5.1 Manipulating Generating Functions

5.1.1 Finding an expansion

Suppose that we have, say \( f(x) = \frac{1}{(1-x)^t} \). How would we expand it into an infinite sum? So far you have learned two methods:

1. Use the Generalized Binomial Theorem: \((1 + z)^\alpha = 1 + \alpha z + \frac{\alpha(\alpha-1)}{2!} z^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!} z^3 + \cdots\), with \( z = -x \) and \( \alpha = -t \).

2. Notice that \( f(x) \) can be obtained by repeatedly taking the derivative of \( \frac{1}{(t-1)! \cdot (1-x)} = \frac{1}{(1-t)!} \left(1 + x + x^2 + \cdots\right) \) for \( t - 1 \) times.

The two approaches above are quite tedious and error-prone. Instead, one can use Table 335 in page 335 of the textbook, and find out that

\[
f(x) = \sum_{n=0}^{\infty} \binom{t+n-1}{n} x^n.
\]

Using the table has another advantage. In particular, occasionally you have to deal with strange functions, say \( f(x) = \ln \left( \frac{1}{1-x} \right) \). The two approaches above will fail in this case, but by looking up the table,

\[
f(x) = \sum_{n=1}^{\infty} \frac{1}{n} x^n.
\]

To see why this is true, note that

\[
f(x) = \int_0^x \frac{1}{1-t} dt = \int_0^x (1 + t + t^2 + \cdots) dt.
\]

5.1.2 Differentiating a generating function

The formulas. Define the operator \( \Delta \) as follows: for any differentiable function \( f \), let \( \Delta(f) = xf'(x) \). In other words, if \( f(x) = F_0 + F_1 x + F_2 x^2 + \cdots \) then

\[
\Delta(f) = F_1 x + 2F_2 x^2 + 3F_3 x^3 + \cdots = \sum_{n=0}^{\infty} nF_n x^n.
\]
We can further define $\Delta^k(f) = \Delta(\Delta^{k-1}(f))$ for any positive integer $k$, with $\Delta^0(f) = f$. Then

$$\Delta^k(f) = \sum_{n=0}^{\infty} n^k F_n x^n .$$

Those operations are linear, meaning $\Delta^k(f + g) = \Delta^k(f) + \Delta^k(g)$.

**AN APPLICATION.** Let’s find the closed-form solution for the familiar sum

$$A_n = 1^2 + \cdots + n^2$$

via generating functions. Again, we want to solve the generating function

$$h(x) = \sum_{n=0}^{\infty} A_n x^n = \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} k^2 \right) x^n .$$

The function $h(x)$ looks like the multiplication of some particular $f(x)$ and $g(x)$. If we pick

$$f(x) = 1 + x + x^2 + \cdots$$

then $g(x) = \sum_{n=0}^{\infty} n^2 x^n = \Delta^2(f)$. On the other hand, recall that $f(x) = \frac{1}{1-x}$ and thus

$$g(x) = \Delta^2 \left( \frac{1}{1-x} \right) = \Delta \left( \frac{x}{(1-x)^2} \right) .$$

Here we need to be a bit careful in performing the $\Delta$ operator on $\frac{x}{(1-x)^2}$, since later we’ll have to expand $f(x)g(x)$ into partial fractions, and we don’t want this step to be overly tedious. Thus we’ll simplify the fraction $\frac{x}{(1-x)^2}$ as follows:

$$\frac{x}{(1-x)^2} = \frac{1-(1-x)}{(1-x)^2} = \frac{1}{(1-x)^2} - \frac{1}{1-x} .$$

Hence

$$g(x) = \Delta \left( \frac{1}{(1-x)^2} - \frac{1}{1-x} \right) = \Delta \left( \frac{1}{(1-x)^2} \right) - \Delta \left( \frac{1}{1-x} \right) = \frac{2x}{(1-x)^3} - \frac{x}{(1-x)^2} .$$

You may stop here, but it’s better to simplify $g(x)$ further

$$g(x) = \frac{2x}{(1-x)^3} - \frac{x}{(1-x)^2} = \frac{2}{(1-x)^3} - \frac{2}{(1-x)^2} - \left( \frac{1}{(1-x)^2} - \frac{1}{1-x} \right) = \frac{2}{(1-x)^3} - \frac{3}{(1-x)^2} + \frac{1}{1-x} .$$

Then

$$h(x) = f(x)g(x) = \frac{2}{(1-x)^4} - \frac{3}{(1-x)^3} + \frac{1}{(1-x)^2}$$

$$= \sum_{n=0}^{\infty} \left( 2 \cdot \binom{n+3}{n} - 3 \cdot \binom{n+2}{n} + \binom{n+1}{n} \right) x^n$$

$$= \sum_{n=0}^{\infty} \frac{(2n+1)n(n+1)}{6} x^n .$$

Hence $A_n = (2n+1)n(n+1)/6$.

**Exercise 5.1** Use the generating-function method to find the closed-form solution of $B_n = 1^3 + \cdots + n^3$ and $C_n = 1^4 + \cdots + n^4$. 
5.2 Randomized QuickSort, Revisited

Recall that the expected running time $C_n$ of Randomized QuickSort on an array of $n$ numbers follows this recurrence

$$ C_n = n + 1 + \frac{2}{n} \sum_{k=0}^{n-1} C_k , $$

for every $n \geq 1$, and $C_0 = 0$. In Scribe 3, we had to use a trick to simplify the recurrence, as it looked quite hopeless to deal with the recurrence above directly. Here with the powerful generating-function method, we now can actually solve the formidable recurrence above. As usual, let

$$ C(x) = C_0 + C_1 x + C_2 x^2 + \cdots $$

By using the recurrence of $C_n$, we have

$$ C(x) = C_0 + \sum_{n=1}^{\infty} \left( n + 1 + \frac{2}{n} \sum_{i=0}^{n-1} C_i \right) x^n $$

$$ = \sum_{n=1}^{\infty} (n + 1) x^n + \sum_{n=1}^{\infty} \frac{2}{n} \left( \sum_{i=0}^{n-1} C_i \right) x^n $$

$$ = \sum_{k=0}^{\infty} (k + 2) x^{k+1} + 2 \sum_{k=0}^{\infty} \frac{1}{k+1} \left( \sum_{i=0}^{k} C_i \right) x^{k+1} . $$

The sum above is rather entangled due to the fraction $\frac{1}{k+1}$, but we can get rid of it if we differentiate both sides. Specifically,

$$ C'(x) = \sum_{k=0}^{\infty} (k + 2)(k + 1) x^k + 2 \sum_{k=0}^{\infty} \left( \sum_{i=0}^{k} C_i \right) x^k . $$

On the one hand, by observing that

$$(k + 2)(k + 1) = 2 \cdot \binom{k + 2}{k} ,$$

we can simplify the first sum as

$$ \sum_{k=0}^{\infty} (k + 2)(k + 1) x^k = 2 \sum_{k=0}^{\infty} \binom{k + 2}{k} x^k = \frac{2}{(1 - x)^3} . $$

On the other hand, using the rule of multiplication for generating functions, we can simplify the second sum as follows:

$$ \sum_{k=0}^{\infty} \left( \sum_{i=0}^{k} C_i \right) x^k = C(x) \cdot \sum_{i=0}^{\infty} x^i = \frac{C(x)}{1 - x} . $$

Hence we obtain an ordinary differential equation

$$ xC'(x) = \frac{2}{(1 - x)^3} + \frac{2C(x)}{1 - x} . $$

One can solve this equation to obtain $C(x) = \frac{2}{(1 - x)^2} \ln \left( \frac{1}{1-x} \right)$; we do not cover the details here. We can expand $C(x)$ as follows:

$$ C(x) = \frac{2}{(1 - x)^2} \ln \left( \frac{1}{1-x} \right) = 2 \left( \sum_{n=0}^{\infty} (n + 1) x^n \right) \cdot x \left( \sum_{n=0}^{\infty} \frac{1}{n+1} x^n \right) = 2x \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{1}{k+1} (n-k+1) x^n . $$
The coefficient of \(x^n\) is complex, but we can simplify it as follows:

\[
\sum_{k=0}^{n} \frac{1}{k+1} (n - k + 1) = \sum_{k=0}^{n} \left(\frac{n + 2}{k+1} - 1\right) = (n + 2)H_{n+1} - (n + 1)
\]

where \(H_{n+1}\) is the \((n+1)\)th harmonic number. Thus

\[
C(x) = 2x \sum_{n=0}^{\infty} \left((n + 2)H_{n+1} - (n + 1)\right)x^n = \sum_{m=1}^{\infty} (2(m + 1)H_m - 2m)x^m.
\]

Hence \(C_n = 2(n + 1)H_n - 2n\).

### 5.3 Exponential Generating Functions

Given a sequence \(\{F_n\}\), previously we define a generating function

\[f(x) = \sum_{n=0}^{\infty} F_n x^n.\] (5.1)

Sometimes, however, it’s more convenient to define the following generating function

\[g(x) = \sum_{n=0}^{\infty} \frac{F_n}{n!} x^n.\] (5.2)

The form of generating functions in Equation 5.1 is called ordinary generating functions, whereas that in Equation 5.2 is called exponential generating functions.

**An Application.** Suppose that a ship arrives at a port, and \(n\) sailors on board go ashore for revelry. Later at night, the sailors return to the ship and, in their state of inebriation, each chooses a random cabin to sleep in. If the ship has \(n\) cabins and if the sailors never share a cabin, what is the probability that every sailor sleeps in a wrong cabin?

To solve this problem, some terminology is needed. We say that a permutation \(\pi\) on \(\{1, \ldots, n\}\) has a fixed point \(i \in \{1, \ldots, n\}\) if \(\pi(i) = i\). A derangement on \(\{1, \ldots, n\}\) is a permutation on \(\{1, \ldots, n\}\) that has no fixed point. Let \(D_n\) denote the number of derangements on \(\{1, \ldots, n\}\). Then the chance that every sailor sleeps in a wrong cabin is

\[
\frac{D_n}{n!}.
\]

We now compute \(D_n\). Again, define

\[f(x) = \sum_{n=0}^{\infty} \frac{D_n}{n!} x^n.\]

Let’s begin by finding a recurrence for \(D_n\). Fix an integer \(0 \leq k \leq n\), and let’s count the number of permutations on \(\{1, \ldots, n\}\) that has exactly \(k\) fixed points. Note that there are \(\binom{n}{k}\) ways to choose the \(k\) fixed points, and then for those fixed points, there are exactly \(D_{n-k}\) corresponding permutations on \(\{1, \ldots, n\}\). Since there are exactly \(n!\) permutations on \(\{1, \ldots, n\}\),

\[n! = \sum_{k=0}^{n} \binom{n}{k} D_{n-k}.\]
In other words,
\[ \sum_{k=0}^{n} \frac{D_{n-k}}{(n-k)!} \cdot \frac{1}{k!} = 1. \] (5.3)

Equation 5.3 above reminds us of a multiplication of generating functions. One factor is of course \( f(x) \); the other should be
\[ g(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x. \]
(The identity above is obtained by looking up Table 335.) From Equation 5.3,
\[ f(x)g(x) = \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} \frac{D_{n-k}}{(n-k)!} \cdot \frac{1}{k!} \right) x^n = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}. \]

Then
\[ f(x) = \frac{e^{-x}}{1-x} = \left( \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!} \right) \left( \sum_{n=0}^{\infty} x^n \right) = \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} \frac{(-1)^k}{k!} \right) x^n. \]

Hence
\[ D_n = n! \cdot \sum_{k=0}^{n} \frac{(-1)^k}{k!}, \]
and thus the chance that every sailor sleeps in a wrong cabin is
\[ \sum_{k=0}^{n} \frac{(-1)^k}{k!}. \]

This probability quickly converges to \( 1/e \) when \( n \to \infty. \)