An Introduction to Brownian Motion, Wiener Measure, and Partial Differential Equations

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Outline of the Lectures

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  Donsker’s Invariance Principal
  Properties of Brownian Motion

The Feynman-Kac Formula
  Explicit Representation of Brownian Motion
  The Karhunen-Loève Expansion
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Introduction to Brownian Motion

- Let \( \Omega = \{ \beta \in C[0, 1]; \beta(0) = 0 \} = C_0[0, 1] \), be an infinitely dimensional space we consider for placing a probability measure.

- Consider \((\Omega, \mathcal{B}, P)\), where \(\mathcal{B}\) is the set of measurable subsets (a \(\sigma\)-algebra) and \(P\) is the probability measure on \(\Omega\).

- We would like to answer questions like \( P \left[ \int_0^1 \beta^2(s)ds \leq \alpha \right] \)?

- We now construct Brownian motion (BM) via some limit ideas.

- **Central Limit Theorem (CLT):** let \(X_1, X_2, \ldots\) be independent, identically distributed (i.i.d.) with \(E[X_i] = 0, \ Var[X_i] = 1\) and define \(S_n = \sum_{i=1}^n X_i\)
  1. Note if \(X_1^*, X_2^*, \ldots\) are i.i.d. with \(E[X_i^*] = \mu, \ Var[X_i^*] = \sigma^2 < \infty\), then \(X_i = \frac{X_i^* - \mu}{\sigma}\) has \(E[X_i] = 0, \ Var[X_i] = 1\)
  2. Then \(\frac{S_n}{\sqrt{n}}\) converges in distribution to \(N(0, 1)\) as \(n \to \infty\).
Introduction to Brownian Motion

- Let $X_1, X_2, \ldots$ be as before, then it follows from the CLT that

$$\lim_{n \to \infty} P \left[ \frac{S_n}{\sqrt{n}} \leq \alpha \right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\alpha} e^{-\frac{u^2}{2}} du.$$

- Erdös and Kac proved (we will find the $\sigma_i(\cdot)$’s):

1. $\lim_{n \to \infty} P \left[ \max \left( \frac{S_1}{\sqrt{n}}, \frac{S_2}{\sqrt{n}}, \ldots, \frac{S_n}{\sqrt{n}} \right) \leq \alpha \right] = \sigma_1(\alpha) = \sqrt{\frac{2}{\pi}} \int_{0}^{\alpha} e^{-\frac{u^2}{2}} du$

2. $\lim_{n \to \infty} P \left[ \frac{S_1^2 + S_2^2 + \cdots + S_n^2}{n} \leq \alpha \right] = \sigma_2(\alpha)$

3. $\lim_{n \to \infty} P \left[ \frac{S_1 + S_2 + \cdots + S_n}{n^{3/2}} \leq \alpha \right] = \sigma_3(\alpha)$

- Let $N_n = \#\{S_1, \ldots, S_n | S_i > 0\}$, then

$$\lim_{n \to \infty} P \left[ \frac{N_n}{n} \leq \alpha \right] = \begin{cases} 0, & \text{if } \alpha \leq 0 \\ \frac{2}{\pi} \arcsin \sqrt{\alpha}, & \text{if } 0 \leq \alpha \leq 1 \\ 1, & \text{if } \alpha \geq 1 \end{cases}$$
Definitions

- $X_1, X_2, \ldots$ are as above, and $\forall n \in \mathbb{N}$ and $t \in [0, 1]$ define

$$\chi^{(n)}(t) = \begin{cases} 
\frac{S_1}{\sqrt{n}}, & t = 0 \\
\frac{S_i}{\sqrt{n}}, & \frac{i-1}{n} < t \leq \frac{i}{n}, \quad i = 1, 2, \ldots, n
\end{cases}$$

- Let $\mathcal{R}$ denote the space of Riemann integrable functions on $[0, 1]$.
- **Theorem**: $F : \mathcal{R} \rightarrow \mathbb{R}$ and with some weak hypotheses, then

$$\lim_{n \rightarrow \infty} P \left[ F \left( \chi^{(n)}(\cdot) \right) \leq \alpha \right] = P_W \left[ F(\beta) \leq \alpha \right],$$

where $P_W$ denotes the probability called “Wiener measure,” and this result is called Donsker’s Invariance Principal.
Examples of Donsker’s Invariance Principal

1. \( F[\beta] = \int_0^1 \beta^2(s) \, ds \), then by the theorem
   \[
   \lim_{n \to \infty} P \left[ \sum_{i=1}^{n} \frac{S_i^2}{n^2} \leq \alpha \right] = P_W \left[ \int_0^1 \beta^2(s) \, ds \leq \alpha \right]
   \]

2. \( F[\beta] = \beta(1) \), then
   \[
   \lim_{n \to \infty} P \left[ \frac{S_n}{\sqrt{n}} \leq \alpha \right] = P_W [\beta(1) \leq \alpha] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\alpha} e^{-\frac{u^2}{2}} \, du
   \]

3. \( F[\beta] = \int_0^1 \frac{1 + \text{sgn} \beta(s)}{2} \, ds \), where \( \text{sgn}(x) = \begin{cases} 1, & x > 0 \\ -1, & x \leq 0 \end{cases} \)
   Then
   \[
   \lim_{n \to \infty} P \left[ \frac{N_n}{n} \leq \alpha \right] = P_W \left[ \int_0^1 \frac{1 + \text{sgn} \beta(s)}{2} \, ds \leq \alpha \right]
   \]
For any integer $n$, any choice of $0 < \tau_1 < \cdots < \tau_n \leq 1$, and any Lebesgue measurable ($\mathcal{L}$-mb) set, $E \in \mathbb{R}^n$ define the "interval"

$$I = I(n; \tau_1; \ldots; \tau_n; E) := \{\beta(\cdot) \in C_0[0, 1]; (\beta(\tau_1), \ldots, \beta(\tau_n)) \in E\}$$

Let $\mathcal{A}$ be the class of intervals containing all the $I$ for all $n, \tau_1, \ldots, \tau_n$ and all $\mathcal{L}$–mb sets $E \in \mathbb{R}^n$, then $\mathcal{A}$ is an algebra of sets in $C_0[0, 1]$

The $I$'s are the cylinder sets upon which we will define Wiener measure, and then standard measure theoretic ideas to extend to all measurable subsets of the infinite dimensional space, $C_0[0, 1]$
Given \( I \), we define its measure as

\[
\mu(I) = \frac{1}{\sqrt{(2\pi)^n \tau_1 (\tau_2 - \tau_1) \cdots (\tau_n - \tau_{n-1})}} \int \cdots \int_{E} e^{-\frac{u_1^2}{2\tau_1} - \frac{(u_2-u_1)^2}{2(\tau_2 - \tau_1)} - \cdots - \frac{(u_n-u_{n-1})^2}{2(\tau_n - \tau_{n-1})}} \, du_1 \cdots du_n.
\]

Let \( B \) be the smallest \( \sigma \)-algebra generated by \( A \), this is the class of Wiener measurable (W-mb) sets in \( C_0[0,1] \).

This extension of Wiener measure, also creates a probability measure on \( C_0[0,1] \), and expectation w.r.t. Wiener measure will be referred to as a

1. Wiener integral or Wiener integration
2. Brownian motion expectation
Examples

- Let $A \in \mathbb{R}^{n \times n}$ with $A_{ij} = \min(\tau_i, \tau_j)$, i.e. for the case $n = 3$, $\tau_1 < \tau_2 < \tau_3$ we have

$$A = \begin{pmatrix}
\tau_1 & \tau_1 & \tau_1 \\
\tau_1 & \tau_2 & \tau_2 \\
\tau_1 & \tau_2 & \tau_3
\end{pmatrix}$$

and in general we can write $U = (u_1, \ldots, u_n)^\top$ and

$$\mu(I) = \frac{1}{\sqrt{(2\pi)^n \det A}} \int_E \cdots \int e^{-U^\top A^{-1}U} \, du_1 \ldots du_n$$

- Let $\beta(\cdot)$ be a BM, and $0 < \tau_1 < \tau_2 < 1$, then

$$P[a_1 \leq \beta(\tau_1) \leq b_1] = \frac{1}{2\pi \tau_1} \int_{a_1}^{b_1} e^{-\frac{u^2}{2\tau_1}} \, du$$

and

$$P[a_1 \leq \beta(\tau_1) \leq b_1 \cap a_2 \leq \beta(\tau_2) \leq b_2]$$

$$= \frac{1}{\sqrt{(2\pi)^2 \tau_1 (\tau_2 - \tau_1)}} \int_{a_2}^{b_2} \int_{a_1}^{b_1} e^{-\frac{u_2^2}{2\tau_2} - \frac{(u_2-u_1)^2}{2(\tau_2-\tau_1)}} \, du_1 \, du_2$$
Useful Properties of Brownian Motion

Theorem: Let \( I = \bigcup_{j=1}^{\infty} I_j \) where \( I_j \cap I_k = \emptyset \forall i \neq k \) and \( I, I_1, I_2, \ldots \in \mathcal{A} \), then \( \mu(I) = \sum_{j=1}^{\infty} \mu(I_j) \)

we will see that the BM, \( \beta(t) \), satisfies:

1. Almost every (AE) path is non-differentiable at every point
2. AE path satisfies a Hölder condition of order \( \alpha < \frac{1}{2} \), i.e.
   \[
   |\beta(s) - \beta(t)| \leq L|s - t|^\alpha
   \]
3. \( E[\beta(t)] = 0 \)
4. \( E[\beta^2(t)] = t \), and so \( \beta(t) \sim N(0, t) \)
5. \( \beta(0) = 0, \beta(t) - \beta(s) \sim N(0, t-s) \)
6. \( E[\beta(t)\beta(s)] = \min(s, t) \)
Brownian Motion

Introduction to Brownian Motion as a Measure

Properties of Brownian Motion

Useful Properties of Brownian Motion

- Let $E \in \mathbb{R}^n (\mathcal{L} - mb)$, $0 < \tau_1 < \cdots < \tau_n < 1$, $l = l(n; \tau_1; \ldots; \tau_n; E)$, then

$$\mu(l) = \int \cdots \int_E p(\tau_1, 0, u_1)p(\tau_2 - \tau_1, u_1, u_2) \cdots p(\tau_N - \tau_{n-1}, u_n, u_{n-1}) \, du_1 \cdots du_n$$

where $p(t, x, y) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{(x-y)^2}{2t}}$

- Note that $p(t, x, y) = \psi(t, x, y)$, the fundamental solution for the initial value problem for the heat/diffusion equation

$$\psi_t = \frac{1}{2} \psi_{yy}, \quad \psi(0, x, y) = \delta(y - x)$$

- $\mu$ is finitely additive since integrals are additive set functions
Useful Properties of Brownian Motion

- **Theorem 1**: Let $a > 0$, $0 < \gamma < \frac{1}{2}$ and define

\[
A_{a,\gamma} = \{ \beta \in C_0[0, 1]; |\beta(\tau_2) - \beta(\tau_1)| \leq a|\tau_2 - \tau_1|^{\gamma} \forall \tau_1, \tau_2 \in [0, 1] \}
\]

For any interval $I \subset C_0[0, 1]$ s.t. $I \cap A_{a,\gamma} = \emptyset$ there is a $K$ independent of $a$ for which

\[
m(I) < Ka^{-\frac{4}{1-2\gamma}}
\]

- **Remark**: $A_{a,\gamma}$ is a compact set in $C_0[0, 1]$ and eventually one can prove that AE $\beta \in C_0[0, 1]$ satisfy some Hölder condition

- **Theorem 2**: $\mu$ is countably additive on $A$, i.e. if $I_n \in A$, $n \in \mathbb{N}$ disjoint ($I_j \cap I_k = \emptyset, j \neq k$) then

\[
l = \bigcup_{n=1}^{\infty} I_n \in A \Rightarrow \mu(l) = \sum_{n=1}^{\infty} \mu(I_n)
\]
Useful Properties of Brownian Motion

▶ Suppose $F : C_0[0, 1] \to \mathbb{R}$ is a measurable functional, i.e. $\{\beta \in C_0[0, 1]; F[\beta] \leq \alpha \}$ is measurable $\forall \alpha$

▶ We can consider

$$E[F] = E_W[F[\beta(\cdot)]] = \int F[\beta(\cdot)]\delta_W, \text{ a Wiener integral}$$

▶ Consider $C_x[0, t] = \{f \in C[0, t]; f(0) = x\}$, then

$$P[\beta(0) = x, \beta(t) \in A] = \frac{1}{\sqrt{2\pi t}} \int_A e^{-\frac{(y-x)^2}{2t}} dy$$

▶ Furthermore

$$E[\beta(\tau)] = \frac{1}{\sqrt{2\pi \tau}} \int_{-\infty}^{\infty} u e^{-\frac{u^2}{2\tau}} du = 0, \forall \tau > 0$$

$$E[g(\beta(\tau_1), \ldots, \beta(\tau_n))] = \frac{1}{\sqrt{(2\pi)^n \tau_1 (\tau_2 - \tau_1) \cdots (\tau_n - \tau_{n-1})}} \times$$

$$\int \cdots \int g(u_1, \ldots, u_n) e^{-\frac{u_1^2}{2\tau_1}} - \frac{(u_2-u_1)^2}{2(\tau_2-\tau_1)} - \cdots - \frac{(u_n-u_{n-1})^2}{2(\tau_n-\tau_{n-1})} du_1 \cdots du_n$$
Let us now consider, without proof, a large deviation result for BM:

**Theorem (The Law of the Iterated Logarithm for BM):** Let $\beta(s) \in C_0[0, \infty)$ be ordinary Brownian Motion, then

(1) \[ P \left( \limsup_{t \to \infty} \frac{\beta(t)}{\sqrt{2t \ln \ln t}} = 1 \right) = 1 \]

(2) \[ P \left( \liminf_{t \to \infty} \frac{\beta(t)}{\sqrt{2t \ln \ln t}} = -1 \right) = 1 \]
Dirac Delta Function

- Let $g$ be Borel measurable (B-mb), then
  \[ E[g(\beta(\tau))] = \frac{1}{\sqrt{2\pi \tau}} \int_{-\infty}^{\infty} g(u) e^{-\frac{u^2}{2\tau}} \, du \]

- Let $g(u) = \delta(u - x)$, using the Dirac delta function, then
  \[ E[\delta(\beta(t) - x)] = \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} \delta(u - x) e^{-\frac{u^2}{2t}} \, du = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} \]
  thus $u(x, t) = E[\delta(\beta(t) - x)] = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}$ is the fundamental solution of the heat equation
  \[ u_t = \frac{1}{2} u_{xx}, \quad u(x, 0) = \delta(x) \]
Consider now $V(x) \geq 0$ continuous and consider the equation

$$u_t = \frac{1}{2} u_{xx} - V(x) u, \quad u(x, 0) = \delta(x),$$

then we can write

$$u(x, t) = E \left[ e^{-\int_0^t V(\beta(s)) \, ds} \delta(\beta(t) - x) \right]$$

This is the Feynman-Kac formula

Example:

$$V(x) = \frac{x^2}{2}, \quad u_t = \frac{1}{2} u_{xx} - \frac{x^2}{2} u, \quad u(x, 0) = \delta(x), \quad \text{then}$$

$$u(x, t) = E \left[ e^{-\frac{1}{2} \int_0^t \beta^2(s) \, ds} \delta(\beta(t) - x) \right]$$
The Feynman-Kac Formula

- The following is clearly true:

\[
P[\beta(\tau) \leq x] = P\left(\{\beta \in C_0[0, \tau]; \beta(\tau) \in E = (-\infty, x]\}\right) = \frac{1}{\sqrt{2\pi \tau}} \int_{-\infty}^{x} e^{-\frac{u^2}{2\tau}} \, du, \text{ and similarly}
\]

With \(0 = \tau_0 \leq \tau_1 \cdots \leq \tau_n\) we have

\[
P[\beta(\tau_1) \leq x_1, \ldots, \beta(\tau_n) \leq x_n] = \frac{(2\pi)^{-n/2}}{\sqrt{(\tau_1 - \tau_0)(\tau_2 - \tau_1) \cdots (\tau_n - \tau_{n-1})}} \times
\]
\[
\int_{-\infty}^{\tau_n} \cdots \int_{-\infty}^{\tau_1} e^{-\frac{u_1^2}{2\tau_1} - \frac{(u_2-u_1)^2}{2(\tau_2-\tau_1)} - \cdots - \frac{(u_n-u_{n-1})^2}{2(\tau_n-\tau_{n-1})}} \, du_1 \cdots du_n
\]

- Hence with \(A_{ij} = \min(\tau_i, \tau_j)\)

\[
E\left[g(\beta(\tau_1), \ldots, \beta(\tau_n))\right] = \frac{1}{\sqrt{(2\pi)^n |A|}} \times
\]
\[
\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} g(u_1, \ldots, u_n) e^{-\frac{1}{2} u^\top A^{-1} u} \, du_1 \cdots du_n
\]
Feynman-Kac Formula: Derivation

Let us consider the Wiener integral below, where expectation is taken over all of $C_0[0, t]$

$$E \left\{ e^{-\int_0^t V(\beta(\tau)) \, d\tau} \right\}$$

We will show that this is equal to the solution of the Bloch equation using an elementary proof of Kac

We assume that $0 \leq V(x) < M$ is bounded from above and non-negative; however, the upper bound will be relaxed

We know

$$e^{-\int_0^t V(\beta(\tau)) \, d\tau} = \sum_{k=0}^{\infty} (-1)^k \left[ \int_0^t V(\beta(\tau)) \, d\tau \right]^k / k!$$

Since $V(\cdot)$ is bounded we also have

$$0 < \int_0^t V(\beta(\tau)) \, d\tau < Mt$$

This allows us to use Fubini’s theorem as follows

$$E \left\{ e^{-\int_0^t V(\beta(\tau)) \, d\tau} \right\} = \sum_{k=0}^{\infty} (-1)^k E \left\{ \left[ \int_0^t V(\beta(\tau)) \, d\tau \right]^k \right\} / k!$$
Feynman-Kac Formula: Derivation

▶ Now let us consider the moments

\[ \mu_k(t) = E \left\{ \left[ \int_0^t V(\beta(\tau)) \, d\tau \right]^k \right\} \]

▶ Consider first \( k = 1 \)

\[
E \left\{ \int_0^t V(\beta(\tau)) \, d\tau \right\} \overset{\text{Fubini}}{=} \int_0^t E \{ V(\beta(\tau)) \} \, d\tau = \int_0^t \int_{-\infty}^{\infty} V(\xi) \frac{1}{\sqrt{2\pi\tau}} e^{-\frac{\xi^2}{2\tau}} \, d\xi \, d\tau
\]

▶ The case \( k = 2 \) is a bit more complicated

\[
E \left\{ \left[ \int_0^t V(\beta(\tau)) \, d\tau \right]^2 \right\} = 2! \int_0^t \int_0^{\tau_2} E \{ V(\beta(\tau_1)) V(\beta(\tau_2)) \} \, d\tau_1 \, d\tau_2 \overset{\text{Fubini}}{=} 2! \int_0^t \int_0^{\tau_2} E \{ V(\beta(\tau_1)) V(\beta(\tau_2)) \} \, d\tau_1 \, d\tau_2 =
\]

\[
2! \int_0^t \int_0^{\tau_2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} V(\xi_1) V(\xi_2) \frac{e^{-\frac{\xi_1^2}{2\tau_1}}}{\sqrt{2\pi\tau_1}} \frac{e^{-\frac{(\xi_2-\xi_1)^2}{2(\tau_2-\tau_1)}}}{\sqrt{2\pi(\tau_2-\tau_1)}} \, d\xi_1 \, d\xi_2 \, d\tau_1 \, d\tau_2
\]
Feynman-Kac Formula: Derivation

- For general $k$ we will proceed by defining the function $Q_n(x, t)$ as follows
  1. $Q_0(x, t) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}$
  2. $Q_{n+1}(x, t) = \int_0^t \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi(\tau-t)}} e^{-\frac{(x-\xi)^2}{2(\tau-t)}} V(\xi)Q_n(\xi, \tau) d\xi d\tau$

- We have that $\mu_k(t) = k! \int_0^t Q_k(x, t) dx$

- By the boundedness of $V(\cdot)$ we also have, by induction, that $0 \leq Q_n(x, t) \leq \frac{(Mt)^n}{n!} Q_0(x, t)$

- Now define $Q(x, t) = \sum_{k=0}^{\infty} (-1)^k Q_k(x, t)$

- This series converges for all $x$ and $t \neq 0$ and $|Q(x, t)| < e^{Mt} Q_0(x, t)$

- One can easily check that the definitions of the $Q_k(x, t)$’s ensures that $Q(x, t)$ satisfies the following integral equation

$$Q(x, t) + \frac{1}{\sqrt{2\pi}} \int_0^t \int_{-\infty}^{\infty} \frac{1}{\sqrt{(t-\tau)}} e^{-\frac{(x-\xi)^2}{2(t-\tau)}} V(\xi)Q(\xi, \tau) d\xi d\tau = Q_0(x, t)$$
Feynman-Kac Formula: Derivation

- It also follows that

\[
E \left\{ e^{-\int_0^t V(\beta(\tau)) \, d\tau} \right\} = \int_{-\infty}^{\infty} Q(x, t) \, dx
\]

- Recall that his Wiener integral is over all of \( C_0[0, t] \), let us restrict this only to \( a < \beta(t) < b \), thus

\[
E \left\{ e^{-\int_0^t V(\beta(\tau)) \, d\tau} ; a < \beta(t) < b \right\} = \int_a^b Q(x, t) \, dx
\]

- This tells us immediately that \( Q(x, t) \geq 0 \)

- Now we will relax the upper bound on \( V(\cdot) \) by considering the function

\[
V_M(x) = \begin{cases} 
V(x), & \text{if } V(x) \leq M \\
M, & \text{if } V(x) \geq M
\end{cases}
\]

and we denote \( Q^{(M)}(x, t) \) as the respective “\( Q \)” function.
By the additivity of Wiener measure we have that
\[
\lim_{M \to \infty} E \left\{ e^{-\int_0^t V_M(\beta(\tau)) \, d\tau}; \ a < \beta(t) < b \right\} = E \left\{ e^{-\int_0^t V(\beta(\tau)) \, d\tau}; \ a < \beta(t) < b \right\}
\]

Furthermore, as \( M \to \infty \) the functions \( Q^{(M)}(x, t) \) form a decreasing sequence with \( \lim_{M \to \infty} Q^{(M)}(x, t) = Q(x, t) \) existing with the resulting limiting function, \( Q(x, t) \) satisfying the (Bloch) equation
\[
\frac{\partial Q}{\partial t} = \frac{1}{2} \frac{\partial^2 Q}{\partial x^2} - V(x)Q
\]
with the initial condition \( Q(x, t) \to \delta(x) \) as \( t \to 0 \)
Feynman-Kac Formula: Derivation Variation

- Recall the integral equation solved by $Q(x, t)$
  \[
  Q(x, t) + \frac{1}{\sqrt{2\pi}} \int_0^t \int_{-\infty}^{\infty} \frac{1}{\sqrt{(t - \tau)}} e^{-\frac{(x - \xi)^2}{2(t - \tau)}} V(\xi) Q(\xi, \tau) d\xi d\tau = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}
  \]

- Let us define $\Psi(x) = \int_{-\infty}^{\infty} Q(x, t) e^{-st} dt$ with $s > 0$, this is the Laplace transform of $Q(x, t)$

- Now multiply the integral equation by $e^{-st}$ and integrate out $t$ to get the equation satisfied by the Laplace transform of $Q(x, t)$
  \[
  \Psi(x) + \frac{1}{\sqrt{2s}} \int_{-\infty}^{\infty} e^{-\sqrt{2s}|x - \xi|} V(\xi) \Psi(\xi) d\xi = \frac{1}{\sqrt{2s}} e^{-\sqrt{2s}|x|}
  \]

- It is easy to verify that $\Psi(x)$ also satisfies the following differential equation
  \[
  \frac{1}{2} \psi'' - (s + V(x)) \psi = 0, \text{ with the following conditions}
  \]

  1. $\psi \to 0$ as $|x| \to \infty$
  2. $\psi'$ is continuous except at $x = 0$
  3. $\psi'(-0) - \psi'(-0) = 2$
Suppose that $F[\beta] = \int_0^t \beta^2(s) \, ds$, then it follows

$$E \left[ \int_0^t \beta^2(s) \, ds \right] \overset{\text{Fubini}}{=} \int_0^t E \left[ \beta^2(s) \right] \, ds = \int_0^t s \, ds = \frac{t^2}{2}$$

To compute $E \left[ e^{\int_0^t \beta(s) \, ds} \right]$, we need to do some classical analysis.

Consider the eigenvalue problem for this integral equation

$$\rho \int_0^t u(s) \min(\tau, s) \, ds = u(\tau)$$

Find eigenvalues $\rho_0, \rho_1, \ldots$ and corresponding orthonormalized eigenfunctions $u_0(\tau), u_1(\tau), \ldots$ with $\int_0^t u_j(\tau) u_k(\tau) \, d\tau = \delta_{jk}, \forall j, k \geq 0$
For \( t > \tau \) we have

\[
\rho \int_0^\tau s u(s) \, ds + \rho \int_{\tau}^{t} \tau u(s) \, ds = u(\tau)
\]

\[
\frac{d}{d\tau} \rho \tau u(\tau) - \rho \tau u(\tau) + \rho \int_{\tau}^{t} u(s) \, ds = u'(\tau)
\]

Thus \( u''(\tau) + \rho u(\tau) = 0 \) and with \( u(0) = 0, \ u'(t) = 0 \) we get

\[
\rho_k = (k + \frac{1}{2})^2 \frac{\pi^2}{t^2}
\]

\[
u_k(s) = \sqrt{\frac{2}{t}} \sin \left((k + \frac{1}{2}) \frac{\pi s}{t}\right)
\]

\[
\left\{ \begin{array}{l}
k = 0, 1, 2, \ldots
\end{array} \right. \]

By the spectral theorem the integral equation kernel can be represented as:

\[
\min(s, \tau) = \sum_{k=0}^{\infty} \frac{u_k(s)u_k(\tau)}{\rho_k}
\]
Explicit Representation of Brownian Motion

Let \( \alpha_0(\omega), \alpha_1(\omega), \ldots \) be i.i.d. \( N(0, 1) \), then we claim that the following is an explicit representation of BM

\[
\sum_{k=0}^{\infty} \frac{\alpha_k(\omega) u_k(\tau)}{\sqrt{\rho_k}} = \beta(\tau)
\]  

(2.1)

This is a Fourier series with random coefficients and we will prove that this converges for AE path \( \omega \) with the following properties

1. We use \( \omega \) to denote an individual sample of i.i.d. \( N(0, 1) \) \( \alpha_i(\omega) \)'s
2. \( E[\alpha_i(\omega)] = 0, \forall i \geq 0 \)
3. \( E[\alpha_i(\omega)\alpha_j(\omega)] = \delta_{ij}, \forall i, j \geq 0 \)

This is the simplest version of the Karhunen-Loève expansion of stochastic processes
Explicit Representation of Brownian Motion (Proof)

We now use the representation (2.1) to compute some expectations w.r.t. the \( \alpha_i \)'s \( \sim N(0, 1) \)

\[
E \left[ \sum_{k=0}^{\infty} \frac{\alpha_k(\omega)}{\sqrt{\rho_k}} u_k(\tau) \right] \quad \text{i.i.d. } N(0,1) \text{ & Fubini}
\]

\[
\sum_{k=0}^{\infty} \frac{E[\alpha_k(\omega)] u_k(\tau)}{\sqrt{\rho_k}} = \sum_{k=0}^{\infty} 0 \times \frac{u_k(\tau)}{\sqrt{\rho_k}} = 0 = E[\beta(\tau)]
\]

We now use the representation (2.1) to compute some expectations

\[
E \left[ \sum_{k=0}^{\infty} \frac{\alpha_k(\omega)}{\sqrt{\rho_k}} u_k(\tau) \sum_{l=0}^{\infty} \frac{\alpha_l(\omega)}{\sqrt{\rho_l}} u_l(\tau) \right] \quad \text{i.i.d. } N(0,1)
\]

\[
\sum_{k=0}^{\infty} \frac{u_k^2(\tau)}{\rho_k} = \min(\tau, \tau) = \tau = E \left[ \beta^2(\tau) \right]
\]
Similarly we compute

\[ E \left[ \sum_{k=0}^{\infty} \frac{\alpha_k(\omega)}{\sqrt{\rho_k}} u_k(\tau) \sum_{l=0}^{\infty} \frac{\alpha_l(\omega)}{\sqrt{\rho_l}} u_l(s) \right] \stackrel{i.i.d.N(0,1)}{=} \]

\[ \sum_{k=0}^{\infty} \frac{u_k(\tau)u_k(s)}{\rho_k} = \min(\tau, s) = E[\beta(\tau)\beta(s)] \]

We have computed the mean, variance, and correlation of the process defined in (2.1), and it is clear that it is \( \sim N(0, \tau) \) and hence Brownian motion, \( \beta(\tau) \).
An Introduction to the Karhunen-Loève Expansion

- Karhunen-Loève (KL) expansion writes the stochastic processes \( Y(\omega, t) \) as a stochastic linear combination of a set of orthonormal, deterministic functions in \( L^2 \), \( \{ e_i(t) \}_{i=0}^{\infty} \)

\[
Y(\omega, t) = \sum_{i=0}^{\infty} Z_i(\omega) e_i(t)
\]

1. Given the covariance function of the random process \( Y(\omega, t) \) as \( C_{YY}(s, \tau) \) the KL expansion is

\[
Y(\omega, t) = \sum_{i=0}^{\infty} \sqrt{\lambda_i} \xi_i(\omega) \phi_i(t)
\]

2. Here \( \lambda_i \) and \( \phi_i(t) \) are the eigenvalues and \( L^2 \)-orthonormal eigenfunctions of the covariance function and \( \xi_i(\omega) \phi_i(t) \) are i.i.d. random variables whose distribution depends on \( Y(\omega, t) \), i.e. \( Z_i(\omega) = \sqrt{\lambda_i} \xi_i(\omega) \), and \( e_i(t) = \phi_i(t) \)

3. It can be shown that such an expansion converges to the stochastic process in \( L^2 \) (in distribution)
An Introduction to the Karhunen-Loève Expansion

4. By the spectral theorem, we can expand the covariance, thought of as an integral equation kernel, as follows

\[ C_{YY}(s, \tau) = \sum_{i=0}^{\infty} \lambda_i \phi_i(s) \phi_i(\tau) \]

5. Here \( \lambda_i \) and \( \phi_i(t) \) are the eigenvalues and eigenfunctions of the following integral equation

\[ \int_{0}^{\infty} C_{YY}(s, \tau) \phi_j(\tau) \, d\tau = \lambda_j \phi_j(s) \]

- For ordinary BM, \( Y(\omega, t) = \beta(t) \), we have from above

1. \( C_{YY}(s, \tau) = C_{\beta\beta}(s, \tau) = \min(s, \tau) \)
2. \( \lambda_j = \frac{1}{\rho_j} \), where \( \rho_j = (j + \frac{1}{2})^2 \frac{\pi^2}{s^2} \)
3. \( \phi_j(t) = u_j(t) = \sqrt{\frac{2}{s}} \sin((j + \frac{1}{2}) \frac{\pi t}{s}) \)
4. \( \xi_j(\omega) = \alpha_j(\omega) \sim \mathcal{N}(0, 1) \)
5. \( Y(\omega, t) = \sum_{j=0}^{\infty} \frac{\alpha_j(\omega) u_j(t)}{\sqrt{\rho_j}} = \beta(t) \)
Explicit Computation of Wiener Integrals

We are now in position to compute

\[
E \left[ e^{\int_0^t \beta(s) \, ds} \right] = E \left[ e^{\int_0^t \sum_{k=0}^{\infty} \frac{\alpha_k u_k(s)}{\sqrt{\rho_k}} \, ds} \right] = \\
E \left[ \prod_{k=0}^{\infty} e^{\frac{1}{2\rho_k} \left( \int_0^t u_k(s) \, ds \right)^2} \right] \quad \text{indep.} \quad \prod_{k=0}^{\infty} E \left[ e^{\frac{\alpha_k}{\sqrt{\rho_k}} \int_0^t u_k(s) \, ds} \right] = \\
e^{\frac{1}{2} \int_0^t \int_0^t \sum_{k=0}^{\infty} \frac{u_k(s)u_k(\tau)}{\rho_k} \, ds \, d\tau} = e^{\frac{1}{2} \int_0^t \int_0^t \min(s, \tau) \, ds \, d\tau}
\]

We have used the following results

1. \( E[e^{\alpha u}] = e^{\frac{\alpha^2}{2}} \), with \( \alpha \sim N(0, 1) \) via moment generating function
2. \( \int_0^t \min(s, \tau) \, ds = \int_0^\tau s \, ds + \int_\tau^t \tau \, ds = \frac{\tau^2}{2} + (\tau(t - \tau)) \)
Moreover

\[
E \left[ e^{-\frac{\lambda^2}{2} \int_0^t \beta^2(s) \, ds} \right] = E \left[ e^{-\frac{\lambda^2}{2} \sum_{k=0}^{\infty} \frac{\alpha_k^2}{\rho_k}} \right]
\]

\[
= \prod_{k=0}^{\infty} E \left[ e^{-\frac{\lambda^2}{2} \frac{\alpha_k^2}{\rho_k}} \right]
= \prod_{k=0}^{\infty} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{\alpha^2}{2} \left(1 + \frac{\lambda^2}{\rho_k}\right)} \, d\alpha
\]

\[
= \prod_{k=0}^{\infty} \frac{1}{\sqrt{1 + \frac{\lambda^2}{\rho_k}}}
= \frac{1}{\sqrt{\prod_{k=0}^{\infty} \left(1 + \frac{\lambda^2 t^2}{(k+\frac{1}{2})^2 + \pi^2}\right)}}
\]

\[
= \frac{1}{\sqrt{\cosh(\lambda t)}}
\]
Let us review the Schrödinger equation from quantum mechanics

1. The “standard,” time-dependent Schrödinger equation

\[ i\hbar \frac{\partial}{\partial t} \psi(x, t) = \left[ -\frac{\hbar^2}{2m} \Delta + V(x, t) \right] \psi(x, t) = \hat{H}(x, t)\psi \]

2. We can make the equation dimensionless as

\[ -i \frac{\partial}{\partial t} \psi(x, t) = \left[ \frac{1}{2} \Delta - V(r, t) \right] \psi(x, t) = H(x, t)\psi \]

3. We also are interested in the spectral properties of the time-independent problem

\[ \left[ \frac{1}{2} \Delta - V(x, t) \right] \psi(x, t) = H(x, t)\psi = \lambda \psi \]
We now arrive at the Bloch equation

1. Consider transformation (analytic continuation) of the Schrödinger to imaginary time, \( \tau = it \), this gives us the Bloch equation, but is sometimes also called the Schrödinger equation (going back to \( u(x, t) \))

\[
\frac{\partial u(x, t)}{\partial \tau} = \frac{1}{2} \Delta u(x, t) - V(x, t)u(x, t)
\]

2. The time dependent Bloch equation can be solved via separation of variables as

\[
u(x, t) = U(x)T(t), \text{ and so we apply this to the Bloch equation}
\]

\[
\frac{\partial u(x, t)}{\partial t} = U(x)T'(t) = \left[ \frac{1}{2} \Delta U(x) - V(x, t)U(x) \right] T(t)
\]
The Schrödinger and Bloch Equations

3. Placing the time and space dependent on different sides of the equation gives

$$\frac{T'(t)}{T(t)} = \lambda = \left[ \frac{1}{2} \Delta - V(x, t) \right] \frac{U(x)}{U(x)}$$, where $\lambda$ is constant

4. Thus we have that $T(t)$ and $U(x)$ satisfy the following equations

$$T'(t) - \lambda T(t) = 0,$$

$$\left[ \frac{1}{2} \Delta - V(x, t) \right] U(x) = \lambda U(x)$$

5. Thus the $\lambda_j$'s and $\psi_j(x, t)$'s are eigenvalues and eigenfunctions of the above eigenvalue problem, and the solution by separation variables is

$$u(x, t) = \sum_{j=1}^{\infty} c_j e^{-\lambda_j t} \psi_j(x), \text{ where, } c_j = \int_{-\infty}^{\infty} u_0(x) \psi_j(x) \, dx$$
Let $\lambda = 1$, as $t \to \infty$, $E \left[ e^{-\frac{1}{2} \int_0^t \beta^2(s) \, ds} \right] = \frac{1}{\sqrt{\cosh(t)}} \sim \sqrt{2} e^{-\frac{t}{2}}$ and

$$
\lim_{t \to \infty} \frac{1}{t} \ln E \left[ e^{-\frac{1}{2} \int_0^t \beta^2(s) \, ds} \right] = -\frac{1}{2}.
$$

**Theorem:** If $V(y) \to \infty$ as $|y| \to \infty$, then

$$
\lim_{t \to \infty} \frac{1}{t} \ln E \left[ e^{-\int_0^t V(\beta(s)) \, ds} \right] = -\lambda_1,
$$

where $\lambda_1$ is the lowest eigenvalue of the Bloch equation

$$
\frac{1}{2} \psi''(y) - V(y) \psi(y) = \lambda \psi(y)
$$
The Schrödinger and Bloch Equations

- Feynmann-Kac: Let $V$ be measurable and bounded below, then the solution of the Bloch equation

$$u_t = \frac{1}{2}u_{xx} - V(x)u, \quad u(x, 0) = u_0(x)$$

is $u(x, t) = E_x \left[e^{-\int_0^t V(\beta(s)) ds} u_0(\beta(t))\right]$

- This equation is the imaginary time analog of the Schrödinger equation

$$\frac{1}{2} \psi''(y) - V(y) \psi(y) = \lambda \psi(y)$$

1. Special case: $V \equiv 0$:

$$E_x \left[u_0(\beta(t))\right] = \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} u_0(y) e^{-\frac{(x-y)^2}{2t}} dy = u(x, t)$$
Another special case

2. For $V(x) = \frac{x^2}{2}$, $u_0 \equiv 1$:

$$u(x, t) = E_x \left[ e^{-\frac{1}{2} \int_0^t \beta^2(s) \, ds} \right] = E_0 \left[ e^{-\frac{1}{2} \int_0^t (\beta(s) + x)^2 \, ds} \right]$$

$$= e^{-\frac{x^2 t}{2}} E \left[ e^{-x \int_0^t \beta(s) \, ds - \frac{1}{2} \int_0^t \beta^2(s) \, ds} \right]$$

$$= e^{-\frac{x^2 t}{2}} E \left[ e^{-x \sum_{k=0}^{\infty} \frac{\alpha_k}{\sqrt{\rho_k}} \int_0^t u_k(s) \, ds - \frac{1}{2} \sum_{k=0}^{\infty} \frac{\alpha_k^2}{\rho_k}} \right]$$

$$= e^{-\frac{x^2 t}{2}} \prod_{k=0}^{\infty} E \left[ e^{-x \frac{\alpha_k}{\sqrt{\rho_k}} \int_0^t u_k(s) \, ds - \frac{1}{2} \frac{\alpha_k^2}{\rho_k}} \right]$$

$$= e^{-\frac{x^2 t}{2}} \prod_{k=0}^{\infty} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x \frac{\alpha}{\sqrt{\rho_k}} \int_0^t u_k(s) \, ds - \frac{\alpha^2}{2} (1 + \frac{1}{\rho_k})} \, d\alpha$$

$$= e^{-\frac{x^2 t}{2}} \frac{1}{\sqrt{\cosh(t)}} \int_0^{\infty} e^{\frac{x^2}{2} \int_0^t \int_0^t \sum_{k=0}^{\infty} \frac{u_k(s) u_k(\tau)}{\rho_k + 1} \, ds \, d\tau}$$
Define $R(s, \tau; -\lambda^2)$ such that

$$\min(s, \tau) = \lambda^2 \int_0^t \min(s, \xi) R(\xi, \tau; -\lambda^2) \, d\xi$$

Note that $R(s, \tau; -1) = -\sum_{k=0}^{\infty} \frac{u_k(s)u_k(\tau)}{\rho_k+1}$.

Consider

$$-\sum_{k=0}^{\infty} \frac{u_k(s)u_k(\tau)}{\rho_k + \lambda^2} + \sum_{k=0}^{\infty} \frac{u_k(s)u_k(\tau)}{\rho_k}$$

$$= \lambda^2 \int_0^t \sum_{k=0}^{\infty} \frac{u_k(s)u_k(\xi)}{\rho_k} \sum_{l=0}^{\infty} \frac{u_l(\xi)u_l(\tau)}{\rho_k + \lambda^2} \, d\xi$$
For $0 \leq s \leq t$ we have

$$R(s, \tau; -\lambda^2) = \begin{cases} \frac{-\cosh(\lambda(t-\tau)) \sinh(\lambda s)}{\lambda \cosh(\lambda t)} & s \leq \tau \\ \frac{-\cosh(\lambda(t-s)) \sinh(\lambda \tau)}{\lambda \cosh(\lambda t)} & s \geq \tau \end{cases}$$

Thus

$$u(x, t) = \frac{1}{\sqrt{\cosh(t)}} e^{-\frac{x^2}{2}} \left( t + \int_0^t \int_0^t R(s, \tau; -\lambda^2) \, ds \, d\tau \right) = \frac{1}{\sqrt{\cosh(t)}} e^{-\frac{x^2 \tanh t}{2}}$$

Exercise: compute $u(x, t)$ for $V(x) = \frac{x^2}{2}$, $u_0(x) = x$. Hint: the solution is

$$u(x, t) = E_x \left[ e^{-\frac{1}{2} \int_0^t \beta^2(s) \, ds} \beta(t) \right] .$$

Calculate

$$\tilde{u}(x, t, \lambda) = E_x \left[ e^{\lambda \beta(t) - \frac{1}{2} \int_0^t \beta^2(s) \, ds} \right] , \quad u(x, t) = \left. \frac{d}{d\lambda} \tilde{u}(x, t, \lambda) \right|_{\lambda=0}.$$
Proof of the Arcsin Law

**Theorem:** Let $X_1, X_2, \ldots$ be i.i.d. r.v.’s with $E[X_i] = 0$, $Var(X_i) = 1$, and $N_n$ is the number of partial sums $S_j = \sum_{i=1}^{j} X_i$ out of $S_1, \ldots, S_n$ which are $\geq 0$:

$$\lim_{n \to \infty} P \left[ \frac{N_n}{n} < \alpha \right] = \Sigma(\alpha) = \begin{cases} 0 & \alpha < 0 \\ \frac{2}{\pi} \arcsin \sqrt{\alpha} & 0 \leq \alpha \leq 1 \\ 1 & \alpha \geq 1 \end{cases}$$

**Proof:** (Using the Feynman-Kac formula and Donsker’s Invariance Principal) Define the random step function

$$X^{(n)}(\tau) = \begin{cases} \frac{S_1}{\sqrt{n}} & \tau = 0 \\ \frac{S_i}{\sqrt{n}} & \frac{i-1}{n} < \tau \leq \frac{i}{n} \end{cases}$$

The invariance principle states that for a large class of functionals $\mathcal{F}$ and $F \in \mathcal{F}$

$$\lim_{n \to \infty} P \left[ F \left[ X^{(n)}(\cdot) \right] \leq \alpha \right] = P_{BM} \left[ F \left[ \beta(\cdot) \right] \leq \alpha \right]$$  \hspace{1cm} (2.2)
Proof of the Arcsin Law

For example, let
\[
F[\beta] = \int_0^t \frac{1 + \text{sgn}[\beta(s)]}{2} \, ds,
\]
where \( \text{sgn}(x) = \begin{cases} 1 & x \geq 0 \\ -1 & x < 0 \end{cases} \)

Then (2.2) says that
\[
\lim_{n \to \infty} P \left[ \frac{N_n}{n} \leq \alpha \right] = P_{BM} \left[ \int_0^1 \frac{1 + \text{sgn}[\beta(s)]}{2} \, ds \leq \alpha \right]
\]
of the Brownian motion that is positive

We drop the \( BM \) from the probabilities as it is understood
Proof of the Arcsine Law

Let
\[ \sigma(\alpha, t) = P \left[ \int_0^t \frac{1 + \text{sgn}[\beta(s)]}{2} \, ds \leq \alpha \right] \]

Then for \( \lambda > 0 \) we can define the Laplace Transform/Moment Generating Function of \( \sigma(\alpha, t) \)
\[ E \left[ e^{-\lambda \int_0^t \frac{1 + \text{sgn}[\beta(s)]}{2} \, ds} \right] = \int_0^\infty e^{-\lambda \alpha} \, d\sigma(\alpha, t) \]

Now define
\[ u(x, t; \lambda) = E \left[ e^{-\lambda \int_0^t \frac{1 + \text{sgn}[\beta(s)]}{2} \, ds} \delta(\beta(t) - x) \right] \]
Proof of the Arcsin Law

By Feynman-Kac this is a solution to the following PDE

\[ u(x, t; \lambda)_t = \frac{1}{2} u(x, t; \lambda)_{xx} - \lambda V(x) u(x, t; \lambda), \quad u(x, 0; \lambda) = \delta(x) \]

where \( V(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases} \)

We also realize that

\[
\int_{-\infty}^{\infty} u(x, t; \lambda) \, dx = \int_{-\infty}^{\infty} E \left[ e^{-\lambda \int_0^t \frac{1 + \text{sgn}[\beta(s)]}{2} \, ds} \delta(\beta(t) - x) \right] \, dx \overset{\text{Fubini}}{=} \\
E \left[ \int_{-\infty}^{\infty} e^{-\lambda \int_0^t \frac{1 + \text{sgn}[\beta(s)]}{2} \, ds} \delta(\beta(t) - x) \, dx \right] = E \left[ e^{-\lambda \int_0^t \frac{1 + \text{sgn}[\beta(s)]}{2} \, ds} \right] = \\
\int_0^\infty e^{-\lambda \alpha} \, d\sigma(\alpha, t)
\]
Proof of the Arcsin Law

- It is known that \( u(x, t; \lambda) \) also solves the following integral equation

\[
\begin{align*}
u(x, t; \lambda) &= \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} - \\
&\quad \lambda \int_0^t d\tau \int_{-\infty}^{\infty} d\xi V(\xi)u(\xi, \tau; \lambda) \frac{1}{\sqrt{2\pi(t-\tau)}} e^{-\frac{(x-\xi)^2}{2(t-\tau)}}
\end{align*}
\]

- Now we apply the heat equation operator, \( \frac{\partial}{\partial t} - \frac{1}{2} \frac{\partial^2}{\partial x^2} \) to this

\[
\frac{\partial u}{\partial t} - \frac{1}{2} \frac{\partial^2 u}{\partial x^2} = 0 - \lambda V(x)u(x, t; \lambda)
\]

- And we the Laplace transform of \( u(x, t; \lambda) \)

\[
\Psi(x, s; \lambda) = \int_{-\infty}^{\infty} e^{-st} u(x, t; \lambda) dt
\]
If we take the Laplace transform of the integral equation we get

\[
\Psi(x, s; \lambda) = \frac{1}{\sqrt{2s}} e^{-\sqrt{2s}|x|} - \lambda \int_{-\infty}^{\infty} d\xi V(\xi) \Psi(\xi, s; \lambda) \frac{1}{\sqrt{2s}} e^{-\sqrt{2s}|x-\xi|}
\]

This is equivalent to the following ordinary differential equation (ODE)

\[
\frac{1}{2} \psi''(x) - (s + \lambda V(x)) \psi(x) = 0, \psi \to 0 \text{ as } |x| \to \infty
\]

\[
\psi(x) \text{ and } \psi'(x) \text{ is continuous at } x \neq 0, \text{ and } \psi'(0^-) - \psi'(0^+) = 2
\]
Proof of the Arcsine Law

The solution to the above ODE is

\[
\psi(x, s; \lambda) = \begin{cases} 
\frac{\sqrt{2}}{\sqrt{s + \lambda} + \sqrt{s}} e^{-\sqrt{2(s + \lambda)}x} & x \geq 0 \\
\frac{\sqrt{2}}{\sqrt{s + \lambda} + \sqrt{s}} e^{-\sqrt{2s}x} & x < 0 
\end{cases}
\]

Thus we have that

\[
\int_{-\infty}^{\infty} \psi(x, s; \lambda) \, dx = \frac{1}{\sqrt{s(s + \lambda)}}
\]

So we have the following

\[
\int_{-\infty}^{\infty} \psi(x, s; \lambda) \, dx = \int_{0}^{\infty} e^{-st} \int_{-\infty}^{\infty} u(x, t; \lambda) \, dx \, ds =
\int_{0}^{\infty} e^{-st} \int_{0}^{\infty} e^{-\lambda \alpha} \, d\sigma(\alpha, t) \, ds = \frac{1}{\sqrt{s(s + \lambda)}}
\]
Proof of the Arcsin Law

- The last line test us that we know the Laplace transform of

\[ F(t) = \int_0^\infty e^{-\lambda \alpha} \, d\sigma(\alpha, t) \]

- The inverse Laplace transform of \( \frac{1}{\sqrt{s(s+\lambda)}} \) tells us that

\[ F(t) = e^{-\frac{\lambda t}{2}} I_0\left(\frac{\lambda t}{2}\right) = \int_0^\infty e^{-\lambda \alpha} \sigma'(\alpha, t) \, d\alpha \]

- Which is itself the Laplace transform of \( \sigma'(\alpha, t) \), so we have

\[ \sigma'(\alpha, t) = \left\{ \begin{array}{ll} \frac{1}{\pi \sqrt{\alpha(t-\alpha)}} & 0 < \alpha < t \\ 0 & \alpha > t \end{array} \right. \]
Proof of the Arcsin Law

We now integrate the previous result

$$\int_{-\infty}^{\alpha} \sigma'(\bar{\alpha}, t) \, d\bar{\alpha} = \sigma(\alpha, t) = \begin{cases} 0 & 0 < \alpha \\ \frac{2}{\pi} \arcsin \sqrt{\frac{\alpha}{t}} & 0 < \alpha < t \\ 1 & \alpha > t \end{cases}$$

Setting $t = 1$ we get the Arcsin Law

$$\sigma(\alpha, 1) = \Sigma(\alpha) = \begin{cases} 0 & 0 < \alpha \\ \frac{2}{\pi} \arcsin \sqrt{t} & 0 < \alpha < 1 \\ 1 & \alpha > 1 \end{cases} \quad \text{Q. E. D.}$$
Another Wiener Integral

- We wish to compute the probability of

\[ P \left\{ \max_{0 \leq s \leq t} \beta(s) \leq \alpha \right\} \]

- By Donsker’s Invariance Principal this is equal to

\[ \lim_{n \to \infty} \left\{ \max \left( \frac{S_1}{\sqrt{n}}, \frac{S_2}{\sqrt{n}}, \ldots, \frac{S_n}{\sqrt{n}} \right) \leq \alpha \right\} = P \left\{ \max_{0 \leq s \leq t} \beta(s) \leq \alpha \right\} = H(\alpha, t) \]

- Consider the step-function potential

\[ V_\alpha(x) = \begin{cases} 1 & x \geq \alpha \\ 0 & x < \alpha \end{cases} \]

- Since \( \beta(\cdot) \) is a continuous function AE, if \( \max_{0 \leq s \leq t} \beta(s) \leq \alpha \) then \( V_\alpha(\beta(s)) = 0 \) on a set of positive measure
Consider the following Wiener integral

$$\lim_{\lambda \to \infty} E \left[ e^{-\lambda \int_0^t V_{\alpha}(\beta(s)) \, ds} \right] = H(\alpha, t)$$

This is because the $\lambda$ limit kills walks that exceed $\alpha$ and only count the walks that satisfy the condition.

For a fixed $\lambda$ this is, by Feynman-Kac, the solution to

$$u(x, t; \lambda) = \frac{1}{2} u(x, t; \lambda)_{xx} - \lambda V(x) u(x, t; \lambda), \quad u(x, 0; \lambda) = 1$$

where $V(x) = \begin{cases} 1 & x \geq \alpha \\ 0 & x < \alpha \end{cases}$

The solution of the PDE is very similar to the solution of the PDE from the Arcsine Law, and is left to the reader.

$$H(\alpha, t) = \sqrt{\frac{2}{\pi}} \int_0^{\alpha \sqrt{t}} e^{-\frac{u^2}{2}} \, du$$
Von Neumann proved that there is no translationally invariant Haar measure in function space; Wiener measure is not translationally invariant.

Consider the following problem where we write our heuristic via a “flat” integral

\[ E \{ F[\beta] \} \approx = \int F[\beta] e^{-\frac{1}{2} \int_0^t [\beta'(\tau)]^2 d\tau} d\beta \]

Here we define the Action as

\[ A[\beta] = -\frac{1}{2} \int_0^t [\beta'(\tau)]^2 d\tau \]

This is obviously a heuristic, as BM is nondifferentiable AE.
Now consider computing the following with Action Asymptotics:

\[ E \left[ e^{\frac{1}{\sqrt{\epsilon}} \int_0^t \beta(s) \, ds} \right] \]

We first compute this using our standard techniques:

\[ E \left[ e^{\frac{1}{\sqrt{\epsilon}} \int_0^t \beta(s) \, ds} \right] = E \left[ e^{\frac{1}{\sqrt{\epsilon}} \sum_{k=0}^{\infty} \frac{\alpha_k u_k(s)}{\sqrt{\rho_k}} \, ds} \right] = E \left[ e^{\frac{1}{\sqrt{\epsilon}} \sum_{k=0}^{\infty} \frac{\alpha_k u_k(s)}{\sqrt{\rho_k}} \, ds} \right] \]

And thus:

\[ \lim_{\epsilon \to 0} \epsilon \ln E \left[ e^{\frac{1}{\sqrt{\epsilon}} \int_0^t \beta(s) \, ds} \right] = \frac{t^3}{6} \]
Let’s “derive” the action asymptotics heuristic with a construction due to Kac and Feynman by considering

\[ F(t) = \mathbb{E} \left\{ e^{- \int_0^t V(\beta(\tau)) \, d\tau} \right\} \]

where \( \beta(\cdot) \in C_0[0, t] \), and the expectation is taken w.r.t. Wiener measure.

Since we assume that \( V(\cdot) \) is continuous and non-negative, and \( \beta(\cdot) \in C_0[0, t] \) is continuous, \( F(t) \) exists as \( \int_0^t V(\beta(\tau)) \, d\tau \) is measurable.

Now let us consider a discrete approximation of this Wiener integral by breaking it up into \( N \) sized time intervals of size \( t/N \), which gives us \( F(t) \) from bounded convergence and the Riemann summability.

\[ F(t) = \lim_{{N \to \infty}} \mathbb{E} \left\{ e^{- \frac{t}{N} \sum_{k=1}^{N} V(\beta(\frac{tk}{N}))} \right\} \]
Action Asymptotics: A Heuristic for Wiener Integrals

If we consider the expectation in the limit we can rewrite it as follows

$$\lim_{N \to \infty} E \left\{ e^{-\frac{t}{N} \sum_{k=1}^{N} V(\beta(\frac{tk}{N}))} \right\} = \lim_{N \to \infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{-h \sum_{k=1}^{N} V(\beta_k)} \times P(0, \beta_1; h) P(\beta_1, \beta_2; h) \cdots P(\beta_{N-1}, \beta_N; h) \, d\beta_1 \, d\beta_2 \cdots d\beta_N$$

where we have

1. $h = \frac{t}{N}$
2. $\beta_k = \beta(kh)$
3. $P(\beta_{k-1}, \beta_k; h) = \frac{1}{\sqrt{2\pi h}} e^{-\frac{(\beta_k - \beta_{k-1})^2}{2h}}$

This limit exists and is equal to the Wiener integral

However, Feynman chose to rewrite the above as (suppressing the limit) with $\beta_0 = 0$

$$\frac{1}{(2\pi h)^{N/2}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{-h \left\{ \sum_{k=1}^{N} V(\beta_k) + \frac{1}{2} \sum_{k=1}^{N} \left( \frac{\beta_k - \beta_{k-1}}{h} \right)^2 \right\}} \, d\beta_1 \, d\beta_2 \cdots d\beta_N$$
Action Asymptotics: A Heuristic for Wiener Integrals

- If we look at the exponent in Feynman’s we notice that

\[
\left\{ \sum_{k=1}^{N} V(\beta_k) + \frac{1}{2} \sum_{k=1}^{N} \left( \frac{\beta_k - \beta_{k-1}}{h} \right)^2 \right\} \overset{h \to 0}{\longrightarrow} \int_{0}^{t} \left\{ \frac{1}{2} \left( \frac{d\beta}{d\tau} \right)^2 + V(\beta(\tau)) \right\} d\tau
\]

- This is the Hamiltonian the along the path, \( \beta(\tau) \), and with the classical action along the path is

\[
\int_{0}^{t} \left\{ \frac{1}{2} \left( \frac{d\beta}{d\tau} \right)^2 - V(\beta(\tau)) \right\} d\tau
\]

- thus Feynman writes the above integral instead as

\[
F(t) = E \left\{ e^{-\int_{0}^{t} V(\beta(\tau)) d\tau} \right\} = \int e^{-\left[ \int_{0}^{t} \left\{ \frac{1}{2} \left( \frac{d\beta}{d\tau} \right)^2 + V(\beta(\tau)) \right\} d\tau \right]} d(path)
\]
How does \( E \left[ e^{\frac{1}{\epsilon} F[\sqrt{\epsilon}\beta]} \right] \) behave as \( \epsilon \to 0 \)?

We can approach this with Action Asymptotics

\[
E \left[ e^{\frac{1}{\epsilon} F[\sqrt{\epsilon}\beta]} \right] = \int e^{\frac{1}{\epsilon} F[\sqrt{\epsilon}\beta]} e^{-\frac{1}{2} \int_0^t [\beta'(s)]^2 ds} \delta \beta
\]

Now let \( \sqrt{\epsilon}\beta = \omega \)

\[
= \int e^{\frac{1}{\epsilon} \left[ F[\omega] - \frac{1}{2} \int_0^t [\omega'(s)]^2 ds \right]} \delta \beta
\]

Using Laplace asymptotics the above will behave like

\[
e^{\frac{1}{\epsilon} \sup_{\omega \in C_0^*[0,t]} \left[ F[\omega] - \frac{1}{2} \int_0^t [\omega'(s)]^2 ds \right]}
\]

Where the space \( C_0^*[0, t] \) is made up functions, \( \omega(t) \), with

1. \( \omega(t) \) continuous in \([0, t]\)
2. \( \omega(0) = 0 \)
3. \( \omega'(t) \in L^2[0, t] \)
A conjecture using Action Asymptotics

\[
\lim_{\epsilon \to 0} \epsilon \ln E \left[ e^{\frac{1}{\epsilon} F[\sqrt{\epsilon} \beta]} \right] = \sup_{\omega \in C_*^0 [0, t]} \left[ F[\omega] - \frac{1}{2} \int_0^t [\omega'(s)]^2 ds \right]
\]

Consider \( F[\beta] = \int_0^t \beta(s) ds \)

\[
E \left[ e^{\frac{1}{\epsilon} F[\sqrt{\epsilon} \beta]} \right] = E \left[ e^{\frac{1}{\epsilon} \int_0^t \beta(s) ds} \right]
\]

From the conjecture we have that

\[
\lim_{\epsilon \to 0} \epsilon \ln E \left[ e^{\frac{1}{\sqrt{\epsilon}} \int_0^t \beta(s) ds} \right] = \sup_{\omega \in C_* [0, t]} \left[ \int_0^t \omega(s) ds - \frac{1}{2} \int_0^t [\omega'(s)]^2 ds \right]
\]
From the calculus of variations we have that the Euler equation for the following maximum principle is

\[ \sup_{\omega \in C^*_0[0,t]} \left[ \int_0^t \omega(s) \, ds - \frac{1}{2} \int_0^t [\omega'(s)]^2 \, ds \right] \Rightarrow \]

1. \( 1 + \omega''(s) = 0 \)
2. \( \omega(0) = 0 \)
3. \( \omega'(t) = 0 \)

The solution is \( \omega(s) = -\frac{s^2}{2} + ts \) and \( \omega'(s) = -s + t \) so

\[ \int_0^t \left(-\frac{s^2}{2} + ts\right) \, ds - \frac{1}{2} \int_0^t [s - t]^2 \, ds = \frac{t^3}{6} \]
Recall some basic properties of the BM, $\beta(\cdot)$ and constant, $c$:

1. $\beta(\tau) \sim N(0, \tau)$
2. $\beta(c\tau) \sim N(0, c\tau)$
3. $\sqrt{c}\beta(\tau) \sim N(0, c\tau)$
4. $E[\beta(\tau)\beta(s)] = \min(\tau, s)$
5. $E[\beta(c\tau)\beta(cs)] = c \min(\tau, s)$
6. $E[\beta(c\tau)\beta(cs)] = E[\sqrt{c}\beta(\tau)\sqrt{c}\beta(s)] = cE[\beta(\tau)\beta(s)] = c \min(\tau, s)$

Now consider the following

$$E \left[ e^{\sup_{0 \leq s \leq t} \beta(s)} \right] = E \left[ e^{\sup_{0 \leq \tau \leq 1} \beta(t\tau)} \right] =$$

$$E \left[ e^{\sup_{0 \leq \tau \leq 1} \sqrt{t}\beta(\tau)} \right] = E \left[ e^{t \sup_{0 \leq \tau \leq 1} \frac{1}{\sqrt{t}} \beta(\tau)} \right] =$$

$$E \left[ e^{\frac{1}{\epsilon} \sup_{0 \leq \tau \leq 1} \sqrt{\epsilon}\beta(\tau)} \right] \quad \text{using the substitution } t = \frac{1}{\epsilon}$$
Action Asympotics: Examples

So we now have that
\[
\lim_{t \to \infty} \frac{1}{t} \ln E \left[ e^{\sup_{s \leq t} \beta(s)} \right] = \lim_{\epsilon \to 0} \epsilon \ln E \left[ e^{\frac{1}{\epsilon} \sup_{0 \leq \tau \leq 1} \sqrt{\epsilon} \beta(\tau)} \right]
\]

By Action Asympotics we have
\[
\lim_{\epsilon \to 0} \epsilon \ln E \left[ e^{\frac{1}{\epsilon} \sup_{0 \leq \tau \leq 1} \sqrt{\epsilon} \beta(\tau)} \right] = \sup_{\omega \in C^0[0,1]} \left[ \sup_{0 \leq \tau \leq 1} \omega(\tau) - \frac{1}{2} \int_0^1 [\omega'(\tau)]^2 d\tau \right] = \max_{a > 0} \left[ a - \frac{a^2}{2} \right] = \frac{1}{2}
\]

The supremum comes on straight lines, that minimize arc-length i.e. the second term, so consider \(\omega(\tau) = a\tau\), and \(a = 1\) is the maximizer.
Consider a more complicated problem for Action Asymptotics is

$$\lim_{\epsilon \to 0} \frac{E \left[ G(\sqrt{\epsilon}\beta(\cdot)) e^{\frac{1}{\epsilon}F(\sqrt{\epsilon}\beta(\cdot))} \right]}{E \left[ e^{\frac{1}{\epsilon}F(\sqrt{\epsilon}\beta(\cdot))} \right]} = \int \left[ E \left[ G(\sqrt{\epsilon}\beta(\cdot)) e^{\frac{1}{\epsilon}F(\sqrt{\epsilon}\beta(\cdot))} - \frac{1}{2} \int_0^t [\beta'(s)]^2 ds \right] \right] \delta \beta$$

We now change variables with $x(\cdot) = \sqrt{\epsilon}\beta(\cdot)$

$$\int \left[ E \left[ G(x(\cdot)) e^{\frac{1}{\epsilon}[F(x(\cdot)) - \frac{1}{2} \int_0^t [x'(s)]^2 ds]} \right] \right] \delta x$$

$$\int \left[ E \left[ e^{\frac{1}{\epsilon}[F(x(\cdot)) - \frac{1}{2} \int_0^t [x'(s)]^2 ds]} \right] \right] \delta x$$
As $\epsilon \to 0$ the exponential term goes to something like a “delta” function in function space and we get

$$= G[\omega^*(\cdot)] \text{ where } \omega^*(\cdot) = \arg\sup_{\omega \in C^*_{[0,t]}} [F[\omega] - A[\omega]]$$

We now apply this to some PDE problems: Burger’s Equation

$$u_t + uu_x = \frac{\epsilon}{2} u_{xx}, \quad -\infty \leq x \leq \infty, \quad t > 0$$

$$u(x, 0) = u_0(x), \quad \int_0^\infty u_0(\eta) \, d\eta = o(x^2) \text{ as } |x| \to \infty$$

We now apply the Hopf-Cole transformation, if we define the solution to Burger’s equation $u(x, t) = -\epsilon \frac{v_x(x, t)}{v(x, t)} = -\epsilon \partial_x [\ln v(x, t)]$ then $v(x, t)$ satisfies

$$v_t = \frac{\epsilon}{2} v_{xx}, \quad v(x, 0) = e^{-\frac{1}{\epsilon} \int_0^x u_0(\eta) \, d\eta}$$
Action Asymptotics: Examples

- So by Feynman-Kac we can write the solution as

\[
v(x, t; \epsilon) = \frac{1}{\sqrt{2\pi t\epsilon}} \int_{-\infty}^{\infty} e^{-\frac{1}{\epsilon} \int_{0}^{y} u_0(\eta) d\eta} e^{-\frac{(x-y)^2}{2\epsilon t}} dy
\]

- We now apply the Hopf-Cole transformation (taking the logarithmic derivative)

\[
u(x, t; \epsilon) = \frac{1}{\sqrt{2\pi t\epsilon}} \int_{-\infty}^{\infty} e^{-\frac{1}{\epsilon} \left[ \int_{0}^{y} u_0(\eta) d\eta + \frac{(y-x)^2}{2t} \right]} dy
\]

- Now let \( F(y) = \int_{0}^{y} u_0(\eta) d\eta + \frac{(y-x)^2}{2t} \), this is the function that Action Asymptotics tells us to minimize (due to the negative sign)

- Note that \( \lim_{|y| \to \infty} \frac{F(y)}{y^2} = \frac{1}{2t} \) by the assumptions, and so there is a minimum, \( y(x, t) = \text{argmin} F(y) \)

- Hopf showed that if at \((x, t)\) there is a single minimizer to \( F(y) \) then

\[
\lim_{\epsilon \to 0} u(x, t; \epsilon) = \frac{x - y(x, t)}{t} = u_0(y(x, t))
\]
Consider the related equation

\[ u_t + uu_x = \frac{\epsilon}{2} u_{xx} - V'(x), \quad -\infty \leq x \leq \infty, \quad t > 0 \]

\[ u(x, 0) = u_0(x), \quad \int_0^\infty u_0(\eta) \, d\eta = o(x^2) \text{ as } |x| \to \infty \]

Again we use the Hopf-Cole transformation to get

\[ v_t = \frac{\epsilon}{2} v_{xx} - \frac{1}{\epsilon} V'(x) v, \quad v(x, 0) = e^{-\frac{1}{\epsilon} \int_0^x u_0(\eta) \, d\eta} \]

And so we can write down the solution to the transformed equation via Feynman-Kac

\[ v(x, t; \epsilon) = E_x \left[ e^{-\frac{1}{\epsilon} \int_0^t V(\sqrt{\epsilon} \beta(s)) \, ds - \frac{1}{\epsilon} \int_0^{\sqrt{\epsilon} \beta(t)} u_0(\eta) \, d\eta} \right] \]

\[ = E_0 \left[ e^{-\frac{1}{\epsilon} \int_0^t V(\sqrt{\epsilon} \beta(s) + x) \, ds \int_0^{\sqrt{\epsilon} \beta(t) + x} u_0(\eta) \, d\eta} \right] \]
We now take apply the Hopf-Cole transformation and get

\[
u(x, t; \epsilon) = \frac{E \left[ G[\sqrt{\epsilon} \beta(\cdot)] e^{-\frac{1}{\epsilon} F[\sqrt{\epsilon} \beta(\cdot)]]} \right]}{E \left[ e^{-\frac{1}{\epsilon} F[\sqrt{\epsilon} \beta(\cdot)]} \right]}\]

where we define

\[
F[\beta(\cdot)] = \int_0^t V(\sqrt{\epsilon} \beta(s)) \, ds - \int_0^{\sqrt{\epsilon} \beta(t)} u_0(\eta) \, d\eta
\]

\[
G[\beta(\cdot)] = \int_0^t V'(\sqrt{\epsilon} \beta(s) + x) \, ds + u_0(\sqrt{\epsilon} \beta(t) + x)
\]
Action Asymptotics: Examples

- By Action Asymptotics we have that
  \[
  \lim_{\epsilon \to 0} u(x, t; \epsilon) = G[\omega^*(\cdot)] \quad \text{where} \quad \omega^*(\cdot) = \underset{\omega \in C^*_0 [0,t]}{\text{arginf}} \ [F[\omega] + A[\omega]]
  \]

- If for \((x, t) \exists!\) minimizer, \(\omega^*\), then the limit exists and is
  \[
  G[\omega^*(t)] = u(x, t) = \int_0^t V'(\omega^*(s) + x) \, ds + u_0(\omega^*(t) + x)
  \]

- Now consider the related variational problem
  \[
  \inf_{\omega \in C^*_0 [0,t]} \left[ \int_0^t V(\omega(s) + x) \, ds \int_0^{\omega(t)+x} u_0(\eta) \, d\eta + \frac{1}{2} \int_0^t [\omega'(s)]^2 \, ds \right]
  \]

- We refer to the functional to be minimized as \(H[\omega(\cdot)]\)
To arrive derive an equivalent system via the Calculus of Variations we need to form the Frechet derivative, in the direction of the arbitrary function, $\Psi$, as follows

$$
\delta H|_{\psi} = \left. \frac{dH[\omega + h\psi]}{dh} \right|_{h=0} = \int_0^t V'(\omega(s) + x)\psi(s) \, ds + u_0(\omega(t) + x)\psi(t)
$$

$$
+ \omega'(t)\psi(t) - \int_0^t \omega''(s)\psi(s) \, ds
$$

Note that the last two terms come from the following computation

$$
J[\omega(\cdot)] \overset{\text{def}}{=} \frac{1}{2} \int_0^t [\omega'(s)]^2 \, ds \quad \Rightarrow \quad \left. \frac{dJ[\omega + h\psi]}{dh} \right|_{h=0}
$$

$$
= \frac{1}{2} \int_0^t [\omega'(s) + h\psi'(s)]^2 \, ds = \int_0^t [\omega'(s) + h\psi'(s)]^2 \, ds
$$

$$
= \int_0^t \omega'(s)\psi'(s) \, ds = \int_0^t \omega'(s) \, d\psi'(s)
$$
We now integrate by parts using the natural boundary conditions

1. \( \omega(0) = 0 \)
2. \( \omega'(0) = 0 \)

\[
\int_0^t \omega'(s) d\Psi'(s) = \omega'(t)\Psi'(s) - \int_0^t \omega''(s)\Psi ds
\]

So the solution to this problem is

1. \( V'(\omega(s) + x) = \omega''(s) \) for \( 0 \leq s \leq t \)
2. \( \omega(0) = 0 \)
3. \( \omega'(t) = -u_0(\omega(s) + x) \)

We can now apply this Hprof's result with \( V \equiv 0 \)

1. \( \omega''(s) = 0 \) for \( 0 \leq s \leq t \)
2. \( \omega(0) = 0 \)
3. \( \omega'(t) = -u_0(\omega(s) + x) \)

The solution is then very simply

1. \( \omega(s) = cs \) for some constant, \( c \)
2. \( \omega'(s) = c = -u_0(ct + x) \)
3. Let \( c = \frac{y(x,t) - x}{t} = -u_0(y(x, t)) \) or \( u_0(t(x, t)) = \frac{x - y(x,t)}{t} \)

With a unique \( y(x, t) \) we get a unique \( \omega^*(s) = \left( \frac{x - y(x,t)}{t} \right)s \)
Action Asymptotics

- We now consider some tools with the “flat integral"
- The Cameron-Martin Translation Formula

\[ E \{ F[\beta + y] \} , \text{ with } y \in C_0[0, t] \]

- We now use the “flat integral"

\[
E \{ F[\beta + y] \} = \int F[\beta + y] e^{-\frac{1}{2} \int_0^t [\beta'(s)]^2 \, ds} \delta \beta, \text{ and let } \omega = \beta + y
\]

\[
= \int F[\omega] e^{-\frac{1}{2} \int_0^t [\omega'(s) - y'(s)]^2 \, ds} \delta \omega
\]

\[
= e^{-\frac{1}{2} \int_0^t [y'(s)]^2 \, ds} \int F[\omega] e^{\frac{1}{2} \int_0^t [\omega'(s)y'(s)] \, ds - \int_0^t [\omega'(s)]^2 \, ds} \delta \omega
\]

\[
= e^{-\frac{1}{2} \int_0^t [y'(s)]^2 \, ds} E \left\{ F[\beta] e^{\int_0^t y'(s) \, d\beta(s)} \right\}
\]

- And so our result is that

\[
E \{ F[\beta + y] \} = e^{-\frac{1}{2} \int_0^t [y'(s)]^2 \, ds} E \left\{ F[\beta] e^{\int_0^t y'(s) \, d\beta(s)} \right\} , \text{ with } y \in C_0[0, t]
\]
Local Time

- Spectral Theory:
  - If $V(x) \geq 0$ and $V(x) \to 0$ as $|x| \to \infty$ then the eigenvalue problem
    \[
    \frac{1}{2} \psi''(x) - V(x)\psi(x) = -\lambda \psi(x)
    \]
    1. Has discrete spectrum: $\lambda_1, \lambda_2, \ldots$
    2. With corresponding eigenfunctions: $\psi_1, \psi_2, \ldots$
- Theorem (1949):
  \[
  \lim_{t \to \infty} \frac{1}{t} E \left[ e^{-\frac{1}{2} \int_0^t V(\beta(s)) \, ds} \right] = -\lambda_1
  \]
  Note: The expectation can start at any $x$ due to ergodicity
- Proof We will first prove this using Feynman-Kac
  \[
  u(x, t) = E_x \left[ e^{-\frac{1}{2} \int_0^t V(\beta(s)) \, ds} \right]
  \]
Local Time

- Satisfies the following PDE
  \[ u_t = \frac{1}{2} u_{xx} - V(x)u, \quad u(x, 0) = 1 \]

- By separation of variables we have
  \[ u(x, t) = \sum_{j=1}^{\infty} c_j e^{-\lambda_j t} \psi_j(x), \text{ where, } c_j = \int_{-\infty}^{\infty} u(x, 0) \psi_j(y) \, dy \]

- But since \( u(x, 0) = 1 \) we have that \( c_j = \int_{-\infty}^{\infty} \psi_j(y) \, dy, \forall j \geq 0 \), and so the two representations must be equal
  \[ u(x, t) = E_x \left[ e^{-\frac{1}{2} \int_{0}^{t} V(\beta(s)) \, ds} \right] = \sum_{j=1}^{\infty} e^{-\lambda_j t} \psi_j(x) \int_{-\infty}^{\infty} \psi_j(y) \, dy \]

- And so the largest eigenvalue, \( \lambda_1 \), controls the behavior
  \[ \lim_{t \to \infty} \frac{1}{t} E \left[ e^{-\frac{1}{2} \int_{0}^{t} V(\beta(s)) \, ds} \right] = -\lambda_1 \]
Local Time

- We also have a variational representation of $\lambda_1$

$$
\lambda_1 = \inf_{\psi \in L^2 \atop ||\psi||=1} \left[ \int_{-\infty}^{\infty} V(y) \psi^2(y) \, dy + \frac{1}{2} \int_{-\infty}^{\infty} [\psi'(y)]^2 \, dy \right]
$$

- Which has a corresponding Euler equation

$$
\frac{1}{2} \psi''(x) - V(x) \psi(x) = -\lambda \psi(x)
$$

- We notice that in the Wiener integral representation, $E \left[ e^{-\frac{1}{2} \int_0^t V(\beta(s)) \, ds} \right]$, since the internal integral is in an negative exponential, the main contribution comes for paths that remain close to where $V(\cdot)$ is smallest, which leads us to dissect this problem as follows

- Let $\beta(s)$, $0 \leq s < \infty$; $\beta(0) = x$ be BM for $t > 0$ and consider the proportion of time that $\beta(\cdot)$ spends in a set $A \subset \mathbb{R}$

$$
\ell_t(\beta(\cdot), \cdot) = \frac{1}{t} \int_0^t \chi_A(\beta(s)) \, ds
$$
Local Time

- Some properties of $L_t(\beta(\cdot), \cdot)$ with $t > 0$, $x$ fixed, and $\beta(\cdot)$ a particular, fixed, path
  1. $L_t(\beta(\cdot), \cdot)$ is a countable additive, non-negative function
  2. $L_t(\beta(\cdot), \mathbb{R}) = 1$
  3. $L_t(\beta(\cdot), \cdot): C_x[0, t] \rightarrow \mathcal{M}$, the space of probability measures on $\mathbb{R}$

- As a set function, $L_t(\beta(\cdot), \cdot)$ for fixed $x \in \mathbb{R}$ and $t > 0$ and for almost all $\beta(\cdot)$ has a density function which we call the normalized local time

$$\ell_t(\beta(\cdot), y) = \frac{1}{t} \int_0^t \delta(\beta(s) - y) \, dy \text{ and }$$

$$L_t(\beta(\cdot), A) = \int_{-\infty}^{\infty} \chi_A(y) \ell_t(\beta(\cdot), y) \, dy$$

- $\ell_t(\beta(\cdot), \cdot) \rightarrow 0$ as $t \rightarrow \infty$ for compact $A$ and almost every $\beta(\cdot)$

- Now consider the following representation

$$E_x \left[ e^{-\int_0^t V(\beta(s)) \, ds} \right] = E_x \left[ e^{-t \int_{-\infty}^{\infty} V(y) \ell_t(\beta(\cdot), y) \, dy} \right]$$
Local Time

- For fixed $x \in \mathbb{R}$ and $t > 0$ we define a probability measure on $\mathcal{M}$, $Q_{x,t} = PL_t^{-1}$, as follows:

$$Q_{x,t}(C) = P \{ \beta(\cdot) \in C_x[0, \infty] : L_t(\beta(\cdot), \cdot) \in C \}$$

- If $C \subset \mathcal{M}$ then we can write

$$E_x \left[ e^{-t \int_0^t V(\beta(s)) \, ds} \right] = E_x \left[ e^{-t \int_0^\infty V(y) \ell_t(\beta(\cdot), y) \, dy} \right] = E_x \left[ e^{-t \int_0^\infty V(y) \, dL_t(\beta(\cdot), y)} \right]$$

- $L_t(\beta(\cdot), \cdot)$ is an occupation measure so we can write

$$E_{Q_{x,t}}^{x} \left[ e^{-t \int_{-\infty}^\infty V(y) \mu(dy)} \right] = E_{Q_{x,t}}^{x} \left[ e^{-t \int_{-\infty}^\infty V(y) f(y) \, dy} \right]$$

- We define $\mathcal{F}$ as the space of probability density functions on $\mathbb{R}$, then this an expected value on $\mathcal{F}$

- To understand how the expected value on $\mathcal{F}$ behaves as $t \to \infty$, we need to understand how $Q_{x,t}$ and therefore also how $L_t(\beta(\cdot), A)$ behaves as $t \to \infty$
Local Time

- Long time behavior of local time measures
  1. \( L_t(\beta(\cdot), A) \to 0 \) as \( t \to \infty \) for \( A \subset \mathbb{R} \), compact, and AE \( \beta(\cdot) \)
  2. \( \ell_t(\beta(\cdot), A) \to 0 \) as \( t \to \infty \) for \( A \subset \mathbb{R} \), compact, and AE \( \beta(\cdot) \) by the ergodic theorem for BM, if \( \beta(\cdot) \) were not BM, then this would converge AE to the invariant measure
  3. \( Q_{x,t}(C) \to 0 \) as \( t \to \infty \) if \( C \subset \mathcal{M}, C \neq \mathcal{M} \), i.e. \( C \) is a reasonable set

- Theorem on Speed of Convergence: We first need to put the Levý topology on \( \mathcal{F} \)
  1. If \( C \in \mathcal{F} \) is closed, then
     \[
     \limsup_{t \to \infty} \frac{1}{t} \ln Q_{x,t}(C) \leq \inf_{f \in C} I(f)
     \]
  2. If \( G \in \mathcal{F} \) is open, then
     \[
     \liminf_{t \to \infty} \frac{1}{t} \ln Q_{x,t}(C) \geq \inf_{f \in G} I(f)
     \]
  3. Where
     \[
     I(f) = \frac{1}{8} \int_{-\infty}^{\infty} \left\{ \left[ f'(y) \right]^2 / f(y) \right\} \, dy
     \]
This is a simple case of what is referred to as “Donsker-Varadhan Asymptotics" and are a large deviation result.

An example, suppose \( f(y) \sim N(0, \sigma^2) \), i.e. \( f(y) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{y^2}{2\sigma^2}} \), then
\[
 f'(y) = -\frac{y}{\sigma^3 \sqrt{2\pi}} e^{-\frac{y^2}{2\sigma^2}} \quad \text{and} \quad f'(y)^2 = \frac{y^2}{\sigma^6 2\pi} e^{-2\left(\frac{y^2}{2\sigma^2}\right)}
\]
and finally we have
\[
 l(f) = \frac{1}{8} \int_{-\infty}^{\infty} \left\{ [f'(y)]^2 / f(y) \right\} \, dy = \frac{1}{8} \frac{1}{\sigma^4} \int_{-\infty}^{\infty} \frac{y^2}{\sigma \sqrt{2\pi}} e^{-\frac{y^2}{2\sigma^2}} \, dy = \frac{\sigma^2}{8\sigma^4} = \frac{1}{8\sigma^2}
\]

Note: the last integral is the variance, \( \sigma^2 \), of a \( N(0, \sigma^2) \) random variable.

We refer to the functional \( l : \mathcal{F} \rightarrow [0, \infty] \) as the entropy, and roughly speaking
\[
 Q_{x,t}(f) \sim e^{-t \inf_{f \in A} l(f)} \quad \text{for “nice”} \ A
\]
Now let us apply the “Entropy Asymptotics" with the “Flat Integral"

\[
E_x \left[ e^{-\frac{1}{2} \int_0^t V(\beta(s)) \, ds} \right] = E_x^{Q_x,t} \left[ e^{-t \int_{-\infty}^\infty V(y)f(y) \, dy} \right] \quad \text{for } t \text{ large}
\]

\[
\quad = \left[ e^{-t \int_{-\infty}^\infty V(y)f(y) \, dy} e^{-tl(f)} \right] \delta f
\]

\[
\quad = \left[ e^{-t [\int_{-\infty}^\infty V(y)f(y) \, dy + l(f)]} \right] \delta f
\]

As \( t \to \infty \) we use Laplace asymptotics to get

\[
\lim_{t \to \infty} \frac{1}{t} \ln E_x \left[ e^{-\frac{1}{2} \int_0^t V(\beta(s)) \, ds} \right] = - \inf_{f \in y} \left[ \int_{-\infty}^\infty V(y)f(y) \, dy + \frac{1}{8} \int_{-\infty}^\infty \frac{[f'(y)]^2}{f(y)} \, dy \right]
\]

Let \( \sqrt{f(y)} = \Psi(y) \), then \( \int_{-\infty}^\infty \Psi^2(y) \, dy = \int_{-\infty}^\infty f(y) \, dy = 1 \) since \( f(y) \) is a p.d.f., and so \( \Psi(\cdot) \in L^2[\mathbb{R}] \) and \( \|\Psi\| = 1 \)
We now transform the “Entropy Asymptotics” expression with some substitutions

1. Let $\sqrt{f(y)} = \Psi(y)$, then $\int_{-\infty}^{\infty} \Psi^2(y) \, dy = \int_{-\infty}^{\infty} f(y) \, dy = 1$ since $f(y)$ is a p.d.f., and so $\Psi(\cdot) \in L^2[-\infty, \infty]$ and $||\Psi|| = 1$

2. Also $\Psi'(y) = \frac{1}{2\sqrt{f(y)}} f'(y)$, and so $[\Psi'(y)]^2 = \frac{1}{4} \left( \frac{f'(y)^2}{f(y)} \right)$

These allow us to write

\[- \inf_{f \in Y} \left[ \int_{-\infty}^{\infty} V(y) f(y) \, dy + \frac{1}{8} \int_{-\infty}^{\infty} \frac{[f'(y)]^2}{f(y)} \, dy \right] = \]

\[- \inf_{\Psi \in L^2, ||\Psi||=1} \left[ \int_{-\infty}^{\infty} V(y) \Psi^2(y) \, dy + \frac{1}{2} \int_{-\infty}^{\infty} [\Psi'(y)]^2 \, dy \right] = -\lambda_1 \]

Theorem: Let $\Phi : \mathcal{F} \to \mathbb{R}$ be bounded and continuous then, by the “general structure theorem”

\[ \lim_{t \to \infty} \frac{1}{t} \ln E_x^{Q_x, t} \left[ e^{-t\Phi(f)} \right] = \lim_{t \to \infty} \frac{1}{t} \ln E_x \left[ e^{-t\Phi(\ell_t(\beta(\cdot), \cdot))} \right] = - \inf_{f \in \mathcal{F}} \left[ \Phi(f) + I(f) \right] \]
This is more subtle than action asymptotics, for example consider

$$\lim_{t \to \infty} \frac{1}{t} \ln E^{Q_{x,t}}_x \left[ e^{t\Phi(f)} \right] = \sup_{f \in \mathcal{F}} \left[ \Phi(f) - I(f) \right]$$

1. There is always a fight between the two terms in the supremum
2. In statistical mechanics we often consider $\alpha \Phi(f)$ and want to compute $\sup_{f \in \mathcal{F}} \left[ \alpha \Phi(f) - I(f) \right] = g(\alpha)$, where $\alpha$ is a convex function of $\alpha$
3. There may be a critical value of $\alpha$, call it $\alpha_0$, where there is a phase transition, this is due to nonuniqueness in the $f$ that maximized the functional
Now we will use “Entropy Asymptotics” to revisit a topic we have already considered.

Recall that

\[ P \left\{ \sup_{0 \leq s \leq t} \beta(s) \leq \alpha \right\} = \sqrt{\frac{2}{\pi t}} \int_{0}^{\alpha} e^{-\frac{u^2}{2t}} \, du, \]

so that we also have

\[ E \left[ e^{\sup_{0 \leq s \leq t} \beta(s)} \right] = h(t) = \int_{0}^{\infty} e^{\alpha} \, dP\{ \sup_{0 \leq s \leq t} \beta(s) \leq \alpha \} = \int_{0}^{\infty} e^{\alpha} \sqrt{\frac{2}{\pi t}} e^{-\frac{\alpha^2}{2t}} \, d\alpha \]

\[ \int_{0}^{\infty} e^{\alpha} \sqrt{\frac{2}{\pi t}} e^{-\frac{\alpha^2}{2t}} \, d\alpha = \sqrt{\frac{2}{\pi t}} \int_{0}^{\infty} e^{-\frac{(\alpha-t)^2}{2t}} e^{\frac{t}{2}} \, d\alpha = \sqrt{\frac{2}{\pi}} e^{\frac{t}{2}} \int_{\sqrt{t}}^{\infty} e^{-\frac{u^2}{2}} \, du \]

with the substitution \( u = \frac{\alpha - t}{\sqrt{t}} \).

Then we have

\[ \lim_{t \to \infty} \frac{1}{t} \ln h(t) = \sqrt{\frac{2}{\pi}} e^{\frac{t}{2}} \int_{\sqrt{t}}^{\infty} e^{-\frac{u^2}{2}} \, du = \frac{1}{2} \]
First we turn the $t \to \infty$ limit into an $\epsilon \to 0$ limit

$$\lim_{t \to \infty} \frac{1}{t} \ln E \left[ e^{\sup_{0 \leq s \leq t} \beta(s)} \right] = \lim_{\epsilon \to 0} \epsilon \ln E \left[ e^{\frac{1}{\epsilon} \sup_{0 \leq \tau \leq 1} \sqrt{\epsilon} \beta(\tau)} \right]$$

Recall that by Action Asymptotics we have

$$\lim_{\epsilon \to 0} \epsilon \ln E \left[ e^{\frac{1}{\epsilon} \sup_{0 \leq \tau \leq 1} \sqrt{\epsilon} \beta(\tau)} \right] = \sup_{\omega \in C^*[0,1]} \left[ \sup_{0 \leq \tau \leq 1} \omega(\tau) - \frac{1}{2} \int_0^1 [\omega'(\tau)]^2 d\tau \right]$$

$$= \max_{a > 0} \left[ a - \frac{a^2}{2} \right] = \frac{1}{2}$$

The supremum comes on straight lines, that minimize arc-length i.e. the second term, so consider $\omega(\tau) = a \tau$, and $a = 1$ is the maximizer.
Now we solve the same problem using Entropy Asymptotics by using a result of Paul Levý that the following have the same probability distributions

\[
P \left\{ \sup_{0 \leq s \leq t} \beta(s) \leq \alpha \right\} = P \{ t \ell_t(\beta(\cdot), 0) \}
\]

Thus we have that

\[
h(t) = E \left[ e^{\sup_{0 \leq s \leq t} \beta(s)} \right] = E \left[ e^{t \ell_t(\beta(\cdot), 0)} \right] = E \left[ e^{t \Phi[\ell_t(\beta(\cdot), 0)]} \right], \text{ where } \Phi[f] = f(0)
\]

So from Entropy Asymptotics we get

\[
\lim_{t \to \infty} \frac{1}{t} \ln h(t) = \lim_{t \to \infty} \frac{1}{t} E \left[ e^{t \Phi[\ell_t(\beta(\cdot), 0)]} \right] = \sup_{f \in \mathcal{F}} \left[ f(0) - \frac{1}{8} \int_{-\infty}^{\infty} \frac{[f'(y)]^2}{f(y)} \, dy \right]
\]

Recall that \( f \in \mathcal{F} \) is a probability distribution, and so the maximizing family of functions (proven below) is \( f_a(y) = ae^{-2a|y|} \)
An Example Using Action and Entropy Asymptotics

- Recall that $f \in \mathcal{F}$ is a probability distribution, and so the maximizing family of functions (proven below) is $f_a(y) = ae^{-2a|y|}$

- We can write

$$f_a(y) = ae^{-2a|y|} = \begin{cases} ae^{-2ay} & y \geq 0 \\ ae^{2ay} & y < 0 \end{cases},$$

so $f'_a(y) = \begin{cases} -2a^2e^{-2ay} & y \geq 0 \\ 2a^2e^{2ay} & y < 0 \end{cases}$, and so

$$[f'_a(y)]^2 = \begin{cases} 4a^4e^{-4ay} & y \geq 0 \\ 4a^4e^{4ay} & y < 0 \end{cases} = 4a^4e^{-4a|y|}$$

- This gives us

$$\sup_{a > 0} \left[ f(0) - \frac{1}{8} \int_{-\infty}^{\infty} \frac{[f'(y)]^2}{f(y)} \, dy \right] = \sup_{a > 0} \left[ a - \frac{1}{8} \int_{-\infty}^{\infty} 4a^3e^{-2a|y|} \, dy \right]$$

$$= \sup_{a > 0} \left[ a - \frac{a^2}{2} \int_{-\infty}^{\infty} ae^{-2a|y|} \, dy \right] = \sup_{a > 0} \left[ a - \frac{a^2}{2} \right] = \frac{1}{2},$$

which occurs at $a = 1$.
Now we find the maximizing family of functions by the same transformation as before
1. \( \sqrt{f(y)} = \psi(y) \) or \( f(y) = \psi^2(y) \), and so
2. \( f(0) = \psi^2(0) \)
3. \( \frac{1}{4} \left( \frac{f'(y)^2}{f(y)} \right) = [\psi'(y)]^2 \)

And so we obtain
\[
\sup_{f \in F} \left[ f(0) - \frac{1}{8} \int_{-\infty}^{\infty} \frac{[f'(y)]^2}{f(y)} \, dy \right] = \sup_{\psi \in L^2 \mid ||\psi||=1} \left[ \psi^2(0) - \frac{1}{2} \int_{-\infty}^{\infty} [\psi'(y)]^2 \, dy \right]
\]

Let \( \psi(0) = a \) we get the following constrained Euler-Lagrange equation
\[
\psi''(y) - 2\lambda \psi(y), \quad \psi(0) = a
\]
This is maximized with a stretched version of \( \psi(y) = e^{-2|y|} \)
Let $\Omega \subset \mathbb{R}^2$ be an open domain with sufficiently smooth boundary, $\partial \Omega$, so that the following problem has a unique solution

$$\frac{1}{2} \Delta u + \lambda u = 0, \text{ with } u = 0 \text{ on } \partial \Omega$$

Under these circumstances we know that

1. $\exists 0 < \lambda_1 < \lambda_2 < \cdots$ a discrete spectrum
2. $\exists u_1(x, y) < u_1(x, y) < \cdots$ corresponding normalized eigenfunctions

Consider

$$C(\lambda) = \sum_{\lambda_j < \lambda} 1 = \# \text{ of eigenvalues } < \lambda$$

$C(\lambda)$ is an increasing function in $\lambda$, and Hermen Weyl proved that

$$C(\lambda) \sim \frac{|\Omega|\lambda}{2\pi} \text{ as } \lambda \to \infty$$

Additionally, Carleman proved that

$$\sum_{\lambda_j < \lambda} u(x, y) \sim \frac{\lambda}{2\pi}, \forall (x, y) \in \Omega \text{ as } \lambda \to \infty$$
Kac’s Drum

- Now consider starting a BM at \((x_0, y_0) \in \Omega\)
- Let \(p(x_0, y_0, x, y, t)\) be the probability density function of a 2D BM starting at \((x_0, y_0)\) reaching \((x, y)\) at time \(t\) without hitting \(\partial \Omega\)
- Einstein-Smoluchowski: Then \(p(x_0, y_0, x, y, t)\) is the solution to
  \[
  \frac{\partial p}{\partial t} = \frac{1}{2} \Delta p \text{ in } \Omega
  \]
  \[p = 0 \text{ on } \partial \Omega, \quad \forall t > 0\]
- We note that as \(t \to 0\)
  \[
  \int_{\Omega} g(x, y)p(x_0, y_0, x, y, t) \, dx \, dy \to g(x_0, y_0)
  \]
- Assume we can find \(p\) using separation of variables:
  \[p(x_0, y_0, x, y, t) = T(t)U(x, y)\], then
  \[
  T'U = \frac{T}{2} \Delta U, \quad U = 0 \text{ on } \partial \Omega, \quad \forall t > 0
  \]
  \[
  \frac{T'}{T} = \frac{\Delta U}{2} = -\lambda \text{ yields}
  \]
  \[T(t) = e^{-\lambda t}, \text{ and } U = \text{ the eigenfunction corresponding to } \lambda\]
So this means that we can write explicitly

\[ p(x_0, y_0, x, y, t) = \sum_{j=1}^{\infty} e^{-\lambda_j t} u_j(x_0, y_0) u_j(x, y), \]  

and so we know

\[ p(x_0, y_0, x_0, y_0, t) = \sum_{j=1}^{\infty} e^{-\lambda_j t} u_j^2(x_0, y_0) \]

Let \( p^*(x_0, y_0, x, y, t) \) be the probability density function of unrestricted 2D BM starting at \((x_0, y_0)\) reaching \((x, y)\) at time \(t\)

\[ p^*(x_0, y_0, x, y, t) = \frac{1}{2\pi t} e^{-\frac{(x-x_0)^2}{2t} - \frac{(y-y_0)^2}{2t}} \]

Thus we conclude that

\[ \sum_{j=1}^{\infty} e^{-\lambda_j t} u_j^2(x_0, y_0) \sim p^*(x_0, y_0, x, y, t) \sim \frac{1}{2\pi t} \text{ as } t \to 0\]
Kac’s Drum

- **Karamata Tauberian Theorem:** Consider

\[ f(t) = \int_0^\infty e^{-\lambda t} \, d\alpha(\lambda), \text{ and assume} \]

1. The above Laplace-Stiltje’s transform exists
2. \( \alpha(\lambda) \) is non-decreasing on \((0, \infty)\)

- If \( f(t) \sim At^{-\gamma} \) as \( t \to 0 \) for \( A \) and \( \gamma \) constants then

\[ \alpha(\lambda) \sim \frac{A\lambda^\gamma}{\Gamma(\gamma + 1)} \text{ as } \lambda \to \infty (\lambda \to 0) \]

- We now apply the Karamata Tauberian Theorem to

\[ f(t) = \int_0^\infty e^{-\lambda t} \, d\alpha(\lambda) = \sum_{j=1}^{\infty} e^{-\lambda_j t} u_j^2(x_0, y_0), \text{ where } \alpha(\lambda) = \sum_{\lambda_j < \lambda} u_j^2(x_0, y_0) \]

- We know \( f(t) \sim \frac{1}{2\pi t} \) as \( t \to 0 \), and so \( \alpha(\lambda) \sim \frac{\lambda}{2\pi} \) as \( \lambda \to \infty \)

- By integrating this over \( \Omega \) we get Weyl’s theorem
1. Let $\Omega \in \mathbb{R}^3$ be a bounded closed domain
2. Let $r(t) \in \mathbb{C}$ be a continuous function starting at the origin
3. Let $\chi_\Omega(\cdot)$ be the indicator function of $\Omega$

Consider the following functional on $\mathbb{C}$

$$T_\Omega (y, r(\cdot)) = \int_0^\infty \chi_\Omega(y + r(\tau)) \, d\tau, \quad y \in \mathbb{R}^3$$

This functional is the total occupations time of $r(\cdot)$, a 3D BM, in $\Omega$ translated by $y$

Now impose Wiener measure on $\mathbb{C}$ and consider the following Wiener integral

$$E \{ T_\Omega (y, r(\cdot)) \} = \int_0^\infty P \{ y + r(\tau) \in \Omega \} \, d\tau$$

Note that because we are using Wiener measure we know

$$P \{ y + r(\tau) \in \Omega \} = \frac{1}{(2\pi \tau)^{3/2}} \int_0^\infty e^{-\frac{|r-y|^2}{2\tau}} \, dr$$
We now use Fubini’s theorem to exchange the order of integration

\[
E \left\{ T_\Omega (y, r(\cdot)) \right\} = \int_{\Omega} dr \int_{0}^{\infty} \frac{1}{(2\pi \tau)^{3/2}} e^{-\frac{|r-y|^2}{2\tau}} d\tau
\]

\[
= \frac{1}{2\pi} \int_{\Omega} \frac{dr}{|r-y|} < \infty \text{ in } \mathbb{R}^3
\]

We see that in \( \mathbb{R}^3 \) AE BM path starting at \( y \) spends a finite amount of time in \( \Omega \)

Now consider the \( k \)th moment of the occupation time

\[
E \left\{ T^k_\Omega (y, r(\cdot)) \right\} = \frac{k!}{(2\pi)^k} \int_{\Omega} \cdots \int_{\Omega} \frac{dr_1}{|r_1 - y|} \frac{dr_2}{|r_2 - r_1|} \cdots \frac{dr_k}{|r_k - r_{k-1}|} \quad k = 1, 2, \ldots
\]

We focus on the second moment, \( k = 2 \)

\[
E \left\{ T^2_\Omega (y, r(\cdot)) \right\} = \int_{0}^{\infty} \int_{0}^{\infty} P \{ y + r(\tau_1) \in \Omega \} P \{ y + r(\tau_2) \in \Omega \} \ d\tau_1 \ d\tau_2
\]
We focus on the second moment, $k = 2$

$$E \left\{ T^2_\Omega (y, r(\cdot)) \right\} = \int_0^\infty \int_0^\infty P \{ y + r(\tau_1) \in \Omega \} P \{ y + r(\tau_2) \in \Omega \} \, d\tau_1 \, d\tau_2$$

$$= 2 \int \int_{0 \leq \tau_1 < \tau_2 < \infty} d\tau_1 \, d\tau_2 \int_{\Omega} \int_{\Omega} \frac{1}{(2\pi \tau_1)^{3/2}} e^{-\frac{|r_1 - y|^2}{2\tau_1}} \frac{1}{[2\pi(\tau_2 - \tau_1)]^{3/2}} e^{-\frac{|r_2 - r_1|^2}{2(\tau_2 - \tau_1)}} \, dr_1 \, dr_2$$

$$= \frac{2}{(2\pi)^2} \int_{\Omega} \int_{\Omega} \frac{dr_1}{|r_1 - y|} \frac{dr_2}{|r_2 - r_1|}$$

The formula for the $k$th moment suggests that we should consider the following eigenvalue problem

$$\frac{1}{2\pi} \int_{\Omega} \frac{\phi(\rho)}{|r - \rho|} \, d\rho = \lambda \phi(r), \quad r \in \Omega$$
The integral kernel in the eigenvalue problem is Hilbert-Schmidt
1. Since the single integral is convergent, we have
\[ \int_{\Omega} \int_{\Omega} \frac{1}{|r - \rho|^2} \, dr \, d\rho < \infty \]
2. We also need to show that the kernel is positive definite:
\[ \int_{\Omega} \int_{\Omega} \frac{\phi(r)\phi(\rho)}{|r - \rho|} \, dr \, d\rho > 0 \quad \forall \phi(\rho) \neq 0 \text{ in } L^2(\Omega) \]

Note that:
\[
\frac{1}{2\pi} \frac{1}{|r - \rho|} = \int_0^\infty \frac{1}{(2\pi \tau)^{3/2}} e^{-\frac{|r-y|^2}{2\tau}} \, d\tau =
\int_0^\infty d\tau \frac{1}{(2\pi \tau)^{3/2}} \frac{\tau^{3/2}}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} e^{i\zeta \cdot (r-\rho)} e^{-\frac{1}{2\tau} |\zeta|^2} \, d\zeta =
\frac{1}{(2\pi)^3} \int_0^\infty d\tau \int_{\mathbb{R}^3} d\zeta e^{i\zeta \cdot (r-\rho)} e^{-\frac{1}{2\tau} |\zeta|^2} \]
Brownian Motion

Advanced Topics

Probabilistic Potential Theory

Probabilistic Potential Theory

So

\[ \int_{\Omega} \int_{\Omega} \frac{\phi(r) \phi(\rho)}{|r - \rho|} \, dr \, d\rho = \]

\[ \frac{1}{(2\pi)^3} \int_{0}^{\infty} d\tau \int_{\mathbb{R}^3} d\zeta e^{-\frac{1}{2} |\zeta|^2 \tau} \left| \int_{\Omega} \phi(\rho) e^{i\zeta \cdot \rho} \, d\rho \right|^2 > 0, \, \forall \phi(\rho) \neq 0 \text{ in } L^2(\Omega) \]

With the kernel being Hilbert-Schmidt, we know that the integral equation has

1. Discrete spectrum: \( \lambda_1, \lambda_2, \cdots \)
2. With corresponding eigenfunctions that form a complete, orthonormal basis for \( L^2(\Omega) \)

Lemma:

\[ \frac{1}{k!} E \left\{ T^k_{\Omega} (y, r(\cdot)) \right\} = \sum_{j=1}^{\infty} \lambda_j^{k-1} \int_{\Omega} \phi_j(r) \, dr \int_{\Omega} \frac{\phi_j(\rho)}{|\rho - y|} \, d\rho \]

1. This holds for all \( y \in \mathbb{R}^3 \)
2. If \( y \in \Omega \), then we note that

\[ \frac{1}{2\pi} \int_{\Omega} \frac{\phi_j(\rho)}{|\rho - y|} \, d\rho = \lambda_j \phi_j(y) \]
Proof: Recall that

\[
\frac{1}{k!} E \left\{ T_{\Omega}^k (y, r(\cdot)) \right\} = \frac{1}{(2\pi)^k} \int_{\Omega} \cdots \int_{\Omega} \frac{dr_1}{|r_1 - y|} \frac{dr_2}{|r_2 - r_1|} \cdots \frac{dr_k}{|r_k - r_{k-1}|}
\]

We recognize this as an iterated integral equation of the form

\[ a(y, r_1) a(r_1, r_2) \cdots a(r_{k-1}, r_k) \]

We can then rewrite this using Mercer’s theorem representation of the kernel of the integral operator

\[
\frac{1}{|\rho - y|} = \sum_{j=1}^{\infty} \lambda_j \phi_j(\rho) \phi_j(y)
\]

Next we apply Mercer’s theorem only to the terms not involving \( y \) to get

\[
\frac{1}{k!} E \left\{ T_{\Omega}^k (y, r(\cdot)) \right\} = \frac{1}{2\pi} \int_{\Omega} \frac{1}{|r_1 - y|} \int_{\Omega} \sum_{j=1}^{\infty} \lambda_j^{k-1} \phi_j(r_1) \phi_j(r_k) dr_1 dr_k
\]
To review we have that
\[
\frac{1}{k!} E \left\{ T^k_{\Omega} (y, r(\cdot)) \right\} = \begin{cases} 
\sum_{j=1}^{\infty} \lambda_j^{k-1} \int_{\Omega} \phi_j(r) \, dr \int_{\Omega} \frac{\phi_j(\rho)}{|\rho - y|} \, d\rho, & y \in \mathbb{R}^3 \\
\sum_{j=1}^{\infty} \lambda_j^k \int_{\Omega} \phi_j(r) \phi_j(y) \, dr, & y \in \Omega 
\end{cases}
\]

Now let us consider the moment generation function (Laplace transform) with \( z \in \mathbb{C} \)
\[
E \left\{ e^{z T_{\Omega}(y, r(\cdot))} \right\} = \sum_{k=0}^{\infty} \frac{z^k}{k!} E \left\{ T^k_{\Omega} (y, r(\cdot)) \right\}
\]

Now we use the above lemma to get
\[
= 1 + \frac{z}{2\pi} \sum_{j=1}^{\infty} \left( \frac{1}{1 - \lambda_j z} \right) \int_{\Omega} \phi_j(r) \, dr \int_{\Omega} \frac{\phi_j(\rho)}{|\rho - y|} \, d\rho
\]

1. This series converges if \( |z| < \frac{1}{\lambda_{\text{max}}} \)
2. The moment generating function is analytic if \( \Re\{z\} < 0 \) since \( T_{\Omega} \geq 0 \)
3. The last series is analytic for \( \Re\{z\} < 0 \), so by analytic continuation this identity holds with \( \Re\{z\} < 0 \)
Let $u > 0$ and define

$$h(y, u) = E \left\{ e^{-uT\Omega(y, r(\cdot))} \right\} = 1 - \frac{u}{2\pi} \sum_{j=1}^{\infty} \left( \frac{1}{1 + \lambda_j u} \right) \int_{\Omega} \phi_j(r) \, dr \int_{\Omega} \frac{\phi_j(\rho)}{|\rho - y|} \, d\rho$$

This series converges on compact sets in $\mathbb{C}$ because

1. $\frac{1}{1 + \lambda_j u} < 1$

2. $\left( \sum_{j=1}^{\infty} \left( \int_{\Omega} \phi_j(r) \, dr \int_{\Omega} \frac{\phi_j(\rho)}{|\rho - y|} \, d\rho \right)^2 \right) \leq \sum_{j=1}^{\infty} \left( \int_{\Omega} \phi_j(r) \, dr \right)^2 \sum_{j=1}^{\infty} \left( \int_{\Omega} \frac{\phi_j(\rho)}{|\rho - y|} \, d\rho \right)^2$

$$\leq |\Omega| \int_{\Omega} \frac{d\rho}{|\rho - y|} < \infty$$

This gives uniform convergence via the Weierstrass M-test and thus this is also analytic.
If \( y \in \Omega \) then we get

\[
h(y, u) = 1 - \sum_{j=1}^{\infty} \left( \frac{\lambda_j u}{1 + \lambda_j u} \right) \int_\Omega \phi_j(r) \, dr \, \phi_j(y)
\]

And so we can multiply both sides by \( \frac{1}{2\pi |y-r|} \) and integrate over \( \Omega \)

\[
\frac{1}{2\pi} \int_\Omega \frac{h(y, u) \, dy}{|y-r|} = \frac{1}{2\pi} \int_\Omega \frac{dy}{|y-r|} - \sum_{j=1}^{\infty} \left( \frac{\lambda_j u}{1 + \lambda_j u} \right) \int_\Omega \phi_j(\rho) \, d\rho \frac{1}{2\pi} \int_\Omega \frac{\phi_j(y) \, dy}{|y-r|}
\]

But we know that

\[
\frac{1}{2\pi} \int_\Omega \frac{dy}{|y-r|} = \sum_{j=1}^{\infty} \int_\Omega \phi_j(\rho) \, d\rho \frac{1}{2\pi} \int_\Omega \frac{\phi_j(y) \, dy}{|y-r|}
\]

Thus we can write that

\[
\frac{1}{2\pi} \int_\Omega \frac{h(y, u) \, dy}{|y-r|} = \sum_{j=1}^{\infty} \left( \frac{1}{1 + \lambda_j u} \right) \int_\Omega \phi_j(\rho) \, d\rho \frac{1}{2\pi} \int_\Omega \frac{\phi_j(y) \, dy}{|y-r|}
\]
We recognize the left hand side of the previous equation from (*), and so we use this to rewrite this as

\[ \frac{1}{2\pi} \int_{\Omega} \frac{h(y, u) \, dy}{|y - r|} = \frac{1}{u} \left( 1 - h(r, u) \right), \quad \forall r \in \mathbb{R}^3 \]

Moreover, if we rename variables we get

\[ \frac{1}{2\pi} \int_{\Omega} \frac{h(\rho, u) \, d\rho}{|y - \rho|} = \frac{1}{u} \left( 1 - h(y, u) \right), \quad \forall y \in \mathbb{R}^3 \] (**)

We now make some important observations
1. From (*) we see that if \( y \notin \Omega \) then \( h(y, u) \) is harmonic in \( y \), and the series in (*) converges uniformly on compact \( \Omega \)'s
2. Again from (*) we get

\[
h(y, u) > 1 - \frac{u}{2\pi} \left\{ \sum_{j=1}^{\infty} \left( \int_{\Omega} \phi_j(\rho) \, d\rho \right)^2 \right\}^{1/2} \left\{ \sum_{j=1}^{\infty} \left( \int_{\Omega} \frac{\phi_j(\rho)}{|\rho - y|} \, d\rho \right)^2 \right\}^{1/2} \]

\[
> 1 - \frac{u}{2\pi} |\Omega|^{1/2} \left( \int_{\Omega} \frac{d\rho}{|\rho - y|} \right)^{1/2}
\]
3. So we now know that \(0 \leq h(y, u) \leq 1\), and so
\[
\lim_{u \to \infty} h(y, u) = 1
\] (***)

4. And for from Courant-Hilbert II, pp. 245–246
\[
\Delta \left( \int_{\Omega} \frac{h(y, u) \, dy}{|y - r|} \right) = -4\pi h(y, u)
\]

› Now apply the Laplacian to both sides of (**) to get
\[
-2h(y, u) = -\frac{1}{u} \Delta h(y, u)
\]
or we get
\[
\frac{1}{2} \Delta h(y, u) - uh(y, u) = 0, \quad y \in \Omega
\]

› Now consider \(U(y) = \lim_{u \to \infty} (1 - h(y, u)) = P \{ T_{\Omega}(y, r(\cdot)) > 0 \}\), this is the capacitory potential (capacitance) and follows easily from the definition of the moment generating function
Example: Let $\Omega$ be a sphere of radius 1 centered at the origin

1. $h(y, u)$ is clearly spherically symmetric
2. $h(y, u)$ is harmonic outside $\Omega$, so we have

$$h(y, u) = \frac{\alpha(u)}{|y|} + \beta(u), \quad y \notin \Omega$$

3. From (***) we see that $\beta(u) = 1$ and so $h(y, u) = \frac{\alpha(u)}{|y|} + 1$ for $y \in \Omega$
4. We also know that for $y \in \Omega$ we have

$$h(y, u) = \gamma(u) \frac{\sinh(\sqrt{2u} |y|)}{|y|}$$

5. If we substitute this into the equation (**) we get that $\gamma(u) = \frac{1}{\sqrt{2u} \cosh(a\sqrt{2u})}$
6. $h(y, u)$ is continuous $\forall y$ so from the uniform convergence of the series, and so

$$\frac{\alpha(u)}{a} + 1 = \frac{1}{\sqrt{2u} \cosh(\sqrt{2u}a)} \frac{\sinh(\sqrt{2ua})}{a}$$

to finally give us

$$h(y, u) = \begin{cases} 1 - \frac{1}{|y|} & y \notin \Omega, \\ \frac{1 - \frac{\tanh(a\sqrt{2u})}{a\sqrt{2u}}}{\sinh(\sqrt{2u}|y|)} \frac{\sinh(\sqrt{2u}a |y|)}{\sqrt{2u} \cosh(\sqrt{2ua}) |y|} & y \in \Omega \end{cases}$$
Recall that

\[ U(y) = \lim_{u \to \infty} (1 - h(y, u)) = P \{ T_{S(0,a)}(y, r(\cdot)) > 0 \} = \begin{cases} \frac{a}{|y|}, & y \notin \Omega \\ 1, & y \in \Omega \end{cases} \]

This is the capacitory potential of \( S(0, a) \)

Now back to the general case, \( \forall y \in \mathbb{R}^3 \) we have

\[
1 - E \left\{ e^{-uT_{\Omega}(y,r(\cdot))} \right\} = \sum_{j=1}^{\infty} \left( \frac{1}{\lambda_j + \frac{1}{u}} \right) \int_{\Omega} \phi_j(r) \, dr \, \frac{1}{2\pi} \int_{\Omega} \frac{\phi_j(\rho) \, d\rho}{|\rho - y|}
\]

1. We note that \( 0 \leq 1 - h(y, u) \leq 1 \)
2. The function \( 1 - h(y, u) \) is non-decreasing in \( u \): \( 1 - h(y, u_1) \leq 1 - h(y, u_2) \)
   if \( u_1 < u_2 \)
3. This is true due to the following
   3.1 \( 0 \leq e^{-uT_{\Omega}(y,r(\cdot))} \leq 1 \) and
   3.2

\[
\lim_{u \to \infty} e^{-uT_{\Omega}(y,r(\cdot))} = \begin{cases} 0, & T_{\Omega} > 0 \\ 1, & T_{\Omega} = 0 \end{cases}
\]
From the previous results and the bounded convergence theorem we have

\[ U(y) = \lim_{u \to \infty} (1 - h(y, u)) = P \{ T_\Omega (y, r(\cdot)) > 0 \} \]

and hence also

\[ U(y) = \lim_{u \to \infty} \sum_{j=1}^{\infty} \left( \frac{1}{1 + \lambda_j} \right) \int_\Omega \phi_j(r) \, dr \frac{1}{2\pi} \int_\Omega \frac{\phi_j(\rho) \, d\rho}{|\rho - y|} \]

and this holds \( \forall y \in \mathbb{R}^3 \)

**Case 1.** Let \( y \in \Omega^o \) (the interior), clearly the continuity of \( r(\cdot) \) immediately implies

\[ U(y) = P \{ T_\Omega (y, r(\cdot)) > 0 \} = 1 \]

**Remark:** with \( y \in \Omega^o \) we have \( U(y) = 1 \) and so we have the following summability result

\[ 1 = \lim_{u \to \infty} \sum_{j=1}^{\infty} \left( \frac{\lambda_j}{\lambda_j + 1} \right) \int_\Omega \phi_j(r) \, dr \frac{1}{2\pi} \int_\Omega \frac{\phi_j(\rho) \, d\rho}{|\rho - y|} \]
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Case 2. Let \( y \not\in \Omega \), we already know that \( 1 - h(y, u) \) is harmonic in \( y \), and it is nondecreasing in \( u \), and the previous limit in \( u \) exists and equals \( P \{ T_\Omega(y, r(\cdot)) > 0 \} \), thus by Harnack’s theorem, \( \mathcal{U}(y) \) is harmonic with \( y \not\in \Omega \). Assume that \( \Omega \subset S(0, a) \), then

\[
P \{ T_\Omega(y, r(\cdot)) > 0 \} \leq P \{ T_{S(0, a)}(y, r(\cdot)) > 0 \}
\]

From the last problem this means

\[
P \{ T_\Omega(y, r(\cdot)) > 0 \} \leq \frac{a}{|y|}, \quad y \not\in S(0, a)
\]

and so \( \lim_{|y| \to \infty} \mathcal{U}(y) = 0 \)

Case 3. Let \( y_o \in \partial \Omega \), and assume that it is regular in the sense of Poincaré: \( \exists \) a sphere \( S(y_*, \epsilon) \) lying completely in \( \Omega \) so that \( y_o \in S(y_*, \epsilon) \) Consider now \( y \not\in \Omega \)

\[
\mathcal{U}(y) = P \{ T_\Omega(y, r(\cdot)) > 0 \} \geq P \{ T_{S(0, a)}(y, r(\cdot)) > 0 \} = \frac{\epsilon}{|y - y_*|}
\]

As \( y \to y_o \) with \( y \not\in \Omega \) we have \( \frac{\epsilon}{|y - y_*|} \to \frac{\epsilon}{|y_o - y_*|} \), and since \( \mathcal{U}(y) \leq 1 \) we have finally that

\[
\lim_{y \to y_o} \mathcal{U}(y) = 1
\]
Thus if \( \Omega \) is a closed and bounded region, each point on the boundary that is regular in the Poincaré sense has \( \mathcal{U}(y) \) as the capacitory potential of \( \Omega \).

Recall that

\[
\mathcal{U}(y) = \lim_{\delta \to 0} \sum_{j=1}^{\infty} \left( \frac{1}{\lambda_j + \delta} \right) \int_{\Omega} \phi_j(r) \, dr \frac{1}{2\pi} \int_{\Omega} \frac{\phi_j(\rho)}{|\rho - y|} \, d\rho
\]

We note that this implies that

\[
\lim_{|y| \to \infty} |y|(1 - h(|y|, u)) = \frac{1}{2\pi} \int_{\Omega} uh(\rho, u) \, d\rho
\]

Again assume that \( \Omega \in S(0, a) \), then \( h(y, u) = E \{ e^{-uT_\Omega} \} \geq \{ e^{-uT_{S(0,a)}} \} \), there for \( y \not\in S(0, a) \) we have \( h(y, u) \geq 1 - \frac{a}{|y|} \) or \( 1 - h(y, u) \leq \frac{a}{|y|} \) and so

\[
\frac{u}{2\pi} \int_{\Omega} h(\rho, u) \, d\rho \leq a
\]
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