A Useful Theorem

We have studied the SVD and the Hermitian eigenvalue problem. A very important result for the nonhermitian eigenvalue problem is the Schur decomposition.1

**Theorem** For any $A \in \mathbb{C}^{n \times n}$ there exist $Q \in \mathbb{C}^{n \times n}$ and $U \in \mathbb{C}^{n \times n}$ where $Q^H Q = QQ^H = I$ and $U$ is upper triangular whose diagonal elements $\lambda_1, \cdots, \lambda_n$ are the eigenvalues of $A$ such that

$$Q^H A Q = U$$

**Problem 1**

Prove that the eigenvalues of a unitary matrix $M$ are complex numbers on the unit circle, i.e., $\lambda = e^{i\theta}$.

**Problem 2**

A matrix $A \in \mathbb{C}^{n \times n}$ is **normal** if $AA^H = A^H A$.

2(a) Prove that an Hermitian matrix is normal.

2(b) Prove the following

A matrix is normal if and only if there exist $Q \in \mathbb{C}^{n \times n}$ and $\Lambda \in \mathbb{C}^{n \times n}$ where $Q^H Q = QQ^H = I$ and $\Lambda$ is a diagonal matrix with elements $\lambda_1, \cdots, \lambda_n$ such that

$$A = Q \Lambda Q^H$$

2(c) Let $A \in \mathbb{C}^{n \times n}$ be a Hermitian matrix. Suppose $A = U \Sigma V^H$ is the SVD of $A$ and $A = Q \Lambda Q^H$ is its eigendecomposition. Given the eigendecomposition can you easily construct the SVD?

**Problem 3**

Consider the following nonhermitian Toeplitz matrix

$$T = \begin{pmatrix} 1 & b & c & d \\ e & 1 & b & c \\ f & e & 1 & b \\ g & f & e & 1 \end{pmatrix}$$

3(a) Show that $T$ has a displacement with rank 2 relative to

$$Z = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

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1The name Schur appears on many diverse results in linear algebra. Do not confuse them. We have seen it twice – the Schur complement, the Schur algorithm for Toeplitz system.
Diagonal or symmetric pivoting is a strategy that maintains symmetry and can be used in some cases for Cholesky-like factorization of symmetric indefinite matrices. The pivot element is chosen from the diagonal. So if you wanted to pivot the element in position \((i, i)\) to position \((1, 1)\) the elementary permutation matrix \(P\) would be defined by swapping row 1 and row \(i\) in the identity matrix and the pivoted matrix would be \(PAP^T\).

Suppose we want to use this technique for symmetric Toeplitz matrices. Does diagonal pivoting preserve the low-rank displacement form of \(T\)?