CNT4406/5412 Network Security

Public Key Algorithms

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Introduction

- Each principal has a pair of public and secret numbers (e, d)
  - ➠ public key is announced to the public
  - ➠ private key is kept secret

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- Each principal has a pair of public and secret numbers (e, d)
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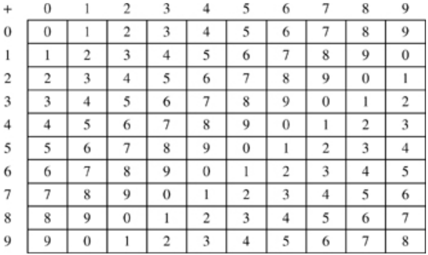
  - ➠ private key is kept secret
- Public key algorithms are different in design
  - ➠ Diffie-Hellman allows establishment of a shared secret

	encryption	signature	key exchange
RSA	y	y	y
Diffie-Hellman	n	n	y
DSA	n	y	n

Modular Addition

- Modular addition is reversible for all numbers $a < n$
  ➠ Caesar cipher uses modular addition



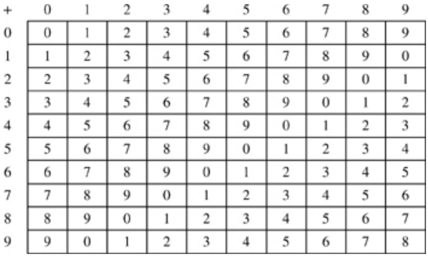
+	0	1	2	3	4	5	6	7	8	9
0	0	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9	0
2	2	3	4	5	6	7	8	9	0	1
3	3	4	5	6	7	8	9	0	1	2
4	4	5	6	7	8	9	0	1	2	3
5	5	6	7	8	9	0	1	2	3	4
6	6	7	8	9	0	1	2	3	4	5
7	7	8	9	0	1	2	3	4	5	6
8	8	9	0	1	2	3	4	5	6	7
9	9	0	1	2	3	4	5	6	7	8

Modular Addition

- Modular addition is reversible for all numbers $a < n$

  ➠ Caesar cipher uses modular addition
- Additive inverse of a is $-a \mod n$

  ➠ e.g., 7 is 3 's additive inverse in $\mod 10$



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5	5	6	7	8	9	0	1	2	3	4
6	6	7	8	9	0	1	2	3	4	5
7	7	8	9	0	1	2	3	4	5	6
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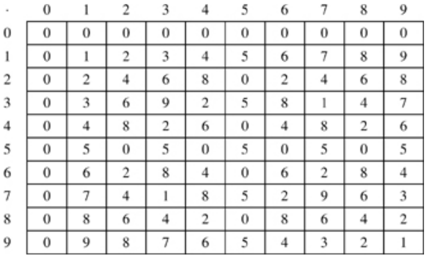
Modular Multiplication

- It is reversible for numbers relatively-prime to and less than *n*
  ➠ multiplicative inverse can be calculated by Euclid's algorithm



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3	0	3	6	9	2	5	8	1	4	7
4	0	4	8	2	6	0	4	8	2	6
5	0	5	0	5	0	5	0	5	0	5
6	0	6	2	8	4	0	6	2	8	4
7	0	7	4	1	8	5	2	9	6	3
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9	0	9	8	7	6	5	4	3	2	1

Modular Multiplication

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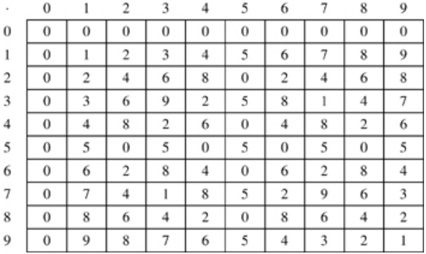
  ➠ multiplicative inverse can be calculated by Euclid's algorithm
- Totient function $\phi(n)$: how many numbers less than and relatively-prime to n
  ➠ if n is prime, $\phi(n) = n - 1$
  ➠ if $n = pq$ and p, q are prime, $\phi(n) = (p-1)(q-1) = \phi(p)\phi(q)$



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5	0	5	0	5	0	5	0	5	0	5
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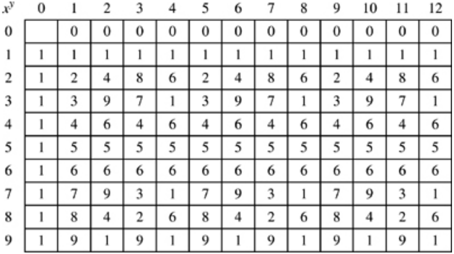
Modular Exponentiation

- Euler's theorem: $x^y \mod n = x^{y \mod \phi(n)} \mod n$ if n is prime or a product of two distinct primes
  - e.g., $n = 10, \phi(n) = 4, x^1 = x^5 = x^9 \mod 10$



x^y	0	1	2	3	4	5	6	7	8	9	10	11	12
0		0	0	0	0	0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	1	2	4	8	6	2	4	8	6	2	4	8	6
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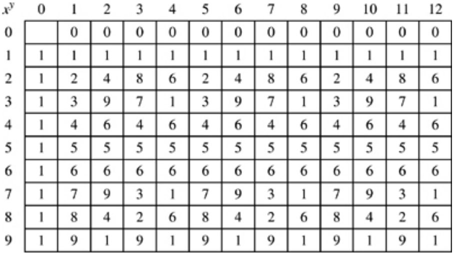
- Exponentiative inverse: $yz = 1 \mod \phi(n) \rightsquigarrow (x^y)^z = x^{yz} = x$
  ➠ $yz = 1 \mod \phi(n)$: z is y 's multiplicative inverse $\mod \phi(n)$



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- RSA is slow, mostly used to encrypt/sign short messages
  - ➠ e.g., shared session keys or message digests

Key Generation

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  ➠ factoring n (512 bit) into p and q is hard
- Public key is $< e, n >$, e relatively prime to $\phi(n)$,
  private key is $< d, n >$, $ed = 1 \mod \phi(n)$

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- Who are the principles of these operations???

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➠ $m = 80$

- encryption: $c = m^e \mod n = 80^{39} \mod 253 = 37$
- decryption: $m = c^d \mod n = 37^{79} \mod 253 = 80$

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- encryption: $c = m^e \mod n = 80^{39} \mod 253 = 37$
- decryption: $m = c^d \mod n = 37^{79} \mod 253 = 80$
- signature: $s = m^d \mod n = 80^{79} \mod 253 = 224$
- verification: $m = s^e \mod n = 224^{39} \mod 253 = 80$

Why RSA Works

$n = pq, \phi(n) = (p-1)(q-1)$, and $de = 1 \mod \phi(n)$

$\rightsquigarrow x^{de} = x \mod n$ (Euler's theorem, $x \in Z_n$)

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➠ encryption: x^e, decryption: $(x^e)^d = x^{ed} = x$

➠ signature and verification are the reverse

Why RSA is Secure

- Public key $< e, n >$ is a public information
- Factoring large number n into $p \times q$ is difficult
  ➠ if factored $\leadsto \phi(n) = (p-1)(q-1)$
  $\leadsto d = e^{-1} \mod \phi(n) \leadsto < d, n >$

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  - ➠ 1024 bits are consider secure for now, 2048 bits are better

Implementation

- Basic operation: exponentiating with big numbers
- Generating RSA keys:
  ➠ finding big primes p and q, and selecting d and e

Exponentiating

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110110_2	$\Lsh$	$123^{54} = 171 \times 171 = 29241 = 87 \mod 678$

Exponentiating

Pseudo code to compute $a^x \mod t$, assuming x has k bits

```
$r = a$
for $i = k - 1$ to 1:
    $r = r \times r \mod t$
    if $x_i == 1$:
        $r = r \times a \mod t$
return $r$
```

Exponentiating: Timing Attacks

- Timing attack: to recover the private key from the running time of the decryption algorithm ($m = c^d \mod n$)

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  - ➠ attackers can determine bits of d by comparing time*
- To mitigate, use *blinding*: multiply the ciphertext by a random number before decryption

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Finding Big Primes

- Infinite number of primes, but thin out when getting bigger
  - ➠ probability of a random number n being prime is $\frac{1}{\ln n}$
  - ➠ e.g., 1 in 23 for a ten-digit number, 1 in 230 for hundred-digits

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  - ➠ probability of a random number n being prime is $\frac{1}{\ln n}$
  - ➠ e.g., 1 in 23 for a ten-digit number, 1 in 230 for hundred-digits
- Method: choose a random number then test if it is a prime

Finding Big Primes

Theorem

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  ➠ Carmichael numbers are special cases

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Miller and Rabin test

If n is a prime, the only $\mod n$ square roots of 1 are 1 and -1, but many square roots if n is not a power of a prime (exercise: why??)

Finding d and e

- Choose a number that is relatively-prime to $\phi(n)$ as e

  ➠ e is public and can be a small number such as 3 or 65537

Finding *d* and *e*

- Choose a number that is relatively-prime to $\phi(n)$ as *e*
  ➠ *e* is public and can be a small number such as 3 or 65537
- Compute *d* using Euclid's algorithm
  ➠ *d* must be big to avoid being searchable

Issues with $e = 3$

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  ➠ to address it, use random/individualized padding

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  ➠ assume message are padded on the right with random numbers
  ➠ to forge signature, digest the message to h, pad it on the right with zeros, then set the signature to $r = h^{\frac{1}{3}}$
  ➠ signature is forged because $r^3 = h$ (padded with random numbers)

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RSA Threats: Smooth Numbers

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  - ➠ "small" primes provide flexible building blocks

    attackers can forge signatures on any product from his collection

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0	2	at least 8 random non-zero octets	0	data

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Diffie-Hellman

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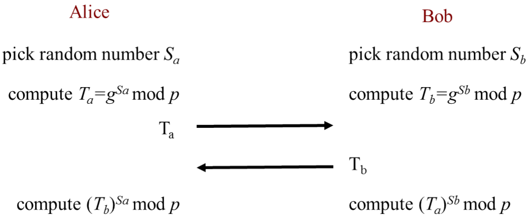
  - ➠ i.e. not suitable for public key encryption
- Diffie-Hellman does not provide authentication of the principles
  - ➠ you could negotiate a key with a complete stranger

Diffie-Hellman Protocol

- Publicly publish two numbers, p and g
  - ➠ p is a large prime (about 512 bits), and $g < p$



Alice

pick random number S_a

compute $T_a = g^{Sa} \bmod p$

$T_a \longrightarrow$

Bob

pick random number S_b

compute $T_b = g^{Sb} \bmod p$

$\longleftarrow T_b$

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Diffie-Hellman Protocol

- Publicly publish two numbers, p and g
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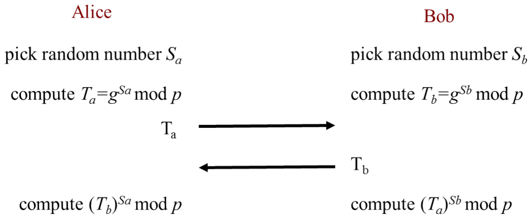
- Alice and Bob exchange two numbers T_a and T_b
- They agree upon $g^{S_a S_b} \mod p$ after DH exchange



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- Alice computes $K = T_b^{S_a} \mod p = 248^{97} \mod 353 = 160$
  Bob computes $K = T_a^{S_b} \mod p = 40^{233} \mod 353 = 160$

Diffie-Hellman Offline Mode

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- Bob computes $K_{ab} = T_a^{S_b}$, and decrypt the message with it

Why DH is Secure

- Discrete logarithms problem is difficult
  - ➠ given $g^S \mod p$, g, and p, it is computationally difficult to get S
  - ➠ no guarantee, but remember Fundamental Tenet of Cryptograph?

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- For “obscure mathematical reasons:”
  - ➠ p and $\frac{p-1}{2}$ should be prime
  - ➠ $g^{\frac{p-1}{2}} = -1 \mod p$

Man-in-the-Middle Attack

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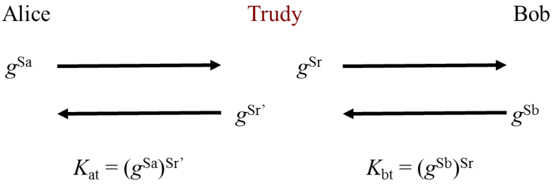
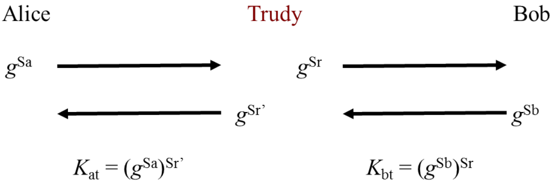


Man-in-the-Middle Attack

- Alice and Bob both negotiated a key with Trudy
- Trudy forwards messages between Alice and Bob
  - Alice → Bob: $E(K_{bt}, D(K_{at}, c_{ab}))$
  - Bob → Alice: $E(K_{at}, D(K_{bt}, c_{ba}))$



Defense Against MITM Attacks

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- Authenticated Diffie-Hellman
  - ➠ share a secret or publish one's public key in advance
  - ➠ there are various ways to mix Diffie-Hellman and authentication

DSS (Digital Signature Standard)

- An algorithm designed by NIST for digital signature
  - ➠ the algorithm is known as DSA (digital signature algorithm)
  - ➠ messages are digested first, the digest is then signed
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  - ➠ the sister function, SHS, is a message digesting function
- Signing is faster than verification
  - ➠ e.g., for use in the smart cards

Digital Signature Algorithm

To generate DSA signature

1. Generate p and q (public)
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4. Choose a per message public/private key pair $< T_m, S_m >$
   - ➠ pick a random S_m, let $T_m = ((g^{S_m} \mod p) \mod q)$

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3. if $z = T_m$, the signature is verified

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➠ $z = g^x T^y = g^{d_m S_m v} g^{S T_m S_m v} = g^{(d_m + ST_m) S_m v}$
$= g^{S_m} = T_m \mod p \mod q$

DSA Pitfalls

Private key (S) can be revealed if

- per-message private key S_m is leaked
  ➠ $X_m = S_m^{-1}(d_m + ST_m) \rightsquigarrow (X_m S_m - d_m) T_m^{-1} \mod q = S \mod q$

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- two messages are signed with the same per-message private key
  ➠ $(X_m - X'_m)^{-1}(d_m - d'_m) \mod q = S_m \mod q$

Summary

- Modular arithmetic
- RSA
- Diffie-Hellman
- DSA

- Next lecture: authentication