This homework will be collected in class on Thursday 1st November. On Wednesday 7th November, solutions will be reviewed and handed out.

1. We have seen that if language $L$ is recursive then so is $\overline{L}$. This problem has to do with the closure properties of r.e. languages and recursive languages.

Tell whether (a) r.e. languages, and (b) recursive languages, are closed under the following operations:

- **union**
  - Suppose that the closure property refers to the languages $L_1 = L(M_1)$ and $L_2 = L(M_2)$. In the first case the closure is $L = L_1 \cup L_2$.
  - Consider the TM, $M$ that accepts $w \in L$ iff, either $M_1$ or $M_2$ accepts $w$. Then $L = L(M)$, so $L$ is r.e. Furthermore, if $M_1, M_2$ halt on all inputs, then so does $M$: so $L$ is recursive if $L_1, L_2$ are recursive.

- **intersection**
  - Use same argument, only this time $M$ accepts $w \in L = L_1 \cap L_2$ iff, both $M_1$ and $M_2$ accept $w$. Then $L = L(M)$, so $L$ is r.e. Furthermore, if $M_1, M_2$ halt on all inputs, then so does $M$: so $L$ is recursive if $L_1, L_2$ are recursive.

- **concatenation**
  - Here $w \in L$ iff $w = w_1 w_2$ with $w_1 \in L_1, w_2 \in L_2$. For this case consider a non-deterministic $M$ which on input $w$ guesses $w_1, w_2$ with $w = w_1 w_2$, and accepts $w$ iff both $w_1 \in L_1$ and $w_2 \in L_2$.
  - Again $L = L(M)$, so $L$ is r.e. Furthermore, if $M_1, M_2$ halt on all inputs, then so does $M$: so $L$ is recursive if $L_1, L_2$ are recursive.

You may give informal but clear constructions to show closure.

2. Informally define multi-tape TMs that enumerate the following sets of integers (in the sense that starting with blank tapes they print out on their tapes $10^6 10^6 1 \ldots$, to represent the set $i_1, i_2, \ldots$).

- The set of perfect squares: $\{1, 4, 9, \ldots\}$.
  - Use a multi-tape TM. On input any string $\neq \varepsilon$; on the first tape list all the integers as: $0^1, 0^2, 0^3, \ldots$; for the second tape square these numbers, and list the squares: $0^1, 0^4, 0^9, \ldots$. For the output use “1” as a separator.
- The set of all primes: $\{2, 3, 5, \ldots\}$.
  - Again use a multi-tape TM, with the difference in this case that for the second tape, a non-deterministic subprocedure is used which tests the numbers $p$ on the first tape for primality, by exhaustively checking all numbers $x$: $1 < x < p$, to see if they are divisors. If none are divisors, and only then, $p$ is copied on the second tape.

3. **The halting problem**: given the pair $(M, w)$, where $M$ is a TM and $w \in \Sigma^*$ is a string, does $M$ halt on input $w$?

Let $L_h$ be the language of the halting problem. Show that $L_h$ is r.e. but not recursive.

(Hint: The universal language $L_u$ is r.e. but not recursive. Mimic the proof.)

$L_h$ is r.e.: Consider a TM, $\widehat{M}$ that on input $(M, w)$ simulates $M$ on input $w$. $\widehat{M}$ halts iff $M$ halts. Then $L(\widehat{M}) = L_h$.

$L_h$ is not recursive: Reduce $\overline{L_d}$ to $L_h$ (we know that $\overline{L_d} = \{w \in (0+1)^* \mid w = \langle M \rangle$ is accepted by $M\}$ is not recursive). For the reduction map $w$ to $(M, w)$ with $w = \langle M \rangle$. Then $w \in \overline{L_d}$ iff $(M, w)$ in $L_h$. 

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4. Show that the following properties of r.e. sets are not decidable. Justify your answers.
   (a) \( L = \emptyset \).
   (b) \( L \) is recursive.
   (c) \( T \) is finite.

_Hint: Apply Rice’s theorem for recursive sets._

_Theorem 1._ Any non-trivial property of r.e. sets is undecidable. (\( \mathcal{P} \) is a trivial property of r.e. sets if it is either empty or consists of all r.e. languages)

Property \( \mathcal{P} = \{\emptyset\} \) is not empty (\( \mathcal{P} \) is a singleton). Similarly for the other two. So in all three cases we have non-decidability.

5. Show that the following properties of r.e. sets are r.e. Justify your answers.
   (a) \( L \neq \emptyset \)
   (b) \( L \) contains at least 2 words.
   (c) \( L \) contains some fixed word \( w \).

_Hint: Apply Rice’s theorem for r.e. sets._

_Theorem 2._ Let \( L_\mathcal{P} = \{L \mid L \text{ has property } \mathcal{P}\} \). Then \( L_\mathcal{P} \) is r.e. if and only if,

- If \( L \) has property \( \mathcal{P} \) and \( L \subseteq L' \) then \( L' \) has property \( \mathcal{P} \).
- If \( L \) is an infinite set of \( \mathcal{P} \) then there exists a finite subset \( L' \) of \( L \) that has property \( \mathcal{P} \).
- The set of finite languages with property \( \mathcal{P} \) is enumerable.

All three conditions of Rice’s theorem are satisfied for (a), (b) and (c).