1. (60 points) Recall that when we run Randomized Quicksort on an array $A[1:n]$, we pick a uniformly random element of the array as the pivot, and the running time for pivot sampling plus the partitioning is $n+1$. Let $X_n$ denote the random variable for the running time of Randomized Quicksort on an array $A[1:n]$. Let $f_n(z)$ denote the probability generating function of $X_n$. Prove that for every $n \geq 1$,

$$f_n(z) = \frac{1}{n} \sum_{r=0}^{n-1} z^{n+1} f_r(z) f_{n-r-1}(z).$$

**Hint:** A common mistake is to claim that

$$X_n = n + 1 + \frac{2}{n} \sum_{r=0}^{n-1} T_r,$$

where each $T_r$ has the same distribution as $X_r$. This is wrong. Instead, let $S$ be the random variable for the size of the left subarray when we quicksort an array of $n$ elements. Then

$$\Pr[X_n = k] = \sum_{r=0}^{n-1} \Pr[X_n = k, S = r] = \sum_{r=0}^{n-1} \Pr[Y_r + Z_r = k - (n + 1), S = r],$$

where $Y_r$ and $Z_r$ are independent random variables of the same distribution as $X_r$ and $X_{n-r-1}$ respectively, and they are also independent of $S$. Intuitively, $Y_r$ and $Z_r$ are the cost of quicksorting the left and right subarrays on $A[1:n]$, if you can magically manage to find a pivot such that the left size is $r$ and the right size is $n - r - 1$.

2. (50 points) A certain grocery store issues coupons of $n$ colors, and a collector wishes to obtain coupons of all colors. Each time the collector buys a product, he will receive a coupon of uniformly random color. The collector will keep buying until he gets all the $n$ colors. Let $X$ be the random variable for number of purchases of the collector. Give an explicit form of the probability generating function $G_X(z)$. 