3.1 Generalization and Specialization

Suppose that we want to solve a problem. One common first step is to study the problem in special cases, so that we understand the problem better—this process is called specialization. Another approach is to generalize the problem, studying a more general problem. You might think that generalizing makes the problem harder, but in several cases, a correct generalization will help us get rid of irrelevant information, and we can clearly see the entire picture.

An example. Suppose that our problem involves a property of triangles. To specialize this problem, we can, for example, consider the case where the triangle is equilateral. There are also several possible directions to generalize the problem. For example, we can generalize this property for polygons, or for tetrahedrons. See Figure 3.1 for an illustration. Note that generalization is mere guessing, and it is well possible that the generalized statement does not hold.

Pythagorean theorem. To illustrate the use of the methods above, consider the classical Pythagorean theorem. Geometrically, it states that on any right triangle, the total area of the two squares on the legs is the same as the area of the square on the hypotenuse. In other words, if the hypotenuse has length $a$, and the two legs have length $b$ and $c$ respectively, then $a^2 = b^2 + c^2$.

Now how should we generalize this problem? The theorem is involved with squares, so can we generalize it to polygons then? Of course the polygons must have some common property, otherwise you can find trivial counter-examples. In Pythagorean theorem, the polygons are of the same shape, specifically, they are all
squares. So let’s generalize the theorem such that instead of 3 squares, we have 3 similar polygons, meaning that those polygons are of the same shape. See Figure 3.2 for an illustration.

The generalized result looks complicated, but in fact it is equivalent to the classical Pythagorean theorem. To see why, suppose that the area of the polygon on the hypotenuse is \(\lambda a^2\). Because the polygons are similar, the area of the polygons on the legs is \(\lambda b^2\) and \(\lambda c^2\) respectively. The generalized result claims that \(\lambda a^2 = \lambda b^2 + \lambda c^2\), for any real number \(\lambda > 0\). Even better, if we can prove the generalized result for a particular shape (instead of all possible shapes), then we also obtain the classical Pythagorean theorem.

Given that we have the freedom to choose the shape, it’s now time for specialization. The simplest polygon shape is certainly a triangle, but what triangle? Looking back in the picture of the classical Pythagorean theorem, there’s already a triangle in front of our noses, so maybe it’s our shape then?

In more details, let \(ABC\) be the original right triangle, with \(BC\) as the hypotenuse. Now, on \(BC\), we draw a triangle of the same shape as \(ABC\); this coincides with \(ABC\) itself. The triangles on the legs are \(ABH\) and \(ACH\), as shown in Figure 3.3, where \(AH\) is the altitude of \(ABC\). The specialized theorem claims that the total area of \(ABH\) and \(ACH\) is the area of \(ABC\), which is obviously true.

### 3.2 Another Example: Tiling

Suppose that we have a board of size \(2^n \times 2^n\), and I mark one particular cell of the board. We want to prove that the remaining cells can be tiled using triminos. See Figure 3.4 for an illustration.
At the first glance, it is a hard problem, since the marked cell is arbitrary. To specialize this problem, we need to find a special location of this marked cell, and it seems that the most likely places are the four corners. To prove that the specialized statement holds, it seems that the most natural approach is via induction. The base case $n = 1$ is trivial. Let’s now consider the inductive case. To deal with a $2^n \times 2^n$ board, the most natural solution is to divide it into 4 sub-boards, each of size $2^{n-1} \times 2^{n-1}$, so that we can exploit the induction hypothesis. Without loss of generality, suppose that the marked cell is at the top left corner. Clearly we can tile the top left sub-board, by the induction hypothesis. For the other sub-boards, we can also tile them with the marked corners positioned as in Figure 3.5. These missing cells can be tiled by another trimino.

Now let’s prove our original statement via induction. Again, the base case $n = 1$ is trivial. For the inductive case, again divide the board into 4 sub-boards, each of size $2^{n-1} \times 2^{n-1}$, and assume that the marked cell is at the top left board. Clearly we can tile the top left sub-board, by the induction hypothesis. For the other sub-boards, we can also tile them with the marked corners positioned as in Figure 3.5, and fill these three cells by another trimino.

3.3 Yet Another Example: Lines In The Plane

Suppose that we have $n$ lines in a plane in a “general position”, meaning that no two lines are parallel, and no three lines meet at the same point. We’d like to know how many regions those lines form. Let’s call this number $F(n)$. 
Let’s try to generalize this problem. What we have here is a problem in 2-dimensional space, so maybe we can generalize it to 3-dimensional space as follows: Suppose that we have \( n \) planes in a general position, how many parts of the space do those planes form? We can even generalize the problem to \( k \)-dimensional space. At this point, the generalized problem seems to be quite scary, so let’s specialize this general problem with \( k = 1 \): For \( n \) points in a line, how many segments do we have? See Figure 3.6 for an illustration. Let’s call this number \( G(n) \). One can easily find \( G(n) = n + 1 \).

Now let’s get back to the problem of computing \( F(n) \). Let’s compute \( F(n) \) for a few first values. The numbers are given in the table below.

<table>
<thead>
<tr>
<th>( n )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>( G(n) )</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
</tr>
<tr>
<td>( F(n) )</td>
<td>2</td>
<td>4</td>
<td>7</td>
<td>11</td>
<td>16</td>
</tr>
</tbody>
</table>

The function \( F(n) \) grows much faster than \( G(n) \), but its rate of growth \( F(n + 1) - F(n) \) seems to be exactly \( G(n) \). So we suspect that we have found a rule \( F(n + 1) = F(n) + G(n) = F(n) + n + 1 \) for every \( n \), and then can find a closed-form solution of \( F(n) \) as follows:

\[
F(n) = n + F(n-1) = n + (n-1) + F(n-2) = \cdots = n + (n-1) + \cdots + 2 + F(1) = \frac{n(n+1)}{2} - 1 + F(1) = \frac{n(n+1)}{2} + 1 .
\]

Let’s try to understand why the recurrence \( F(n + 1) = F(n) + G(n) \) holds. Suppose that we have \( n \) lines in a general position, and they form \( F(n) \) regions. Now let’s draw a new line. This intersects the prior \( n \) lines, forming \( G(n) = n + 1 \) segments. But these segments correspond to the new \( n + 1 \) regions that the new line creates, so it’s why \( F(n + 1) = F(n) + G(n) \). See Figure 3.7 for an illustration.

Here we don’t solve the generalized problem in \( k \)-dimension, but you can also find their recurrences using the argument above.