x+2}{4} \right] + \left[ \frac{x+3}{4} \right]
\]

\( n \mod 4 = 0: \frac{n}{4}, \frac{2n}{4}, \frac{3n}{4} \) are integers
\[
\left[ \frac{x}{4} \right] + \left[ \frac{x+1}{4} \right] + \frac{n}{4} + \left[ \frac{x+2}{4} \right] + \frac{2n}{4} + \left[ \frac{x+3}{4} \right] + \frac{3n}{4} = 4 \left\lfloor \frac{x}{4} \right\rfloor + \frac{3n}{2}
\]

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1: \( \frac{n-1}{q}, \frac{2n-2}{q}, \frac{3n-3}{q} \) are integers

\[
\left\lfloor \frac{x}{q} \right\rfloor + \left\lfloor \frac{x+1}{q} \right\rfloor + \frac{n-1}{q} + \left\lfloor \frac{x+2}{q} \right\rfloor + \frac{2n-2}{q} + \left\lfloor \frac{x+3}{q} \right\rfloor + \frac{3n-3}{q} = Lx + \frac{3n}{2} - \frac{3}{2}
\]

3: \( \frac{n+2}{q}, \frac{2n+3}{q}, \frac{3n+3}{q} \) are integers

\[
\left\lfloor \frac{x}{q} \right\rfloor + \left\lfloor \frac{x-1}{q} \right\rfloor + \frac{n+1}{q} + \left\lfloor \frac{x-2}{q} \right\rfloor + \frac{2n+2}{q} + \left\lfloor \frac{x-3}{q} \right\rfloor + \frac{3n+3}{q} =
\]

\[
\left\lfloor \frac{x}{q} \right\rfloor + \left\lfloor \frac{x-1}{q} \right\rfloor + \left\lfloor \frac{x-2}{q} \right\rfloor + \left\lfloor \frac{x-3}{q} \right\rfloor + \frac{3n}{2} + \frac{3}{2}
\]

But this is shifted by 4

\[
Lx + 3, \quad Lx + \frac{3n}{2} - \frac{3}{2}.
\]

2: \( \frac{n-2}{q}, \frac{2n-1}{q}, \frac{3n-2}{q} \) are integers

\[
\left\lfloor \frac{x}{q} \right\rfloor + \left\lfloor \frac{x+2}{q} \right\rfloor + \frac{n-2}{q} + \left\lfloor \frac{x}{q} \right\rfloor + \frac{2n}{q} + \left\lfloor \frac{x+2}{q} \right\rfloor + \frac{3n-2}{q} =
\]

\[
2 \left( \left\lfloor \frac{x}{q} \right\rfloor + \left\lfloor \frac{x+2}{q} \right\rfloor \right) + \frac{3n}{2} - 1 = 2 \left\lfloor \frac{x}{q} \right\rfloor + \frac{4n}{2} - 1
\]

\[
\begin{array}{c|c|c|c|c|c}
\text{mod } n & 0 & 1 & 2 & 3 & 4 \\
\hline
1 & Lx & - & - & - & - \\
2 & 2Lx + \frac{3}{2} & Lx + \frac{n}{2} - \frac{1}{2} & - & - & - \\
3 & 3Lx + n & Lx + n - 1 & Lx + n - 1 & - & - \\
4 & 4Lx + \frac{3n}{2} & Lx + \frac{3n}{2} - \frac{3}{2} & 2Lx + \frac{3n}{2} - 1 & Lx + \frac{3n}{2} - \frac{3}{2} & - \\
\end{array}
\]
Seems to be of the form
\[ a \left\lfloor \frac{x}{a} \right\rfloor + bn + c \]
where \( b = \frac{m-1}{2} \) is a good guess.

\( a \) seems to be \( \gcd(m,n) \).

We have always rewritten as \( \left\lfloor \frac{x + kn}{m} \right\rfloor = \left\lfloor \frac{x}{m} \right\rfloor + \frac{kn}{m} - \frac{kn \mod m}{m} \)

because \( \frac{kn - kn \mod m}{m} \) is an integer:

\[
\left\lfloor \frac{x}{m} \right\rfloor + \frac{0}{m} - \frac{0 \mod m}{m}
\]

\[
+ \left\lfloor \frac{x + n \mod m}{m} \right\rfloor + \frac{1}{m} - \frac{n \mod m}{m}
\]

\[
+ \left\lfloor \frac{x + 2n \mod m}{m} \right\rfloor + \frac{2}{m} - \frac{2n \mod m}{m}
\]

\[
\vdots
\]

\[
+ \left\lfloor \frac{x + (m-1)n \mod m}{m} \right\rfloor + \frac{(m-1)n}{m} - \frac{(m-1)n \mod m}{m}
\]

\( b \) is from
\[
\frac{1}{m} \sum_{i=1}^{m-1} i = \frac{n}{m} \frac{(m-1)m}{2} = n \cdot \frac{m-1}{2} \Rightarrow b.
\]

To find an \( c \) we must consider the sequence
\( 0 \mod m, n \mod m, 2n \mod m, \ldots, (m-1)n \mod m \).
Look at $m=12$, $n=5$ (time on a clock)

$0, 5, 10, 3, 8, 1, 6, 11, 4, 9, 2, 7$

(full period)

$m=12$, $n=8$

$0, 8, 4, 0$ (short period)

Note $\gcd(12, 5) = 1$ but $\gcd(12, 4) = 4$

So we get period of length $\frac{12}{\gcd(12, 4)}$, and we will see we get $0, d, 2d, \ldots, m-d$

in some order followed by $d-1$ more copies with

d = \gcd(m, n)$. Thus the 1st column contains

d copies of $\lfloor \frac{x}{m} \rfloor, \lfloor \frac{x+d}{m} \rfloor, \ldots, \lfloor \frac{x+m-d}{m} \rfloor$ so

it sums to

$$d \left( \lfloor \frac{x}{m} \rfloor + \lfloor \frac{x+d}{m} \rfloor + \lfloor \frac{x+2d}{m} \rfloor + \ldots + \lfloor \frac{x+m-d}{m} \rfloor \right)$$

$$= d \left( \lfloor \frac{x}{m/d} \rfloor + \lfloor \frac{x+d}{m/d} \rfloor + \ldots + \lfloor \frac{x+d+m/d-1}{m/d} \rfloor \right)$$

$$= d \lfloor \frac{x}{d} \rfloor$$

so $a = \gcd(m, n)$.

But what about $c$? We can now compute

the last column: it is $d$ copies of $\frac{0}{m}, \frac{1}{m}, \ldots, \frac{m/d-1}{m}$
This is \( \frac{m}{m} + \frac{1}{m} + \ldots + \frac{m-d}{m} = \)

\[
\frac{1}{2} \left( \frac{m}{m} + \frac{m-d}{m} \right) \cdot \frac{m}{d} = \frac{m-d}{2d}
\]

\[
\uparrow \quad \uparrow \quad \# \text{ of terms}
\]

so \( d \cdot \frac{m-d}{z} = \frac{m-d}{2} = -C \).

\[
\sum \left\lfloor \frac{nk+x}{m} \right\rfloor = d \left\lfloor \frac{x}{d} \right\rfloor + \frac{(m-1)n}{2} + \frac{d-m}{2}
\]

\( \leq k < m \)

\( d = \text{gcd} (m, n) \)

\[
= d \left\lfloor \frac{x}{d} \right\rfloor + \frac{(m-1)(n-1)}{2} + \frac{m-1}{2} + \frac{d-m}{2}
\]

\[
= d \left\lfloor \frac{x}{d} \right\rfloor + \frac{(n-1)(n-1)}{2} + \frac{d-1}{2}
\]

thus \( m \) and \( n \) can be interchanged and so

\[
\sum \left\lfloor \frac{nk+x}{m} \right\rfloor = \sum \left\lfloor \frac{mk+x}{n} \right\rfloor
\]

\( \leq k < m \)

\( \forall \ m, n \in \mathbb{Z}^+ \)

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