Oblivious Routing in Fat-Tree Based System Area Networks with Uncertain Traffic Demands*

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***Abstract*—We study oblivious routing in fat-tree based system area networks with deterministic routing under the assumption that the traffic demand is uncertain. The performance of a routing algorithm under uncertain traffic demands is characterized by the oblivious performance ratio that bounds the relative performance of the routing algorithm with respect to the optimal algorithm for any given traffic demand. We consider both single path routing where only one path is used to carry the traffic between each source-destination pair, and multi-path routing where multiple paths are allowed. For single path routing, we derive lower bounds of the oblivious performance ratio for different fat-trees and develop routing schemes that achieve the optimal oblivious performance ratios for commonly used topologies. Our evaluation results indicate that the proposed oblivious routing schemes not only provide the optimal worst case performance guarantees, but also outperform existing schemes in average cases. For multi-path routing, we show that it is possible to obtain an optimal scheme for all traffic demands (an oblivious performance ratio of 1). These results quantitatively demonstrate the performance difference between single path routing and multi-path routing in fat-trees.**

***Index Terms*—Oblivious routing, fat-tree, system area networks**

I. Introduction

The fat-tree topology has many properties that make it attractive for large scale interconnects and system area networks [17], [18]. Most importantly the bisection bandwidth of the fat-tree topology scales linearly with the number of network ports.[1] The topology is also inherently highly resilient with a large number of redundant paths. The fat-tree topology is very popular for building medium and large system area networks [15], [21]. In particular, it has been widely adopted in high performance computing (HPC) clusters that employ the off-the-shelf high speed system area networking technology, such as Myrinet [22] and InfiniBand [16].

Although the fat-tree topology provides rich connectivity, having a fat-tree topology alone does not guarantee high performance: the routing mechanism also plays a crucial role. Historically, adaptive routing, which dynamically builds the path for a packet based on the network condition, has been used with the fat-tree topology to achieve load balancing [18]. However, the routing in the current major system area networking technology, such as InfiniBand and Myrinet, is deterministic [16], [22]. For a fat-tree based system area network with deterministic routing, it is important to employ an efficient load balanced routing scheme in order to fully exploit the rich connectivity provided by the fat-tree topology.

Traditional load balanced routing schemes usually optimize the network usage for a given traffic demand. Such *demand specific* schemes may not be effective for system area networks where the traffic demand is uncertain and changing. Consider, for example, the traffic demand in a large HPC cluster. Since many users share such a system and can run different applications, the traffic demand depends both on how the processing nodes are allocated to different applications and on the communication requirement within each application. Hence, an ideal routing scheme should provide load balancing across all possible traffic patterns. This requirement motivates us to study *demand-oblivious routing schemes* that determine routes independent of the traffic demand. Our focus is on oblivious routing with routes being set deterministically since most current system area networks use deterministic routing. It has recently been demonstrated that oblivious routing can promise excellent performance guarantees with uncertain traffic demands in the Internet environment [1], [2], [28].

We investigate oblivious routing in fat-tree networks with deterministic routing under the assumption that the traffic demand is uncertain. For a given traffic demand, the performance of a routing scheme is measured by the *maximum link load* metric. The performance of a routing algorithm under uncertain traffic demands is characterized by the *oblivious performance ratio* [1]. The formal definition of the oblivious performance ratio will be introduced in the next section. Informally, a routing algorithm r has an oblivious performance ratio of x if for *any* traffic demand, the performance (maximum link load) of r is at most x times that of the optimal routing algorithm for this demand. An oblivious performance ratio of 1 means that the algorithm is optimal for all traffic demands.

This study focuses on the fat-tree topologies that are formed with m-port switches, where m is a parameter that is restricted to be a multiple of 2. Although the results are obtained for this type of fat-trees, the results, as well as our analysis techniques, can be extended to other types of fat-tree topologies. We consider both single path routing where only one path is used to carry the traffic between each source-destination pair, and

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[1] As originally defined [17], fat-trees are a very broad class of networks with substantial flexibility regarding the bisection bandwidth. The fat-tree topologies used in the current system area networks, which are also known as constant bisectional bandwidth (CBB) networks [21], scale bisection bandwidth linearly with the number of network ports. This paper focuses on this type of fat-trees.

multi-path routing where multiple paths are allowed. Let H be the height of the fat-tree. The major results of this paper include the following.

- For single path routing, we prove that (1) when $H = 3$, the oblivious performance ratio of any single path routing algorithm is at least $\sqrt{\frac{m}{2}}$; (2) when $H = 4$, the oblivious performance ratio of any single path routing algorithm is at least $\frac{m}{2}$; and (3) when $H > 4$, the oblivious performance ratio of any single path routing algorithm is at least $(\frac{m}{2})^{\lfloor \frac{H-2}{3} \rfloor}$. We develop optimal single path oblivious routing schemes that achieve the oblivious performance ratio lower bounds for the cases when $H = 3$ and $H = 4$, which implies that the lower bounds for these two cases are tight. These routing schemes are sufficient for most practical cases since the heights of most practical fat-trees are no more than 4. The results of our performance study indicate that the proposed optimal oblivious routing schemes not only provide the optimal worst case performance guarantees among all single path routing schemes, but also outperform existing schemes in average cases.
- For multi-path routing, we show that it is possible to obtain a scheme with an oblivious performance ratio of 1, that is, an optimal scheme for any traffic demand. This suggests that multi-path routing is much more effective in balancing network loads than single path routing in fat-trees. Note that although it is well known that single path routing is simple, but not as effective as multi-path routing in balancing network loads in general, the performance difference between single path routing and multi-path routing in fat-trees is not well understood. Without a clear understanding of the performance difference, it is difficult to make a wise decision about whether a system should use single path routing for its simplicity or multi-path routing for its performance. This paper quantifies the performance difference and resolves this issue.

The rest of the paper is organized as follows. In Section II, we formally define routing and the metrics for evaluating routing schemes, and specify the fat-tree topology. In Section III, we present the results for multi-path routing. In Section IV, we derive the lower bounds of the oblivious performance ratio for any single path routing scheme in fat-trees. In Section V, we give the optimal oblivious routing schemes for commonly used fat-trees. Section VI reports the results of our performance study. Section VII describes the related work. Finally, Section VIII concludes the paper.

II. Background

A. Routing and its performance metrics

Let the system have N processing nodes, numbered from 0 to $N-1$. The traffic demand is described by an $N \times N$ Traffic Matrix, TM. Each entry $tm_{i,j}$ in TM, $0 \leq i \leq N-1$, $0 \leq j \leq N-1$, denotes the amount of traffic from node i to node j. Let A be a set and $|A|$ be the size of the set.

A *routing* specifies how the traffic for each Source-Destination (SD) pair is routed across the network. We consider single path routing where only one path can be used for each SD pair, and multi-path routing where multiple paths can be used. In multi-path routing, each path for an SD pair routes a fraction of the traffic for the SD pair.

A multi-path routing is characterized by a set of paths $MP_{i,j} = \{MP^1_{i,j}, MP^2_{i,j}, ..., MP^{|MP_{i,j}|}_{i,j}\}$ for each SD pair (i,j), and the fraction of the traffic routed through each path $f_{i,j} = \{f^k_{i,j} | k = 1, 2, ..., |MP_{i,j}|\}$. $\sum_{k=1}^{|MP_{i,j}|} f^k_{i,j} = 1$. Let link $l \in MP^k_{i,j}$. The contribution of the traffic $tm_{i,j}$ to link l through path $MP^k_{i,j}$ is thus $tm_{i,j} \times f^k_{i,j}$. Single path routing is a special case of multi-path routing where $|MP_{i,j}| = 1$ and all traffic from node i to node j is routed through $MP^1_{i,j}$ ($f^1_{i,j} = 1$). Hence, a single path routing can be specified by a path $MP^1_{i,j}$ for each SD pair (i,j).

For a given traffic matrix, the performance of a routing is measured by the maximum link load. Since all links in a fat-tree network have the same capacity, the maximum link load is equivalent to the maximum link utilization. Let $Links$ denote the set of all links in the network. For a multi-path routing mr, the maximum link load is given by

$$MLOAD(mr, TM) = \max_{l \in Links} \{ \sum_{i,j,k \text{ such that } l \in MP^k_{i,j}} tm_{i,j} \times f^k_{i,j} \}.$$

For a single path routing sr, the formula simplifies to

$$MLOAD(sr, TM) = \max_{l \in Links} \{ \sum_{i,j \text{ such that } l \in P^1_{i,j}} tm_{i,j} \}.$$

An optimal routing for a given traffic matrix TM is a routing that minimizes the maximum link load. Formally, the optimal load for a traffic matrix TM is given by

$$OPTU(TM) = \min_{\text{r is a routing}} \{MLOAD(r, TM)\}.$$

The *performance ratio* of a routing r on a traffic matrix TM measures how far r is from being optimal on TM. It is defined as the maximum link load of r divided by the smallest possible maximum link load on TM [1].

$$PERF(r, TM) = \frac{MLOAD(r, TM)}{OPTU(TM)}$$

$PERF(r, TM)$ is at least 1. It is exactly 1 if and only if the routing is optimal for TM. The definition of the performance ratio follows the "competitive analysis" framework where performance guarantees of a certain solution are provided relative to the best possible solution. As such, the performance ratio is not directly related to the absolute network performance: the performance ratio of a routing scheme may be better when the network is under heavy loads and worse when the network is under light loads. The definition of performance ratio of a routing is extended to be with respect to a set of traffic matrices [1]. Let Γ be a set of traffic matrices, the performance ratio of a routing r on Γ is defined as

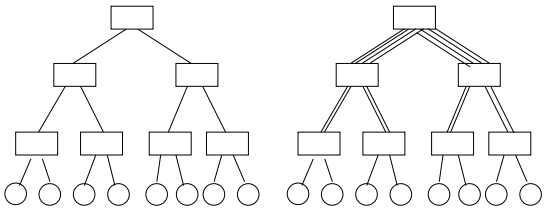
$$PERF(r, \Gamma) = \max_{TM \in \Gamma} \{PERF(r, TM)\}.$$

When the set Γ includes all possible traffic matrices, the performance ratio is referred to as the **oblivious performance ratio** [1]. The oblivious performance ratio of a routing r is denoted by $PERF(r)$. The oblivious performance ratio is the

worst performance ratio that a routing obtains with respect to all traffic matrices. A routing with a minimum oblivious ratio is an *optimal oblivious routing* scheme and its oblivious ratio is the optimal oblivious ratio of the network.

B. Fat-tree topology

In a fat-tree network, all links are bidirectional with the same capacity. Fig. 1 compares a binary tree with a binary fat-tree. In the binary tree, the number of links (and thus the aggregate bandwidth) is reduced by half at each level from the leaves to the root. This can cause congestion towards the root. The binary fat-tree topology remedies this situation by maintaining the same bandwidth at each level of the network.



(a) Binary tree (b) Binary fat−tree

Fig. 1. Binary tree and binary fat-tree topologies

The fat-tree topology shown in Fig. 1 (b) is not practical for building large networks due to the large nodal degree of the root. Alternatives were proposed to approximate such a topology using multi-stage networks that are formed by nodes with small nodal degrees [8], [15], [26]. For example, the fat-tree in Fig. 1 (b) can be approximated by the topology in Fig. 2. These alternatives trade connectivity with implementation simplicity. In this paper, we focus on one of such alternative: the fat-tree topologies formed by m-port switches, where m is a parameter that is restricted to a multiple of 2. Let an *internal node* in the fat-tree topology be a node with a degree more than 1. All internal nodes in our fat-tree topology has a degree of m (so that they can be realized by m-port switches). Such a topology is a minor generalization of the topology proposed in [15]. The technique we developed for this topology can easily be extended for other fat-tree variations.

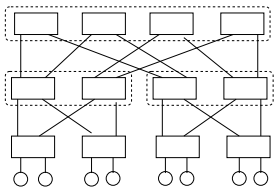


Fig. 2. Approximate the topology in Fig. 1 (b)

We will follow the naming convention in [15]: the fat-tree is called m-port n-tree and denoted as $FT(m,n)$. The parameter m in $FT(m,n)$, which must be a multiple of 2, specifies the nodal degree of all internal nodes in the topology. The parameter n specifies the number of levels of internal nodes in the topology. Thus, the height of $FT(m,n)$ is $n+1$, that is, $FT(m,n)$ is an $n+1$ level tree. In the rest of this paper, internal nodes in $FT(m,n)$ may also be referred to as *switches* since each of the internal nodes is realized by a switch when the topology is constructed. Similarly, leaf nodes may also be referred to as *processing nodes*. A 4-port 3-tree, $FT(4,3)$, is shown in Fig. 3.

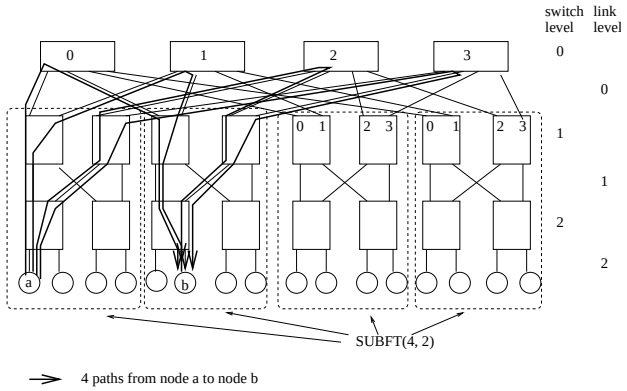


Fig. 3. The 4-port 3-three ($FT(4,3)$)

Next, we will describe how $FT(m,n)$ is formed. More details can be found in [15]. $FT(m,n)$ is formed by connecting the root level switches to m sub-fat-trees with $n-1$ levels of switches. We will use the notation $SUBFT(m,n-1)$ to denote the sub-fat-trees with $n-1$ levels of switches. $SUBFT(m,l)$ is different from $FT(m,l)$ in that $SUBFT(m,l)$ must provide (open ended) up links for the sub-fat-tree while $FT(m,l)$ does not have up links. $SUBFT(m,l)$ is recursively constructed as follows.

When $l=1$, $SUBFT(m,1)$ contains 1 m-port switch. $\frac{m}{2}$ of the ports in the switch connect to $\frac{m}{2}$ processing nodes, and $\frac{m}{2}$ ports remain open. We will call these opened ports *up-link ports* since they will be used to connect to the upper level switches. We denote the number of up-link ports in $SUBFT(m,l)$ as $nu(m,l)$. $nu(m,1)=\frac{m}{2}$. As will be shown later, $nu(m,l)=(\frac{m}{2})^l$. The up-link ports in $SUBFT(m,l)$ are numbered from 0 to $nu(m,l)-1$.

$SUBFT(m,l)$ is formed by connecting $nu(m,l-1)=(\frac{m}{2})^{l-1}$ m-port top level (of the sub-fat-tree) switches with $\frac{m}{2}$ $SUBFT(m,l-1)$'s. Each of the top level switches uses $\frac{m}{2}$ ports to connect to all $\frac{m}{2}$ of the $SUBFT(m,l-1)$'s. Let us number top level switches from 0 to $nu(m,l-1)-1$. The up-link ports i, $0 \le i < nu(m,l-1)$, in all of the $SUB(m,l-1)$'s are connected to top level switch i. The remaining $\frac{m}{2}$ ports in a top level switch are up link ports of $SUBFT(m,l)$. Top level switch i provides up-link ports $\frac{m}{2} \times i$ to $\frac{m}{2} \times (i+1)-1$ for $SUBFT(m,l)$. Fig. 4 (a) shows $SUBFT(m,1)$ and (b) shows the structure of $SUBFT(m,l)$. Clearly, $nu(m,l)=nu(m,l-1) \times \frac{m}{2}=(\frac{m}{2})^l$. Hence, each $SUBFT(m,l)$ has $(\frac{m}{2})^l$ up-link ports and connects to $(\frac{m}{2})^l$ processing nodes.

$FT(m,n)$ is formed by having $nu(m,n-1)=(\frac{m}{2})^{n-1}$ root level switches connecting with m $SUBFT(m,n-1)$'s. Let us number top level switches from 0 to $(\frac{m}{2})^{n-1}-1$. The up-link port i, $0 \le i < nu(m,n-1)$, in all of the $SUB(m,n-1)$'s is connected to top level switch i. Each of the m ports in the root level switch connects to one $SUBFT(m,n-1)$. The structure of $FT(m,n)$ is shown in

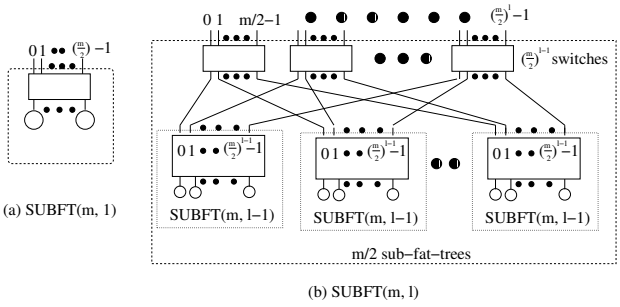


Fig. 4. $SUBFT(m,1)$ and $SUBFT(m,n)$

Fig. 5. $FT(m,n)$ supports $m(\frac{m}{2})^{n-1}$ processing nodes. The root level contains $nu(m,n-1)=(\frac{m}{2})^{n-1}$ switches and each of the other $n-1$ layers has $2\times(\frac{m}{2})^{n-1}$ switches. Hence, the total number of switches in $FT(m,n)$ is $(2n-1)\times(\frac{m}{2})^{n-1}$.

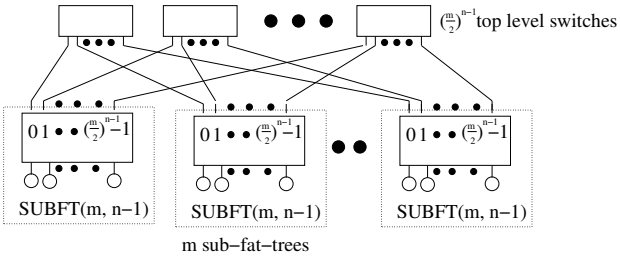


Fig. 5. The structure of $FT(m,n)$

Let us number the n levels of switches in $FT(m,n)$ from 0 to $n-1$ (root level switches being level 0 switches). We will classify the links according to their levels. For $0\le i<n-1$, the links connecting level i switches with level $i+1$ switches are called level i links. The links connecting level $n-1$ switches with the processing nodes are level $n-1$ links. All links in $FT(m,n)$ are bi-directional links: with an up channel for communication from a lower level switch to an upper level switch, and a down channel from an upper level switch to a lower level switch. We will use the term *level i up link* to denote an up channel from a level $i+1$ switch to a level i switch, and *level i down link* to denote a down channel from a level i switch to a level $i+1$ switch. A path between two processing nodes in $FT(m,n)$ has two phases: the first phase contains only up channels and the second phase contains only down channels.

From the definition of $FT(m,n)$, one can easily derive the following properties.

Property 1: $FT(m,n)$ contains m $SUBFT(m,n-1)$'s, $m\times\frac{m}{2}$ $SUBFT(m,n-2)$'s, ..., $m\times(\frac{m}{2})^{n-2}$ $SUBFT(m,1)$'s.

Level 0 (root level) switches do not belong to any sub-fat-tree. Each level 1 switch is in a $SUBFT(m,n-1)$; each level 2 switch is in a $SUBFT(m,n-1)$ and a $SUBFT(m,n-2)$; and so on. A level i switch, $1\le i\le n-1$, is in a $SUBFT(m,n-1)$, a $SUBFT(m,n-2)$, ..., and a $SUBFT(m,n-i)$. In $FT(m,n)$, we will call switches in levels 0, 1, ..., and $n-i-1$ the upper level switches for $SUBFT(m,i)$. The upper level switches for $SUBFT(m,i)$ provide connectivity among all $SUBFT(m,i)$'s.

Property 2: Through upper level switches for $SUBFT(m,i)$, $1\le i\le n-1$, an up-link port in a $SUBFT(m,i)$ is only connected to an up-link port of the same port number in any other $SUBFT(m,i)$. More specifically, up-link port j, $0\le j\le nu(m,i)-1$ in one $SUBFT(m,i)$ is only connected with the up-link port j of other $SUBFT(m,i)$'s (but not other ports) through upper level switches for $SUBFT(m,i)$.

It is clear that this property is true for $SUBFT(m,n-1)$. The property for general $SUBFT(m,i)$, $1\le i<n-1$, can be formally proven by induction on i (with base case $i=n-1$) and by examining how the top level switches in $SUBFT(m,i)$'s are connected.

Property 3: Let $SUBFT(m,i)$ be the smallest sub-fat-tree in $FT(m,n)$ that contains two processing nodes a and b, there exist $(\frac{m}{2})^{i-1}$ different shortest paths from a to b. If such a sub-tree does not exist, there are $(\frac{m}{2})^{n-1}$ different shortest paths from a to b. In this case, a and b are in different top level sub-fat-trees ($SUBFT(m,n-1)$'s).

Fig. 3 shows an example. From node a to node b in $FT(4,3)$, there are $(\frac{m}{2})^{n-1}=2^2=4$ shortest paths. In both cases in Property 3, the number of shortest paths between any two nodes can be represented as $(\frac{m}{2})^x$ with the value of x, $0\le x\le n-1$, depending on the positions of the source and the destination.

Property 4: In $FT(m,n)$, let there exist $(\frac{m}{2})^x$ different shortest paths from processing node s to processing node d. Each of the level $n-1-i$ up/down links that carry traffic from s to d is used by $(\frac{m}{2})^{x-i}$ shortest paths, $0\le i\le x$.

This property is intuitive. For example, level $n-1$ links are the links connecting processing nodes. Hence, all paths from the processing node connected by a level $n-1$ link must use the link. This is the case when $i=0$: all $(\frac{m}{2})^x$ shortest paths use the link. For the next level ($i=1$), a source will have $\frac{m}{2}$ choices (the fan-out from the first switch) to go to another node. Thus, each of such links will be used by $(\frac{m}{2})^x/\frac{m}{2}=(\frac{m}{2})^{x-1}$ shortest paths. The cases for links in other levels are similar. Consider the 4 paths from node a to node b in Fig. 3, all 4 paths use the level 2 up/down links (the link connecting the processing node), 2 paths use each of the level 1 up/down links that carries traffic from a to b, and 1 path uses each of the level 0 up/down links.

Property 5: In $FT(m,n)$, a level i, $0\le i\le n-1$, up link carries traffic from at most $(\frac{m}{2})^{n-1-i}$ source nodes. A level i down link carries traffic to at most $(\frac{m}{2})^{n-1-i}$ nodes.

This property is also intuitive. For example, when $i=n-1$, a level $i=n-1$ link directly connects to a processing nodes. So such a link carries traffic to/from at most $(\frac{m}{2})^{n-1-(n-1)}=1$ node. When $i=n-2$, the link connects to a level $n-1$ switch; and such a link carries traffic to/from the $(\frac{m}{2})^{n-1-(n-2)}=\frac{m}{2}$ nodes directly connected to that switch.

III. Multi-path oblivious routing

Let the traffic matrix be TM with entries $tm_{i,j}$, $0\le i\le N-1$ and $0\le j\le N-1$, specifying the amount of traffic from node i to node j. The total traffic sent from node i is $\sum_j tm_{i,j}$ and the total traffic received by node i is $\sum_j tm_{j,i}$. Since there is only one link connecting each processing node to the network, such traffic must be carried on that link regardless

of the routing scheme. Hence, for any routing scheme the load on the link (which has two directions) connecting to node i is $\max\{\sum_j tm_{i,j}, \sum_j tm_{j,i}\}$. We define the base load of a traffic matrix TM as

$$baseload(TM) = \max_{0 \le i \le N-1}\{\max\{\sum_j tm_{i,j}, \sum_j tm_{j,i}\}\}.$$

Clearly, for any TM and any $FT(m,n)$, the minimum maximum link load with any routing scheme, single path or multi-path, is at least $baseload(TM)$:

$$OPTU(TM) \ge baseload(TM).$$

We will give a multi-path routing algorithm, $OMRMN$, such that $MLOAD(OMRMN, TM) = baseload(TM)$ for any $FT(m,n)$ and any TM ($PERF(OMRMN) = 1$). $OMRMN$ works as follows: Let X be the number of shortest paths between processing nodes i and j (since processing nodes are in the same level of the fat-tree, X is fixed for all pairs of nodes) and let the X different shortest paths between nodes i and j be $P^1_{i,j}$, $P^2_{i,j}$, ..., $P^X_{i,j}$ (From Property 3 in Section II-B, these paths can be easily found). $OMRMN$ makes use of all the paths and allocates exactly the same amount of traffic on each path. That is, $MP_{i,j} = \{P^1_{i,j}, P^2_{i,j}, ..., P^X_{i,j}\}$ and $f^1_{i,j} = ... = f^X_{i,j} = \frac{1}{X}$.

Theorem 1: For any $FT(m,n)$, $PERF(OMRMN) = 1$.

Proof: Since in $FT(m,n)$ the up links and down links are symmetrical, it is sufficient to show that, for any traffic matrix TM, the load on each up link l is no more than $baseload(TM)$. Consider a source s in $FT(m,n)$. Let us denote $LD(s) = \sum_{j \ne s} tm_{s,j}$ the total amount of traffic sent from node s. For each source node s, using $OMRMN$, each level $n-1-i$ link carries at most $LD(s)/(\frac{m}{2})^i$ traffic since the traffic is evenly distributed among the $(\frac{m}{2})^i$ links at level $n-1-i$ that can carry traffic from node s (Property 4). In addition, each level $n-1-i$ link carries traffic from at most $(\frac{m}{2})^i$ source nodes (Property 5). Let the nodes be s_0, s_1, ..., $s_{(\frac{m}{2})^i-1}$ and the load on a level $n-1-i$ link l be $load(l)$, we have $load(l) \le \sum_{j=0}^{(\frac{m}{2})^i-1} \frac{LD(s_j)}{(\frac{m}{2})^i}$. Since $LD(s_j) \le baseload(TM)$, $0 \le j \le (\frac{m}{2})^i - 1$, we have $load(l) \le baseload(TM)$. Note that there is no restriction on the link l and traffic matrix TM. Hence, for all links and all traffic matrices, we have $load(l) \le baseload(TM) \le OPTU(TM)$. Hence, $PERF(OMRMN) = 1$. □

Corollary 1 For any $FT(m,n)$ and any traffic matrix TM, $OPTU(TM) = baseload(TM)$. □

Theorem 1 states that $OMRMN$ is optimal for all traffic patterns. However, $OMRMN$ uses all of the shortest paths between two processing nodes. We will refer to it as an *unrestricted* multi-path routing scheme. Practical system area networks cannot support such unrestricted multi-path routing in large fat-trees, and the performance of more restricted forms of routing must be studied.

IV. Lower bounds of oblivious performance ratio for single path routing

We will first introduce some concepts that will be used later in the lower bound derivation. Let $A = \{(s_1, d_1), (s_2, d_2), ...\}$ be a set of source-destination (SD) pairs.

Definition 1: The set of SD pairs, $A = \{(s_1, d_1), (s_2, d_2), ...\}$, is said to be **node disjoint** if for any $(s_i, d_i) \in A$ and $(s_j, d_j) \in A$ with $i \ne j$, we have $s_i \ne s_j$ and $d_i \ne d_j$.

Basically, in a node disjoint set of SD pairs, each source (in the source-destination pairs) appears in the set as a source exactly once; and each destination appears as a destination exactly once. A node may appear as a source and as a destination in a node disjoint set. For example, $\{(1,2),(1,3)\}$ is not a node disjoint set while $\{(1,2),(3,1)\}$ is.

Definition 2: For a set of SD pairs A, a set of SD pairs B is said to be a **node disjoint subset** of A when (1) $B \subseteq A$; and (2) B is a node disjoint set.

Definition 3: For a given set of SD pairs A, a set of SD pairs B is said to be a **largest node disjoint subset** of A when (1) B is a node disjoint subset of A; and (2) for any node disjoint subset C of A, $|B| \ge |C|$. We let $L(A)$ be the size of a largest node disjoint subset of A.

Let $S^A_s = \{(s,x)|(s,x) \in A\}$ be the set of SD pairs in A with source node s and $D^A_d = \{(x,d)|(x,d) \in A\}$ be the set of SD pairs in A with destination node d. $SRC(A) = \{s|\exists(s,d) \in A\}$ is the set of source nodes in A and $DST(A) = \{d|\exists(s,d) \in A\}$ is the set of destination nodes in A. We denote by $LS(A)$ the largest number of SD pairs in A either with the same source or with the same destination. Formally,

$$LS(A) = \max\{\max_{s \in SRC(A)} |S^A_s|, \max_{d \in DST(A)} |D^A_d|\}.$$

For any node i, $|S^A_i| \le LS(A)$ and $|D^A_i| \le LS(A)$.

Consider for example $A = \{(1,2),(1,3),(2,1),(2,4),(3,1)\}$. The set $\{(1,2),(2,1)\}$ is a node disjoint subset of A, but not a largest node disjoint subset. Both $\{(1,2),(2,4),(3,1)\}$ and $\{(1,3),(2,4),(3,1)\}$ are largest node disjoint subsets of A. Hence, $L(A) = 3$. $SRC(A) = \{1,2,3\}$; and $DST(A) = \{1,2,3,4\}$. $S^A_1 = \{(1,2),(1,3)\}$; $S^A_2 = \{(2,1),(2,4)\}$; and $S^A_3 = \{(3,1)\}$. $D^A_1 = \{(2,1),(3,1)\}$; $D^A_2 = \{(1,2)\}$; $D^A_3 = \{(1,3)\}$; and $D^A_4 = \{(2,4)\}$. Hence, $LS(A) = 2$.

Lemma 1: Let A be a set of SD pairs, $|SRC(A)| \ge L(A)$ and $|DST(A)| \ge L(A)$.

Proof: Straight-forward from the largest node disjoint subset definition. □

Lemma 2: Let A and B be two sets of SD pairs, $L(A) + L(B) \ge L(A \bigcup B)$.

Proof: Let C be a largest node disjoint subset of $A \bigcup B$. $|C| = L(A \bigcup B)$. Each element in C must either be in A, or in B (or in both A and B). Let $C_A = \{(s,d)|(s,d) \in C \cap A\}$ and $C_B = \{(s,d)|(s,d) \in C \cap B\}$. We have $|C_A| + |C_B| \ge |C| = L(A \bigcup B)$. Since C_A is a node disjoint subset of A and C_B is a node disjoint subset of B, by definition, $L(A) \ge |C_A|$ and $L(B) \ge |C_B|$. Hence, $L(A) + L(B) \ge L(A \bigcup B)$. □

Lemma 3: Let A be a set of SD pairs. If there is a source node s such that $|S^A_s| > L(A)$, then $L(A - S^A_s) = L(A) - 1$.

Proof: Since S^A_s has only one source node, $L(S^A_s) = 1$. From Lemma 2, we have $L(A - S^A_s) + L(S^A_s) \ge L((A - S^A_s) \bigcup S^A_s) = L(A)$. Hence, $L(A - S^A_s) \ge L(A) - 1$.

Next, we will prove $L(A - S^A_s) \le L(A) - 1$ by contra-

diction. Let $B = \{(s_1, d_1), (s_2, d_2), ..., (s_k, d_k)\}$ be a largest node disjoint subset of $A - S_s^A$. Assume that $|B| = k = L(A - S_s^A) > L(A) - 1$. Since $A - S_s^A$ is a subset of A, $k \leq L(A)$. Hence, k must be exactly equal to $L(A)$. Since $|S_s^A| > L(A) = k$, there exists at least one $(s, d) \in S_s^A$ such that $d \neq d_i$, $1 \leq i \leq k$. Hence, the set $C = B \bigcup \{(s, d)\}$ is node disjoint and $|C| = L(A) + 1$. Since C is a node disjoint subset of A, $|C| \leq L(A)$. This is the contradiction. Hence, $L(A - S_a^A) = L(A) - 1$. □

Lemma 3a: Let A be a set of SD pairs. If there is a node d such that $|D_d^A| > L(A)$, then $L(A - D_d^A) = L(A) - 1$. □

Lemma 4: Let A be a set of SD pairs. If there exist k source nodes s_i, $1 \leq i \leq k$, such that $|S_{s_i}^A| > L(A)$, and l destination nodes d_j, $1 \leq j \leq l$, such that $|D_{d_j}^A| > L(A)$, then $L(A - \bigcup_{i=1}^{k} S_{s_i}^A - \bigcup_{j=1}^{l} D_{d_j}^A) = L(A) - k - l$.

Proof: The conclusion in this lemma is obtained by repeatedly applying Lemma 3 and Lemma 3a. □.

Lemma 5: Let A be a set of SD pairs. $|A| \leq L(A) \times LS(A)$.

Proof: See Appendix. □

We use a topology, called extended 2-layer fat-tree ($EFT2(m, k)$), in the derivation of the lower bounds. $EFT2(m, k)$ contains two levels of switches. The top level contains $\frac{m}{2}$ k-port switches. The bottom level contains k m-port switches. Half of the m ports in the bottom level switches are used to connect to processing nodes and the other half connect to top level switches. There is a link between each top level switch and each bottom layer switch. The structure of $EFT2(m, k)$ is similar to $FT(m, 2)$ (shown in Fig. 6) except that $EFT2(m, k)$ allows different types of switches in the two levels. $FT(m, 2)$ is the same as $EFT2(m, m)$. A sub-graph of $EFT2(m, k)$, called $SEFT2(m, k)$, contains all lower level switches and processing nodes in $EFT2(m, k)$, but only one top level switch. Fig. 7 shows the $SEFT2(m, k)$ topology, which is basically a regular tree topology with the root having k children and each level 1 switch having $\frac{m}{2}$ children. In Fig. 7, we separate the two directional channels.

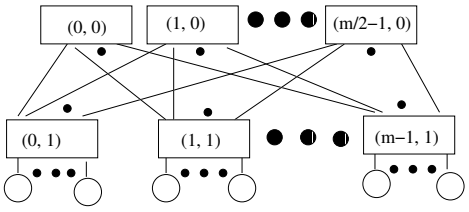


Fig. 6. $FT(m, 2)$ topology

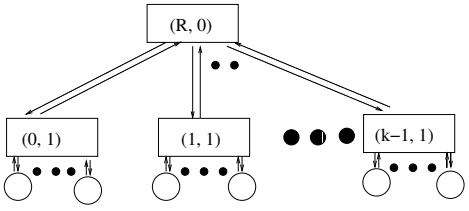


Fig. 7. $SEFT2(m, k)$ topology

In $EFT2(m, k)$, there are $\frac{m}{2}$ upper level switches and each of the k lower level switches connects directly to each of the upper level switches. Hence, by routing different SD pairs through different upper level switches, this topology can provide $\frac{m}{2}$ link disjoint paths for any $\frac{m}{2}$ node disjoint SD pairs. As a result, if a scheme routes X, $X \leq \frac{m}{2}$, node disjoint SD pairs on a link, the performance ratio of this routing scheme is at least X since the X node disjoint SD pairs can be routed using a set of link disjoint paths. Lemmas 6 and 7 formally capture this intuition.

Lemma 6: Let the processing nodes in $EFT2(m, k)$ be numbered from 0 to $N-1$. Let $A = \{(s_1, d_1), ..., (s_{|A|}, d_{|A|})\}$ be a set of node disjoint SD pairs (for $1 \leq i \leq |A|$, $s_i \in \{0, ..., N-1\}$ and $d_i \in \{0, ..., N-1\}$). When $|A| \leq \frac{m}{2}$, the SD pairs in A can be routed in $EFT2(m, k)$ with $|A|$ link disjoint paths.

Proof: Since $|A| \leq \frac{m}{2}$, we can assign a different top level switch SW_i for each SD pair $(s_i, d_i) \in A$. For each (s_i, d_i), if s_i and d_i connect to the same (lower level) switch SW, the path from s_i to d_i is $s_i \rightarrow SW \rightarrow d_i$. If s_i and d_i are not in the same switch, let s_i connect to $SW1$ and d_i to $SW2$. The path for (s_i, d_i) is $s_i \rightarrow SW1 \rightarrow SW_i \rightarrow SW2 \rightarrow d_i$. Since A is node disjoint, all of the paths are link disjoint. □

Lemma 7: Let sr be a single path routing on $EFT2(m, k)$. Assume that under routing sr, there exists a link l that carries traffic for a set A of node disjoint SD pairs, $|A| \leq \frac{m}{2}$, Then, $PERF(sr) \geq |A|$.

Proof: Consider a traffic matrix TM where $tm_{i,j} = 1$ for all $(i, j) \in A$ and all other entries are 0. From Lemma 6, there exists a routing scheme sr' that routes the SD pairs in A using link disjoint paths. Hence, $MLOAD(sr', TM) = 1$ and $OPTU(TM) \leq 1$. Since using routing sr, the load on link l is $|A|$ and $MLOAD(sr, TM) \geq |A|$. Hence,

$$PERF(sr) \geq \frac{MLOAD(sr, TM)}{OUTU(TM)} \geq \frac{|A|}{1} = |A|. \square$$

For a single path routing r, let us define the *maximum disjoint size* on link l, $mds(r, l)$, to be the size of the largest node disjoint subset of the set of SD pairs routed on l. In $EFT2(m, k)$ and $SEFT2(m, k)$, a level 1 link directly connected to a processing node: the SD pairs on that link has at most one source node or one destination node. From Lemma 1, the maximum disjoint size on such a link is at most 1. The *maximum disjoint size* of routing r, $mds(r)$, is defined as $mds(r) = \max_{l \in Links} mds(r, l)$.

Lemma 8 and Theorem 2 prove the lower bound of the oblivious performance ratio for single path routing on $EFT2(m, k)$. The proof follows the following logic. To realize routes for all SD pairs in $EFT2(m, k)$, there are $k(k-1)(\frac{m}{2})^2$ SD pairs that must be routed through upper level switches. Since there are $\frac{m}{2}$ upper level switches in $EFT2(m, k)$, at least one upper level switch must carry at least $k(k-1)\frac{m}{2}$ SD pairs. This particular upper level switch, the lower switches, and processing nodes form an $SEFT2(m, k)$. Lemma 8 proves a more general result, implying that to carry traffic for $k(k-1)\frac{m}{2}$ SD pairs through the root in the $SEFT2(m, k)$, at least one link must carry traffic for $\sqrt{\frac{m}{2}}$ node disjoint SD pairs. Combining these results with Lemma 7, we derive the lower bound for $EFT2(m, k)$.

Lemma 8: Consider using the $SEFT2(m,k)$ topology to route a subset of all possible SD pairs. If the largest of the maximum disjoint sizes of all links is at most X, the number of SD pairs routed through the root is at most $k(k-1)X^2$ when $X \geq \frac{m}{k}$.

Proof: Let (s,d) be a SD pair. The pair must be routed through the root only when nodes s and d are connected to different switches. We will call the root switch in $SEFT2(m,k)$ switch $(R,0)$ and the k level 1 switches $(0,1)$, $(1,1)$, ..., $(k-1,1)$ as shown in Fig. 7. Let S be a largest set of SD pairs that are routed through the root when the largest of the maximum disjoint sizes of all links is at most X. Let $S_{i,j}$, $0 \leq i \neq j \leq k-1$, be the set of SD pairs in S with source nodes in switch $(i,1)$ and destination nodes in switch $(j,1)$. $S = \bigcup_{i,j \text{ such that } 0 \leq i \neq j \leq k-1} S_{i,j}$. Let us denote

$$LX_{i,j}^{src} = \bigcup_{a \text{ such that } a \in SRC(S_{i,j}) \text{ and } |S_a^{S_{i,j}}| > X} S_a^{S_{i,j}}$$

Let $E_{i,j} = |SRC(LX_{i,j}^{src})|$. For the SD pairs in $S_{i,j}$, $E_{i,j}$ is the number of source nodes in switch $(i,1)$, each of which having more than X destination nodes in switch $(j,1)$. $LX_{i,j}^{src}$ contains all such SD pairs. Similarly, we will denote

$$LX_{i,j}^{dst} = \bigcup_{d \text{ such that } d \in DST(S_{i,j}) \text{ and } |D_d^{S_{i,j}}| > X} D_d^{S_{i,j}}$$

Let $F_{i,j} = |DST(LX_{i,j}^{dst})|$. For the SD pairs in $S_{i,j}$, $F_{i,j}$ is the number of destination nodes in switch $(j,1)$, each of which having more than X source nodes in switch $(i,1)$. $LX_{i,j}^{dst}$ contains all such SD pairs.

All SD pairs in $S_{i,j}$ must pass through links $(i,1) \rightarrow (R,0)$ and $(R,0) \rightarrow (j,1)$. First, let us consider link $(i,1) \rightarrow (R,0)$. Let all SD pairs with source nodes in $(i,1)$ be $All_{(i,1)\rightarrow(R,0)} = \bigcup_{j \neq i} S_{i,j}$. All SD pairs in $All_{(i,1)\rightarrow(R,0)}$ must go through link $(i,1) \rightarrow (R,0)$. Hence, $L(All_{(i,1)\rightarrow(R,0)}) \leq X$. From Lemma 4, $L(All_{(i,1)\rightarrow(R,0)} - \bigcup_{x \neq i} LX_{i,x}^{src}) \leq X - \sum_{x \neq i} E_{i,x}$. Since $S_{i,j} - LX_{i,j}^{src} \subseteq All_{(i,1)\rightarrow(R,0)} - \bigcup_{x \neq i} LX_{i,x}^{src}$, we have $L(S_{i,j} - LX_{i,j}^{src}) \leq L(All_{(i,1)\rightarrow(R,0)} - \bigcup_{x \neq i} LX_{i,x}^{src}) \leq X - \sum_{x \neq i} E_{i,x}$. Hence, applying Lemma 4,

$$L(S_{i,j} - LX_{i,j}^{src} - LX_{i,j}^{dst}) \leq X - \sum_{x \neq i} E_{i,x} - F_{i,j}.$$

Using the similar logic, by considering link $(R,0) \rightarrow (j,1)$, we can obtain

$$L(S_{i,j} - LX_{i,j}^{src} - LX_{i,j}^{dst}) \leq X - E_{i,j} - \sum_{x \neq j} F_{x,j}.$$

Combining these two inequalities, we obtain $L(S_{i,j} - LX_{i,j}^{src} - LX_{i,j}^{dst}) \leq X - (\sum_{x \neq i} E_{i,x} + F_{i,j} + E_{i,j} + \sum_{x \neq j} F_{x,j})/2$. Each source or destination node in $S_{i,j} - LX_{i,j}^{src} - LX_{i,j}^{dst}$ can have no more than X SD pairs in the set (otherwise, these SD pairs would be included in either $LX_{i,j}^{src}$ or $LX_{i,j}^{dst}$). Hence, $LS(S_{i,j} - LX_{i,j}^{src} - LX_{i,j}^{dst}) \leq X$. From Lemma 5, $|S_{i,j} - LX_{i,j}^{src} - LX_{i,j}^{dst}| \leq L(S_{i,j} - LX_{i,j}^{src} - LX_{i,j}^{dst}) \times LS(S_{i,j} - LX_{i,j}^{src} - LX_{i,j}^{dst}) \leq (X - (\sum_{x \neq i} E_{i,x} + F_{i,j} + E_{i,j} + \sum_{x \neq j} F_{x,j})/2) \times X$. Hence,

$$\begin{aligned}
&|\bigcup_{i=0}^{k-1} \bigcup_{j \neq i} (S_{i,j} - LX_{i,j}^{src} - LX_{i,j}^{dst})| \\
&\leq \sum_{i=0}^{k-1} \sum_{j \neq i} |S_{i,j} - LX_{i,j}^{src} - LX_{i,j}^{dst}| \\
&\leq \sum_{i=0}^{k-1} \sum_{j \neq i} X \times (X - (\sum_{x \neq i} E_{i,x} + F_{i,j} \\
&\qquad + E_{i,j} + \sum_{x \neq j} F_{x,j})/2) \\
&= k(k-1)X^2 - \frac{X}{2} \sum_{i=0}^{k-1} \sum_{j \neq i} (\sum_{x \neq i} E_{i,x} + E_{i,j}) \\
&\qquad - \frac{X}{2} \sum_{i=0}^{k-1} \sum_{j \neq i} (\sum_{x \neq j} F_{x,j} + F_{i,j})
\end{aligned}$$

The values of j and x are in the range $0..k-1$ and are not equal to i. Thus, $\sum_{j \neq i} \sum_{x \neq i} E_{i,x} = (k-1) \sum_{j \neq i} E_{i,j}$. This can be seen by expanding the summation form: each term $E_{i,j}$, $j \neq i$, happens $k-1$ times in the summation. Hence, $\sum_{j \neq i} (\sum_{x \neq i} E_{i,x} + E_{i,j}) = k \sum_{j \neq i} E_{i,j}$. Similarly, $\sum_{j \neq i} (\sum_{x \neq j} F_{x,j} + F_{i,j}) = k \sum_{j \neq i} F_{i,j}$. Hence,

$$\begin{aligned}
&|\bigcup_{i=0}^{k-1} \bigcup_{j \neq i} (S_{i,j} - LX_{i,j}^{src} - LX_{i,j}^{dst})| \\
&\leq k(k-1)X^2 - \frac{kX}{2} \sum_{i=0}^{k-1} \sum_{j \neq i} E_{i,j} - \frac{kX}{2} \sum_{i=0}^{k-1} \sum_{j \neq i} F_{i,j}
\end{aligned}$$

Since each switch connects to $\frac{m}{2}$ processing nodes, $LX_{i,j}^{src} \leq E_{i,j} \times \frac{m}{2}$ and $LX_{i,j}^{dst} \leq F_{i,j} \times \frac{m}{2}$. Hence,

$$\begin{aligned}
|S| &= |\bigcup_{i=0}^{k-1} \bigcup_{j \neq i} S_{i,j}| \\
&\leq |\bigcup_{i=0}^{k-1} \bigcup_{j \neq i} ((S_{i,j} - LX_{i,j}^{src} - LX_{i,j}^{dst}) \bigcup LX_{i,j}^{src} \bigcup LX_{i,j}^{dst})| \\
&\leq |\bigcup_{i=0}^{k-1} \bigcup_{j \neq i} (S_{i,j} - LX_{i,j}^{src} - LX_{i,j}^{dst})| \\
&\quad + |\bigcup_{i=0}^{k-1} \bigcup_{j \neq i} LX_{i,j}^{src}| + |\bigcup_{i=0}^{k-1} \bigcup_{j \neq i} LX_{i,j}^{dst}| \\
&\leq k(k-1)X^2 - \frac{kX}{2} \sum_{i=0}^{k-1} \sum_{j \neq i} E_{i,j} - \frac{kX}{2} \sum_{i=0}^{k-1} \sum_{j \neq i} F_{i,j} \\
&\quad + \frac{m}{2} \sum_{i=0}^{k-1} \sum_{j \neq i} E_{i,j} + \frac{m}{2} \sum_{i=0}^{k-1} \sum_{j \neq i} F_{i,j} \\
&= k(k-1)X^2 \\
&\quad - (\frac{kX}{2} - \frac{m}{2})(\sum_{i=0}^{k-1} \sum_{j \neq i} E_{i,j} + \sum_{i=0}^{k-1} \sum_{j \neq i} F_{i,j})
\end{aligned}$$

When $X \geq \frac{m}{k}$, $\frac{kX}{2} \geq \frac{m}{2}$. Thus, $|S| \leq k(k-1)X^2$. □

Let us denote by $T(X)$ the maximum number of SD pairs routed through $SEFT2(m,k)$ when the largest of the maximum disjoint sizes of the links in $SEFT2(m,k)$ is X. Obviously, when $X > Y$ and $T(Y)$ is less than the total number of SD pairs that can be routed, $T(X) > T(Y)$ regardless of the relation among X, m, and k. Lemma 8 states that when $X \geq \frac{m}{k}$, $T(X) \leq k(k-1)X^2$. Hence, when $X < \frac{m}{k}$, $T(X) < T(\frac{m}{k}) \leq k(k-1)(\frac{m}{k})^2$.

Theorem 2: Let r be a single path routing algorithm on $EFT2(m,k)$. If $k \geq \sqrt{2m}$, $PERF(r) \geq \sqrt{\frac{m}{2}}$.

Proof: Regardless of the single path routing algorithm r used, $k(k-1)(\frac{m}{2})^2$ SD pairs must be routed through top level switches. Since there are $\frac{m}{2}$ top level switches in $EFT2(m,k)$, at least one top level switch must carry $\frac{k(k-1)(\frac{m}{2})^2}{\frac{m}{2}} = k(k-1)\frac{m}{2}$ SD pairs. Consider the $SEFT2(m,k)$ formed by this particular top level switch (with $k(k-1)\frac{m}{2}$ SD pairs passing through) with all level 1 switches and all processing nodes. Let the maximum disjoint size of the links connecting to this switch be X. Under the assumption $k \geq \sqrt{2m}$, if $X < \frac{m}{k} \leq \frac{m}{\sqrt{2m}} = \sqrt{\frac{m}{2}}$, $T(X) < k(k-1)(\frac{m}{k})^2 \leq k(k-1)\frac{m}{2}$. Since there are $k(k-1)\frac{m}{2}$ SD pairs that must be routed through the switch ($T(X) \geq k(k-1)\frac{m}{2}$), $X < \sqrt{\frac{m}{2}}$ cannot be true. Thus, $X \geq \sqrt{\frac{m}{2}}$. Since $\frac{m}{2} \geq \sqrt{\frac{m}{2}}$, from Lemma 7, $PERF(r) \geq \sqrt{\frac{m}{2}}$. □

Theorem 3: Let r be a single path routing algorithm for $FT(m,2)$, $m \geq 2$, $PERF(r) \geq \sqrt{\frac{m}{2}}$.

Proof: $FT(m,2)$ is equivalent to $EFT2(m,m)$. Since in $FT(m,2)$, $m \geq 2$ and $m \geq \sqrt{2m}$. From Theorem 2, $PERF(r) \geq \sqrt{\frac{m}{2}}$. □

Theorem 4: Let r be a single path routing algorithm for $FT(m,3)$, $PERF(r) \geq \frac{m}{2}$.

Proof: $FT(m,3)$ is composed by top level switches (m-port) with m $SUBFT(m,2)$'s. Let us consider the maximum disjoint sizes on the links that connect the $SUBFT(m,2)$'s with the root level switches (Level 0 links). By treating each $SUBFT(m,2)$ as one $2(\frac{m}{2})^2$-port switch, $FT(m,3)$ is approximated as $EFT2(2(\frac{m}{2})^2, m)$. The top level links in $EFT2(2(\frac{m}{2})^2, m)$ correspond to the top level links in $FT(m,3)$. For any routing algorithm r on $FT(m,3)$, there is a routing algorithm r' on $EFT2(2(\frac{m}{2})^2, m)$ such that the corresponding top level links are used exactly the same. Following the proof of Theorem 2, since $m \geq \sqrt{2 \times 2(\frac{m}{2})^2}$, for any r' on $EFT2(2(\frac{m}{2})^2, m)$, the largest of the maximum disjoint size of the level 0 links is at least $\sqrt{\frac{2\times(\frac{m}{2})^2}{2}} = \frac{m}{2}$. Hence, for the any routing r on $FT(m,3)$, there exists a link carrying at least $\frac{m}{2}$ node disjoint SD pairs. Let this set of node disjoint SD pairs be A. Consider the traffic matrix TM where $tm_{i,j} = 1$ for all $(i,j) \in A$ and all other entries are 0. Clearly, $MLOAD(r,TM) \geq \frac{m}{2}$. Since A is node disjoint, each node sends and receives at most 1 unit of traffic and $baseload(TM) = OPTU(TM) = 1$ (Corollary 1). Thus,

$$PERF(r) \geq \frac{MLOAD(r,TM)}{OPTU(TM)} = \frac{m}{2}.\square$$

Theorem 5: Let r be a single path routing algorithm for $FT(m,n)$, $PERF(r) \geq (\frac{m}{2})^{\lfloor \frac{n-1}{3} \rfloor}$.

Proof: Let us consider the maximum disjoint sizes on links connecting to up-link ports of $SUBFT(m,i)$'s, $1 \leq i \leq n-1$, in $FT(m,n)$. From Property 1 and Property 2 of $FT(m,n)$, the connectivity in $FT(m,n)$ can be partitioned into two levels (with respective to such links): the lower level connectivity provided by $SUBFT(m,i)$'s and the upper level connectivity provided by the upper level switches for $SUBFT(m,i)$'s. The connectivity in $SUBFT(m,i)$ can be approximated as a $2(\frac{m}{2})^i$-port switch; and the upper level switches that connect the up-link ports with the same port number in each of the $SUBFT(m,i)$ (Property 2), which approximates a $m(\frac{m}{2})^{n-1-i}$-port switch. Consider the case when $i = \lfloor \frac{2(n-1)}{3} \rfloor$, the topology can be approximated by $EFT2(2(\frac{m}{2})^{\lfloor \frac{2(n-1)}{3} \rfloor}, m(\frac{m}{2})^{\lceil \frac{n-1}{3} \rceil})$. Following the same logic as the proof of Theorem 4, for any r on $FT(m,n)$, there exists a link carrying at least $\sqrt{\frac{2(\frac{m}{2})^{\lfloor \frac{2(n-1)}{3} \rfloor}}{2}} = (\frac{m}{2})^{\lfloor \frac{n-1}{3} \rfloor}$ node disjoint pairs and

$$PERF(r) \geq (\frac{m}{2})^{\lfloor \frac{n-1}{3} \rfloor}.\square$$

V. Optimal single path oblivious routing for $FT(m,2)$ and $FT(m,3)$

Most practical fat-trees have no more than three levels of switches since such topologies can already support thousands of processing nodes. For example, $FT(24,3)$ supports 3456 process nodes; $FT(32,3)$ supports 8192 processing nodes; and $FT(48,3)$ can support 27648 processing nodes. Hence, developing routing schemes for $FT(m,2)$ and $FT(m,3)$ bears most practical significance. Moreover, the development of these algorithms also has theoretical significance by making the lower bounds of the oblivious performance ratio for $FT(m,2)$ and $FT(m,3)$ (Theorem 3 and Theorem 4) tight bounds. The proposed optimal single path oblivious routing schemes are based on the following lemma.

Lemma 9: If a single path routing scheme r routes SD pairs such that the SD pairs in each of the links in $FT(m,n)$ are either from at most X sources or towards at most X destinations, then $PERF(r) \leq X$.

Proof: As discussed earlier, for any traffic demand TM, on $FT(m,n)$, $OPTU(TM) \geq baseload(TM)$. Since each link carries traffic either from at most X sources or towards X destinations, the load of the link is no more than $X \times baseload(TM)$, hence, $PERF(r,TM) \leq \frac{X \times baseload(TM)}{baseload(TM)} = X$. Since this applies for any traffic demand TM, $PERF(r) \leq X$. $\square$

Existing routing schemes for the fat-tree topology [9], [15] try to balance the link load by spreading the traffic among different links. However, all of these schemes spread the traffic with a "locally optimal" heuristic: they make sure that the traffic from one node to all other nodes are spread out uniformly among all possible links. However, such a locally optimal heuristic is not globally optimal in the sense that a link can potentially carry traffic from many sources and to many destinations, which can potentially make the link a hot-spot for some particular traffic patterns. The proposed optimal oblivious routing schemes achieve global optimality by routing traffics either from at most $\sqrt{\frac{m}{2}}$ sources or to at most $\sqrt{\frac{m}{2}}$ destinations on each link in $FT(m,2)$, and traffics either from at most $\frac{m}{2}$ sources or to at most $\frac{m}{2}$ destinations on each link in $FT(m,3)$. From Lemma 9, this ensures that the performance ratios of our schemes are at most $\sqrt{\frac{m}{2}}$ for $FT(m,2)$ and $\frac{m}{2}$ for $FT(m,3)$, which are optimal (Theorems 3 and 4).

A. Optimal oblivious routing for $FT(m,2)$

To describe the oblivious routing algorithm, we will give a non-recursive description of $FT(m,2)$. $FT(m,2)$ contains $\frac{3m}{2}$ switches and supports $\frac{m^2}{2}$ processing nodes. In this topology, $\frac{m}{2}$ switches are in the level 0 and m switches are in level 1. The $\frac{m}{2}$ top level switches are labeled switches $(0,0)$, $(1,0)$, ..., $(\frac{m}{2}-1,0)$. The m level 1 switches are labeled switches $(0,1)$, $(1,1)$, ..., $(m-1,1)$. Each level 1 switch $(i,1)$, $0 \leq i \leq m-1$, is connected with $\frac{m}{2}$ processing nodes numbered as $(i,0)$, $(i,1)$, ..., $(i,\frac{m}{2}-1)$. Notice that the process nodes and switches are numbered independently. There is a link between switch $(i,0)$, $0 \leq i \leq \frac{m}{2}-1$, and switch $(j,1)$, $0 \leq j \leq m-1$. For $0 \leq i \leq m-1$, there is a link between processing node (i,x), $0 \leq x \leq \frac{m}{2}-1$, and switch $(i,1)$. Fig. 8 depicts the $FT(8,2)$ topology as well as the switch and processing node labeling.

To ease exposition, let us assume that $\sqrt{\frac{m}{2}}$ is an integer. Our algorithm $OSRM2$, described in Fig. 9, can also handle the case when $\sqrt{\frac{m}{2}}$ is not an integer. In $FT(m,2)$, each level 1 link connects to 1 processing node and can only carry traffic to and from one node (Property 5). Thus, We only need to make sure that the traffic on each level 0 link has no more than $\sqrt{\frac{m}{2}}$ sources or destinations.

Let $Z = \sqrt{\frac{m}{2}}$. $OSRM2$ partitions the $\frac{m}{2} = Z^2$ processing nodes in each bottom level switch into Z groups, each group having Z nodes. More specifically, the $\frac{m}{2}$ processing nodes connected to switch $(i, 1)$, $0 \leq i \leq m-1$, are partitioned into $Z = \sqrt{\frac{m}{2}}$ groups: group j, $0 \leq j \leq Z-1$, includes nodes $(i, j*Z)$, $(i, j*Z+1)$, ..., $(i, j*Z+Z-1)$. Let us denote $g_i \rightarrow g_j$ as SD pairs from nodes in group i in the one switch to nodes in group j in all other switches. Fig. 8 shows how the routing algorithm works on $FT(8, 2)$. In $FT(8, 2)$, $\frac{m}{2} = 4$ and $\sqrt{\frac{m}{2}} = 2$. As shown in the figure, the $\frac{m}{2} = 4$ nodes attached to each lower level switch are partitioned into two groups: group 0 and group 1 with each group having 2 nodes. The SD pairs are scheduled such that $g_0 \rightarrow g_0$ goes through switch (0, 0); $g_0 \rightarrow g_1$ goes through switch (1, 0); $g_1 \rightarrow g_0$ goes through switch (2, 0); and $g_1 \rightarrow g_1$ goes through switch (3, 0). Since each upper level switch carries SD pairs from nodes in one group to nodes in another group, each up link (to a top level switch) carries SD pairs with exactly $\sqrt{\frac{m}{2}}$ sources and each down link (to a lower level switch) carries SD pairs with exactly $\sqrt{\frac{m}{2}}$ destinations.

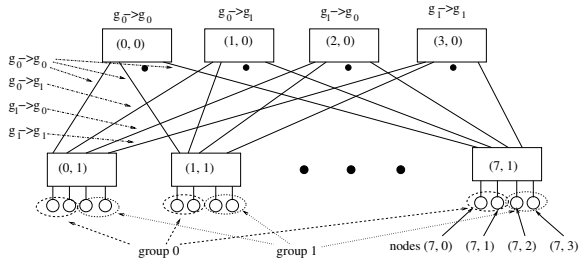


Fig. 8. Optimal oblivious routing for $FT(8, 2)$

For a general $FT(m, 2)$, the SD pairs are scheduled as follows. The up-link $(i, 1) \rightarrow (j, 0)$, $0 \leq i \leq m-1$ and $0 \leq j \leq \frac{m}{2} - 1$, carries SD pairs with source nodes in group j/Z in switch $(i, 1)$ and destination nodes in group $j \ mod \ Z$ in all other switches. $OSRM2$ is shown in Fig. 9. When $\sqrt{\frac{m}{2}}$ is an integer, the algorithm works exactly as just described. When $\sqrt{\frac{m}{2}}$ is not an integer, the algorithm partitions the $\frac{m}{2}$ sources attached with each of the level 1 switch into $Z_s = \lceil \sqrt{\frac{m}{2}} \rceil$ groups and the $\frac{m}{2}$ destinations into $Z_d = \lfloor \sqrt{\frac{m}{2}} \rfloor$ groups. It then uses the same logic as the cases when $\sqrt{\frac{m}{2}}$ is an integer to schedule the SD pairs.

Theorem 6: When $\sqrt{\frac{m}{2}}$ is an integer, $PERF(OSRM2) = \sqrt{\frac{m}{2}}$.

Proof: As discuss earlier, using $OSRM2$, each link carries traffic either from $\sqrt{\frac{m}{2}}$ sources or to $\sqrt{\frac{m}{2}}$ destinations. From Lemma 9, $PERF(OSRM2) \leq \sqrt{\frac{m}{2}}$. From Theorem 3, $PERF(OSRM2) \geq \sqrt{\frac{m}{2}}$. Hence, $PERF(OSRM2) = \sqrt{\frac{m}{2}}$ and $OSRM2$ is an optimal oblivious routing algorithm for $FT(m, 2)$ when $\sqrt{\frac{m}{2}}$ is an integer. □

B. Optimal oblivious routing for $FT(m, 3)$

We will now consider $FT(m, 3)$. $FT(m, 3)$ contains three levels of switches, with the top level having $nu(m, 2) = \frac{m}{2} \times \frac{m}{2}$ switches and each of the other levels having $m \times \frac{m}{2}$ switches (m $SUBFT(m, 2)$'s, each $SUBFT(m, 2)$ having $\frac{m}{2}$ switches at each level). We label the switches by $((i_0, i_1), level)$: the top level switches are labeled as $((i_0, i_1), 0)$, $0 \leq i_0 \leq \frac{m}{2} - 1$ and $0 \leq i_1 \leq \frac{m}{2} - 1$; the level 1 switches are labeled as $((i_0, i_1), 1)$, $0 \leq i_0 \leq m-1$, and $0 \leq i_1 \leq \frac{m}{2} - 1$; the level 2 switches are labeled as $((i_0, i_1), 2)$, $0 \leq i_0 \leq m-1$ and $0 \leq i_1 \leq \frac{m}{2} - 1$. Notice that in the switch labeling, for levels 1 and 2, i_0 identifies the columns corresponding to the i_0-th $SUBFT(m, 2)$ and i_1 identifies the column corresponding to the i_1-th $SUBFT(m, 1)$ within the i_0-th $SUBFT(m, 2)$. A $FT(m, 3)$ has $m \times \frac{m}{2} \times \frac{m}{2}$ processing nodes, which are labeled as (p_0, p_1, p_2), $0 \leq p_0 \leq m-1$, $0 \leq p_1 \leq \frac{m}{2} - 1$, and $0 \leq p_2 \leq \frac{m}{2} - 1$. A processing node (p_0, p_1, p_2) is attached to switch $((p_0, p_1), 2)$, $0 \leq p_0 \leq m-1$, $0 \leq p_1 \leq \frac{m}{2} - 1$, and $0 \leq p_2 \leq \frac{m}{2} - 1$. A level 2 switch $((i_0, i_1), 2)$, $0 \leq i_0 \leq m-1$ and $0 \leq i_1 \leq \frac{m}{2} - 1$, has a link to each of the level 1 switches $((i_0, X), 1)$, $0 \leq X \leq \frac{m}{2} - 1$. A level 1 switch $((i_0, i_1), 1)$, $0 \leq i_0 \leq m-1$ and $0 \leq i_1 \leq \frac{m}{2} - 1$, has a link to each of the level 0 switches $((i_1, X), 0)$, $0 \leq X \leq \frac{m}{2} - 1$.

Algorithm OSRM2:

Route from node (s_0, s_1) to node (d_0, d_1)
Let $m2 = \frac{m}{2}$;
Let $Z_s = \lceil \sqrt{m2} \rceil$, $Z_d = \lfloor \sqrt{m2} \rfloor$;
Let $N_s = \lceil \frac{m2}{Z_s} \rceil$, $N_d = \lceil \frac{m2}{Z_d} \rceil$;
if $(s_0 == d_0)$
  use route: $node(s_0, s_1) \rightarrow switch(s_0, 1) \rightarrow node(d_0, d_1)$
if $(s_0 != d_0)$
  use route: $node(s_0, s_1) \rightarrow switch(s_0, 1)$
  $\rightarrow switch(s_1/N_s * N_d + d_1/N_d, 0)$
  $\rightarrow switch(d_0, 1) \rightarrow node(d_0, d_1)$

Fig. 9. Optimal oblivious routing for $FT(m, 2)$

Like in the $FT(m, 2)$ case, our optimal routing algorithm ensures that the SD pairs on each link are either from at most $\frac{m}{2}$ sources or towards at most $\frac{m}{2}$ destinations. From Property 5 in Section II-B, each level 1 or level 2 link in $FT(m, 3)$ carries traffic either from no more than $\frac{m}{2}$ sources or to no more than $\frac{m}{2}$ destinations. Hence, routing within each $SUBFT(m, 2)$ does not affect the performance oblivious ratio. Hence, we only need to focus on level 0 links. The idea is similar to that in $OSRM2$: the routing algorithm ensures that each up link out of the sub-fat-tree $SUBFT(m, 2)$ carries traffic from $\frac{m}{2}$ sources and each down link to a $SUBFT(m, 2)$ carries traffic to $\frac{m}{2}$ destinations. Basically, we can treat each $SUBFT(m, 2)$ as if it is a $2(\frac{m}{2})^2$-port switch that connects to $(\frac{m}{2})^2$ processing nodes and has $(\frac{m}{2})^2$ up-links. The routing algorithm partitions the $\frac{2(\frac{m}{2})^2}{2} = Z^2$ processing nodes in a $SUBFT(m, 2)$ into $Z = \frac{m}{2}$ groups, each group having $Z = \frac{m}{2}$ nodes. Node (p_0, p_1, p_2) is in group p_2 of the p_0-th $SUBFT(m, 2)$. The routing for links between $SUBFT(m, 2)$ and the top level switch is similar to that for links between level 1 switches to level 0 switches in $FT(m, 2)$: the up-link $((i_0, 0), 1) \rightarrow ((0, 0), 0)$ carries traffic from group 0 processing nodes (in the i_0-th $SUBFT(m, 2)$)to group 0 processing nodes in other $SUBFT(m, 2)$'s; $((i_0, 0), 1) \rightarrow ((0, 1), 0)$ carries traffic from group 0 processing nodes to group 1 processing nodes in other $SUBFT(m, 2)$'s; and so on. The detailed routing algorithm, called $OSRM3$, is shown in Fig. 10

Algorithm OSRM3:

Route from node (s_0, s_1, s_2) to (d_0, d_1, d_2):
if ($s_0 == d_0$ and $s_1 == d_1$)
  /* within one $SUBFT(m, 2)$*/
  /* routing won't affect the oblivious ratio */
  Use route: $node(s_0, s_1, s_2) \rightarrow switch((s_0, s_1), 2)$
  $\rightarrow node(d_0, d_1, d_2)$
else if ($s_0 == d_0$)
  /* within one $SUBFT(m, 2)$ */
  /* routing won't affect the oblivious ratio */
  Use route: $node(s_0, s_1, s_2) \rightarrow switch((s_0, s_1), 2)$
  $\rightarrow switch((s_0, s_2), 1)$
  $\rightarrow switch((s_0, d_1), 2)$
  $\rightarrow node(d_0, d_1, d_2)$
else
  /* must be careful about links to/from level 0 switches */
  Use route: $node(s_0, s_1, s_2) \rightarrow switch((s_0, s_1), 2)$
  $\rightarrow switch((s_0, s_2), 1)$
  $\rightarrow switch((s_2, d_2), 0)$
  $\rightarrow switch((d_0, s_2), 1)$
  $\rightarrow switch((d_0, d_1), 2)$
  $\rightarrow node(d_0, d_1, d_2)$

Fig. 10. Optimal oblivious single routing for $FT(m, 3)$

Theorem 7: $PERF(OSRM3) = \frac{m}{2}$.
Proof: From above discussion, using $OSRM3$, the SD pairs on each link have either at most $\frac{m}{2}$ source nodes or at most $\frac{m}{2}$ destination nodes. From Lemma 9, $PERF(OSRM3) \leq \frac{m}{2}$. From Theorem 4, an performance oblivious ratio of $\frac{m}{2}$ is the low bound for any single path routing scheme on $FT(m, 3)$. Hence, $PERF(OSRM3) = \frac{m}{2}$ and $OSRM3$ is an optimal oblivious routing algorithm for $FT(m, 3)$. □

The proposed oblivious routing algorithms, $OSRM2$, $OSRM3$, and $OMRMN$, are quite simple. The paths between two nodes can basically be enumerated, and thus, can be established with either a centralized or a distributed scheme. Detailed realization of the routing schemes is beyond the scope of this paper. Interested readers can refer to other literature such as [9] for more details.

In fat-trees with uncertain traffic demands, the performance of (unrestricted) multi-path routing is much better than that of single path routing. These results argue strongly that in a large fat-tree based system area network, the unrestricted multi-path routing should be used to alleviate the network contention problem. Moreover, these results raise questions in the current system area networks that only support a limited form of multi-path routing. One example is the InfiniBand, where only a limited number of paths (128) between any two processing nodes are supported. With such a restriction, it is difficult to achieve optimal load balancing with multi-path routing in fat-trees.

VI. Performance study

We compare the performance of several known single path routing algorithms, including the Multiple LID algorithm ($MLID$) in [15] and the widest shortest routing (WSR) algorithm. WSR was designed to achieve load balancing in the Internet environment. It works as follows. We first generate a traffic matrix where each SD pair has one unit of traffic. All links in the network are initialized with the same weight. The algorithm then computes routes for each SD pair in the following order $(0, 1)$, $(0, 2)$, ..., $(0, N-1)$, $(1, 0)$, ..., $(1, N-1)$, ..., $(N-1, 1)$, $(N-1, 1)$, ..., $(N-1, N-2)$. Every time a route is computed, the weight of each of the links along the route is increased by 1. When computing the route for each SD pair, the path with the smallest accumulated weight is selected. Note that the "shortest" heuristic enforces that only the shortest paths between two nodes are selected; and the "widest" heuristic spreads traffic from the same source among all links in the fat-tree. A recently proposed scheme [9] yields exactly the same routes as WSR.

	$FT(m, 2)$	$FT(m, 3)$
$OMRMN$	1	1
$OSRM2$	$\sqrt{\frac{m}{2}}$	-
$OSRM3$	-	$\frac{m}{2}$
$MLID$	$\frac{m}{2}$	$m-1$
WSR	$\frac{m}{2}$	$m-1$

TABLE I
Oblivious performance ratios of different routing algorithms

Table I shows the oblivious performance ratios for different routing algorithms. The worst case oblivious performance ratios for $MLID$ and WSR are obtained by analyzing the paths computed by the algorithms. This table shows (1) that our optimal single path oblivious routing algorithms provide better performance guarantees than other existing single path routing algorithms; and (2) that multi-path routing ($OMRMN$) is significantly better than single path routing.

We design experiments to investigate the performance of single path routing algorithms with practical traffic patterns. In particular, our optimal oblivious routing algorithms ($OSRM2$ and $OSRM3$) group SD pairs in a particular way so as to guarantee the best performance in the worst case condition. This, however, might yield lower performance on typical uniform traffic demands (average case performance). In fact, both $MLID$ and WSR fully spread traffic among all links in the fat-tree topology and should perform well (among single path routing schemes) for typical traffic demands. In this section, we will use the average case performance in the comparison and show that the proposed optimal oblivious algorithms not only provide the optimal worst case performance guarantees, but also often perform better in average cases. We will report results for $FT(32, 2)$, $FT(8, 3)$, and $FT(16, 3)$ that support 512, 128, and 1024 processing nodes respectively.

We will show the results for four types of traffic patterns: random uniform traffic, regular traffic, clustered traffic, and hot-spot traffic. In a random uniform traffic demand, each entry in the traffic matrix has an equal probability to send 1 unit of traffic (or not send any traffic). The regular traffic consists five different patterns on all nodes in the system: ring, 2-dimensional mesh, 3-dimensional mesh, hypercube, and binary tree. These communication patterns are frequently used in high performance applications. In a clustered traffic demand, the processing nodes are partitioned into groups of the same

size (size = 2, 4, 8, 16, 32, 64, 128 nodes). Each processing node in the system is in one group. The members in each group are randomly selected from all processing nodes. 1 unit of data is communicated between each pair of nodes in one group (all-to-all communication pattern within each group). This patterns represent the cases when the nodes in the system are allocated to different jobs with each job having the all-to-all communication. The hot-spot traffic is created as follows: the system contains a number of hot-spots, which are a group of processing nodes performing the all-to-all communication (1 unit of data between each pair in a hot-spot). The rest of the system is quiet. The number of hot-spots and the size of the hot-spots are parameters of this traffic pattern. For each data point, we produce 32 random instances and report the average performance ratio for the 32 instances. For regular traffic patterns, the node assignment is randomly generated for each instance. For example, in different ring patterns, the nodes can be in different positions of the ring.

The results for the random uniform traffic on $FT(16,3)$ are depicted in Fig. 11. The results on $FT(32,2)$ and $FT(8,3)$ are very similar. For the random traffic with different probability values, all of the single path routing algorithms achieve a similar performance and their performance ratios are very close to 1. This indicates that single path routing is effective in dealing with such demands. All of the routing schemes considered are able to evenly distribute the load when all pairs are communicating. When the load is high (the probability of the communication between two nodes is larger than 0.75), all schemes have a performance ratio of 1. When the communication is more sparse, since MLID, OSRM3, and WSR are deterministic and demand oblivious, they may not perform as good as the optimal routing scheme, which is typically demand-dependent. Hence, their performance ratios are slightly higher (when the network load is lower). Note again that the performance ratio is relative and is not directly related to the absolute network performance.

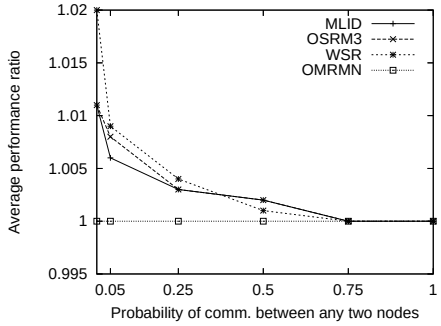


Fig. 11. Random uniform traffic on $FT(16,3)$

Tables II and III show the performance of different routing algorithms for regular traffic patterns. The regular traffic patterns are imposed on all the nodes in the system. The $MLID$ and WSR always have exactly the same performance ratio for symmetrical traffics on the fat-tree topology even though paths between a pair of nodes are different under the two routing schemes. They give different performance ratios only when the traffic matrix is asymmetrical, like the random traffic in Fig. 11. Thus, for all results presented in the rest of this section where the traffic matrices are symmetrical, the results for $MLID$ and WSR will be given together. As can be seen from Table II, $OSRM2$ offers a fairly large performance improvement over $MLID/WSR$ for the regular traffics on $FT(32,2)$, ranging from 6.8% for the hypercube pattern to 16.8% for the ring pattern. For $FT(16,3)$ and $FT(8,3)$, $OSRM3$ has slightly better performance, but the difference is statistically insignificant. This shows that the proposed optimal oblivious routing schemes do not sacrifice the average case performance for these traffic patterns.

Traffic Pattern	Average performance ratio	
	MLID/WSR	OSRM2
Ring	3.47	2.97
2D Mesh	1.88	1.74
3D Mesh	2.34	2.14
Hypercube	2.03	1.90
Binary Tree	2.37	2.20

TABLE II
AVERAGE PERFORMANCE RATIO FOR REGULAR TRAFFIC ON THE WHOLE $FT(32,2)$

Topology	Traffic Pattern	Average performance ratio	
		MLID/WSR	OSRM3
$FT(8,3)$	Ring	2.84	2.80
	2D Mesh	1.64	1.63
	3D Mesh	2.04	1.99
	Hypercube	1.90	1.90
	Binary Tree	2.07	2.03
$FT(16,3)$	Ring	3.78	3.78
	2D Mesh	2.83	2.83
	3D Mesh	2.48	2.43
	Hypercube	2.11	2.10
	Binary Tree	2.69	2.67

TABLE III
AVERAGE PERFORMANCE RATIO FOR REGULAR TRAFFIC ON THE WHOLE $FT(8,3)$ AND $FT(16,3)$

Fig. 12 shows the results for clustered traffic on $FT(32,2)$ and Fig. 13 shows the results for $FT(8,3)$. Results for $FT(16,3)$ are similar to those in $FT(8,3)$. As can be seen from the figures, the single path routing algorithms are not effective in dealing with such traffic demands: the average performance ratios for all single path routing schemes are much larger than 1, especially when the group size is small. This indicates that with single path routing, the network contention can be a problem with such traffic demands. The advantage of our optimal oblivious routing scheme is manifested in this experiment: $OSRM2$ performs noticeable better than $MLID/WSR$. Notice that when the group size is equal to 2, the average performance ratio for $MLID/WSR$ is larger than 4 on $FT(32,2)$. $OSRM2$ guarantees that the performance ratio for any traffic pattern is no more than $\sqrt{\frac{32}{2}} = 4$. Our schemes improve the performance noticeably on $FT(32,2)$, but only slightly on $FT(8,3)$ and $FT(16,3)$.

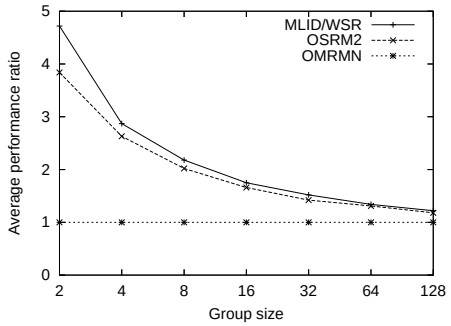


Fig. 12. Clustered traffic on $FT(32, 2)$

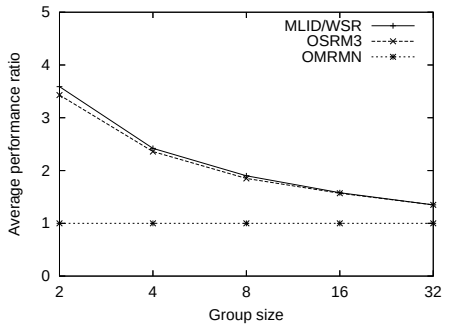


Fig. 13. Clustered traffic on $FT(8, 3)$

Fig. 14 and Fig. 15 show the results for hop-spot traffics. In the experiments, there are four hot-spots in the system with the size of the hot-spots varying. The performance ratios of all schemes are close to 1, which indicates that all schemes can handle this traffic pattern effectively. The performance of $OSRM3$ is similar to other single path routing schemes in $FT(16, 3)$ while $OSRM2$ is slightly (about 3% on average among all cases in the experiment) worse than WSR/DLID in $FT(32, 2)$. This is the only case in all of our experiments that $OSRM2$ performs worse on average: there exist traffic patterns that our proposed algorithms cannot handle as good as other algorithms. This is a common limitation of all deterministic routing schemes.

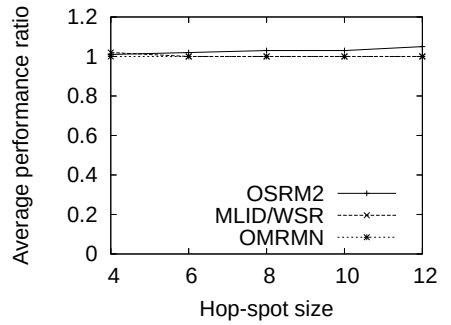


Fig. 14. Results for 4 hot-spots with different hot-spot sizes on $FT(32, 2)$

We have performed many other experiments with different traffic demands: including clustered traffic with random group sizes, clustered traffic with a random regular pattern in each group, and clustered traffic with random traffic in each group. We have also carried out experiments on other

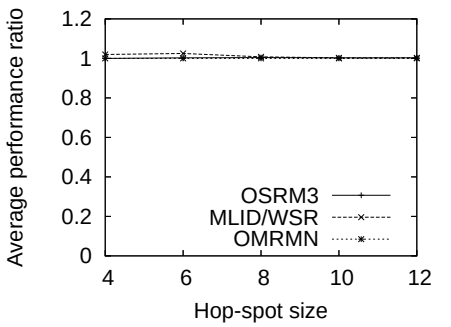


Fig. 15. Results for 4 hot-spots with different hot-spot sizes on $FT(16, 3)$

$FT(m, 2)$'s and $FT(m, 3)$'s. All of these experiments have similar results: our optimal oblivious routing algorithms are either comparable to or better than $MLID$ and WSR on average. Moreover, the improvement on $FT(m, 2)$ is quite noticeable in most cases as we showed in the figures and tables, while the improvement on $FT(m, 3)$ is small. This is because $OSRM2$ improves the worst-case performance ratio over $MLID/WSR$ on $FT(m, 2)$ by a factor of $\frac{\frac{m}{2}}{\sqrt{\frac{m}{2}}} = \sqrt{\frac{m}{2}}$ while $OSRM3$ improves the worst-case performance ratio over $MLID/WSR$ on $FT(m, 3)$ by a factor of $\frac{m-1}{\frac{m}{2}} \approx 2$. The hot-spot traffic is the only case in all of our experiments that the proposed schemes perform slightly worse than other schemes on average. These results indicate that our optimal single path oblivious routing algorithms can provide performance guarantees without sacrificing the average case performance and they often provide better performance in average cases.

VII. Related Work

The research most related to this work falls into three areas: the development of system area networks, routing on fat-tree, and oblivious routing. System area networks with the off-the-shelf networking technology such as InfiniBand [16] and Myrinet [22] have become more common recently. The load balancing problems in such networks motivated this research. Most routing research for system area networks (see for example, [5], [6], [19], [24], [25]) has focused on developing techniques for computing and establishing routes. In [9], [15], routing algorithms were developed for fat-tree based InfiniBand networks. We show that algorithms in [9], [15] are not optimal oblivious routing schemes. Routing performance with various routing algorithms, such as randomized routing and adaptive routing, and various performance metrics on the fat-tree topology has also been studied [8], [17], [18]. However, we are unaware of any work studying the routing performance on fat-trees with deterministic routing when the traffic demand is uncertain and changing.

Oblivious routing has recently attracted much attention due to its effectiveness in guaranteeing routing performance under uncertain and changing traffic demands in the Internet environment [1], [2], [28]. The bounds of the competitive ratios of oblivious routing on the general directed and undirected topologies have been analyzed [3], [10], [11], [12], [23]. In

[23], it was shown that for any undirected network, there exists an oblivious randomized routing algorithm that can achieve a polylogarithmic competitive ratio. The results in [3] show that for directed network, the optimal competitive ratio of oblivious routing algorithm is at least $O(\sqrt{n})$, where n is the number of nodes in the network. Works in [10], [11] considered the cases when demands are randomly chosen from a known distribution, and showed that routing algorithms with polylogarithmic competitive ratios can be obtained for directed networks and that the competitive ratio cannot be significantly improved. These lower bounds for general networks are much higher than the lower bound for fat-trees. Various polynomial time algorithms for computing the optimal oblivious routing were also developed [1], [2], [3], [4], [13]. These algorithms are still too computational intensive to be applied to find the optimal oblivious routing scheme for large fat-trees. Moreover, these methods can only be used to compute optimal oblivious routing for unrestricted multi-path routing, but cannot be used to compute optimal deterministic oblivious single path routing. In this paper, we consider deterministic oblivious routing in fat-trees. We show that in fat-trees, competitive ratio for multi-path oblivious routing is 1, which is much smaller than that lower bounds obtained for general networks. Moreover, optimal oblivious routing for fat-trees with unrestricted multi-path routing is given ($OMRMN$) without running the high complexity algorithms. We further give the bounds for the competitive ratios of single-path oblivious routing and develop optimal single path oblivious routing schemes for $FT(m, 2)$ and $FT(m, 3)$. Oblivious routing on specific topologies has also been studied [7], [14], [27]. However, the fat-tree topology has not been investigated.

VIII. Conclusion

We study oblivious routing in fat-tree based system area networks with deterministic routing under the assumption that the traffic demand is uncertain and changing. We show that single path routing cannot provide good performance guarantees while unrestricted multi-path routing is effective in balancing network loads in fat-trees. We develop optimal single path oblivious routing schemes for $FT(m, 2)$ and $FT(m, 3)$ and demonstrate that these optimal oblivious routing schemes can not only provide the optimal worst-case performance guarantees, but also offer better performance than existing single path routing schemes in average cases. These results may directly influence the design of systems with large scale fat-tree based networks such as large HPC clusters.

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APPENDIX

Lemma 5: Let A be a set of SD pairs. $|A| \leq L(A) \times LS(A)$.
Proof: We will prove this lemma by induction on $L(A)$.

Base case: when $L(A) = 1$, $|A| \leq LS(A)$. Let $B = \{(s_1, d_1)\}$ be one largest node disjoint subset of A. If A only contains (s_1, d_1), the case is proven. Otherwise, there exists another SD pair (s_2, d_2) in A. Since $L(A) = 1$, either $s_1 = s_2$ or $d_1 = d_2$. We will show that if $s_1 = s_2$, $A = S^A_{s_1}$. Similar logic can be used to show that if $d_1 = d_2$, $A = D^A_{d_1}$. In both case, $|A| \leq L(A) \times LS(A) = LS(A)$.

Let us now prove that if $s_1 = s_2$, $A = S^A_{s_1}$. Assume that $A \neq S^A_{s_1}$, there exists a SD pair (s, d) such that $s \neq s_1$. In this case, if $d \neq d_1$, then $\{(s_1, d_1), (s, d)\}$ is a node disjoint subset of A; otherwise, $d = d_1$ and $\{(s_1, d_2), (s, d)\}$ is a node disjoint subset of A. Hence, $L(A) \geq 2$ (contradiction).

Induction case: Assume that $|A| \leq L(A) \times LS(A)$ when $L(A) \leq k$ (induction hypothesis), we will prove that $|A| \leq L(A) \times LS(A)$ when $L(A) = k + 1$.

Let $B = \{(s_1, d_1), (s_2, d_2), ..., (s_{k+1}, d_{k+1})\}$ be a largest node disjoint subset of A. If $SRC(A) = \{s_1, s_2, ..., s_{k+1}\}$, $|A| \leq (k+1) \times LS(A)$ (since each source nodes can at most have $LS(A)$ SD pairs in A and there are $k+1$ source nodes) and the theorem is proven.

If $SRC(A) \supset \{s_1, s_2, ..., s_{k+1}\}$, there must exist a source node $s \in SRC(A)$ such that $s \neq s_i$, $1 \leq i \leq k+1$. Let $(s, d) \in A$. We have $d \in DST(B)$ (Otherwise, $(s, d) \bigcup B$ is node disjoint and B is not a largest node disjoint subset). Without loss generality, let $d = d_1$. We have $\{(s, d_1), (s_1, d_1)\} \subseteq D^A_{d_1}$. Obviously $L(D^A_{d_1}) = 1$, $LS(D^A_{d_1}) \leq LS(A)$, and $LS(A - D^A_{d_1}) \leq LS(A)$.

Next, we will show that $L(A - D^A_{d_1}) = k$. From Lemma 2, $L(A - D^A_{d_1}) \geq L(A) - L(D^A_{d_1}) = k + 1 - 1 = k$. Since $L(A) = k + 1 \geq L(A - D^A_{d_1})$, to show that $L(A - D^A_{d_1}) = k$, we only need to show that $L(A - D^A_{d_1}) \neq k + 1$. We prove this by contradiction. Assume that $L(A - D^A_{d_1}) = k + 1$. Let $C = \{(s'_1, d'_1), (s'_2, d'_2), ..., (s'_{k+1}, d'_{k+1})\}$ be a largest node disjoint subset of $A - D^A_{d_1}$. We have $s_1 \in SRC(C)$ (otherwise, $C \bigcup \{(s_1, d_1)\}$ is a node disjoint subset of A and $L(A) \geq k + 2$). Similarly, $s \in SRC(C)$. Let us assume that $s_1 = s'_1$ and $s = s'_2$. d'_2 must be in $DST(B) - \{d_1\}$ (otherwise, $(s, d'_2) \bigcup B$ is node disjoint and B is not the largest node disjoint subset). Similarly, d'_1 must be in $DST(B) - \{d_1\}$. Let $d_{k+1} = d'_1$ and $d_2 = d'_2$. We have $s_2 \in SRC(C)$ (otherwise, $C - \{(s'_2, d'_2)\} + \{(s_2, d_2), (s, d_1)\}$ is a node disjoint subset of A and $L(A) \geq k+2$). This process (finding that a source node s_i in B belongs to $SRC(C)$ and then finding that destination node d' such that $(s_i, d') \in C$ belongs to $DST(B)$) can be repeated. Once the process cannot continue, one can construct a node disjoint subset of A whose size is $k + 2$. Since there are a finite number of elements in B and C, this process will stop at some point (in the worst case, one of B or C runs out of elements). Thus, $L(A) \geq k + 2$, which contradicts the assumption that $L(A) = k + 1$. Hence, $L(A - D^A_{d_1}) = k$. By the induction hypothesis,

$$|A - D^A_{d_1}| \leq L(A - D^A_{d_1}) \times LS(A - D^A_{d_1}) \leq k \times LS(A)$$

$$|D^A_{d_1}| \leq L(D^A_{d_1}) \times LS(D^A_{d_1}) \leq LS(A).$$

Hence, $|A| = |A - D^A_{d_1}| + |D^A_{d_1}| \leq (k+1) \times LS(A)$. □

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