Brief Review

Introduction to Probability Theory

Introduction to Queing Theory

NOTES

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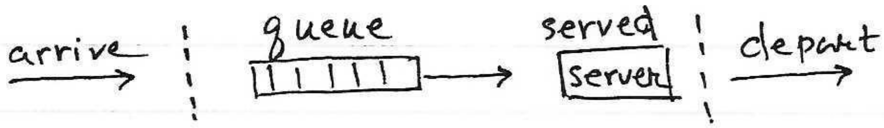
Review : Probability Theory
Intro : Queuing Theory

Probability Theory : formalizing "random events"
games of "chance"

probability spaces and measures ; random variables.
discrete and continuous distributions

Queuing Theory : related to : waiting in line!



Example: arrival rate $\lambda = 3$ /sec. service rate $\mu = 4$/sec.

$\lambda/\mu = 3/4 < 1$

Under very general assumptions : no queue develops that is not "finite size" on the average.

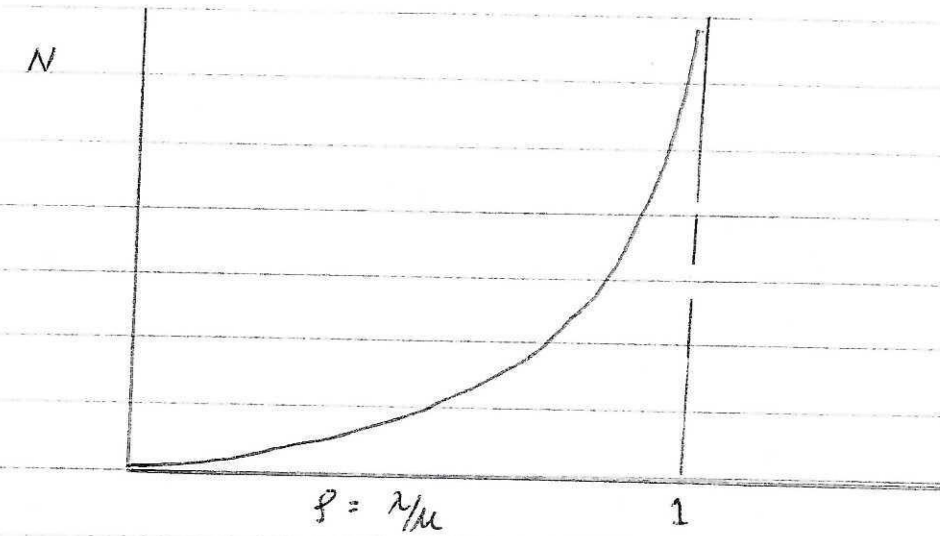
Little's Theorem : $N = \lambda T$ where N is the number in the system and T is the average time in the system.

Under addition restrictions , ie. stronger conditions on the arrival & service processes get : If $\rho = \frac{\lambda}{\mu} < 1$ (ρ is called utilization)

Average delay (queue + service) $T = \frac{1}{\mu - \lambda}$

Average wait in queue $W = \frac{\rho}{\mu - \lambda}$

Average # customers in system $N = \frac{\rho}{1-\rho}$



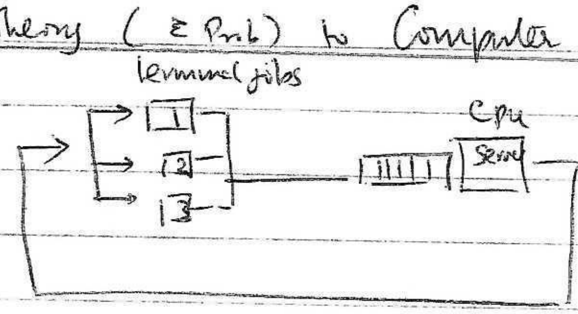
{Probability Theory
Stochastic Processes}

Applications of Queueing Theory and Prob. Theory to Networks

Delay models over communication links
Routing Models
Multiaccess Communication
Reliable Communication (Noise)
Routing (Flow models) Flow control.

Applications of Queueing Theory (& Prob) to Computer Performance Eval.

Modeling Workload
Resource Modeling



Limitations: Assumptions often not valid. / Closed form solutions very complex.

Ω Sample Space : possible outcomes of an experiment

$\Omega = \{1, 2, 3, 4, 5, 6\}$ "toss of a die"

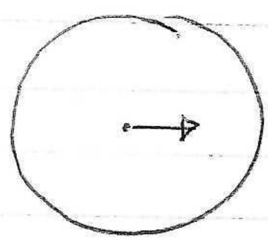
↳ elementary outcome

$\Omega = \{(a,b) \mid a \in 1, \dots 6, b \in 1, \dots 6\}$ "toss two dice"

# of phone calles coming into an exchange / central office in 1 minute

assume no upper bound

$\Omega = \{0, 1, 2, 3, \dots\}$



Spinning a pointer

$\Omega = \{\theta : 0 \leq \theta < 2\pi\}$

$\Omega = \{x : 0 \leq x \leq 1\}$

$\Omega =$ the real line.

Events

subsets of "interest" of Ω. $\mathcal{F}$ collection of subsets

$\Omega = \{1, 2, \dots 6\}$ Event $E =$ even #.

$\Omega = \{0, 1, 2, \dots\}$ Event. # calles ≤ 8.

Spinner Event $0 \leq \theta \leq \frac{\pi}{2}$.

Properties of $\mathcal{F}$:

σ - algebra.

$$\begin{cases} \text{closed under union (countable)} \\ \text{closed under intersection (countable)} \\ \text{closed under complement.} \\ \text{includes } \emptyset \text{ empty set.} \\ \text{includes } \Omega \text{ full set.} \end{cases}$$

<u>Probability space</u> (Kolmogorov Axioms)

$(\Omega, \mathcal{F}, P)$

P is a probability measure on events $E \in \mathcal{F}$.

(1) $0 \leq P(E) \leq 1$

(2) $P(\Omega) = 1$

additive (3) $P(E_1 \cup E_2) = P(E_1) + P(E_2)$ if $E_1 \cap E_2 = \phi$.

σ-additive (3') $P(\bigcup_i E_i) = \sum_i P(E_i)$ for any countable disjoint set of events.

Finite Ω can always use power set for events.

Suppose $\Omega = [0, 1]$

No σ-additive measure on <u>power</u> set of $[0,1]$.

However σ-algebra generated by intervals of the ~~form~~ form $[0 \leq x]$ $0 \leq x \leq 1$ is <u>OK</u>.

Suppose $E_1 \subset E_2 \subset E_3 \ldots$ ascending sequence.

it can be shown that

$$P(\bigcup_1^\infty E_i) = \lim_{n\to\infty} P(E_n)$$

Examples with previous sample spaces.

$\{0, 1, 2, \ldots\}$ $\Omega =$

Prob. ~~[illegible]~~ measure. $\rightarrow$

Conditional Probabilities

$$P(E \mid F) = \frac{P(E \cap F)}{P(F)} \qquad P(F) > 0$$

$$P(E \mid F) \cdot P(F) = P(E \cap F)$$

Bayes Theorem

Let $P_1 \ldots P_n$ be a partition of Ω.

$$P(F \mid E) = \frac{P(E \mid F) \cdot P(F)}{P(E)} = \frac{P(E \mid F) \cdot P(F)}{\sum P(E \mid P_i) \cdot P(P_i)}$$

Independence

$$P(E \cap F) = P(E) \cdot P(F)$$

Independent experiments

E_1 E_2

tossing coin, drawing a card.

Tossing 3 dice.

$\Omega = \{(x, y, z)\}$ $x = 1, 2, \ldots 6$, $y = 1, \ldots 6$, $z = 1, \ldots 6$

Event : First dice is a 2

3rd dice is a 3

Random Variables

Given $(\Omega, \mathcal{F}, P)$

$X : \Omega \to \mathbb{R}$ is a random variable

if X is a measurable function wrt Borel Field $\mathcal{F}$ (the σ-algebra)

Equivalently: $\{\omega : X(\omega) \le r\} \in \mathcal{F}$ for all r.

or $X^{-1}(-\infty, r] \in \mathcal{F}$.

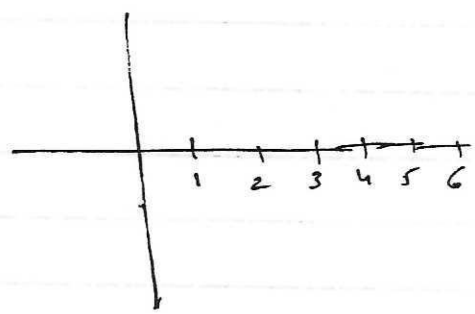
Example

$\Omega = \{1, 2, 3, 5, 6\} \to \mathbb{R}$

$X(1) = 1$

$\vdots$

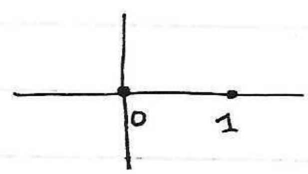
$X(6) = 6$



$\Omega = \{H, T\} \to \mathbb{R}$

$X(H) = 0$

$X(T) = 1$



$\Omega = \{\overset{T}{\cancel{H}}, HT, HHT, \dots\}$

$X(T) = 0$

$X(HT) = 1$

$X(HHT) = 2$

$\vdots$

Spinner ?

We usually just refer to the r.v. X and forget about Ω & the function $X(\omega)$.

Thus we simply consider the induced probability distribution on R. (Prob $(-\infty, x]$)

Cumulative Distribution Function (c.d.f)

A random variable X is characterized by its c.d.f. F.

$$F_X(x) = P[\{\omega : X(\omega) \leq x\}]$$

or

$$F_X(x) = P[X \leq x]$$

or

$$F(x) = P(X \leq x)$$

Properties of a c.d.f.

(1.) $F(\infty) = 1$; $F(-\infty) = 0$

(2) $F(x)$ is monotone non-decreasing

i.e. $b > a \Rightarrow F(b) \geq F(a)$

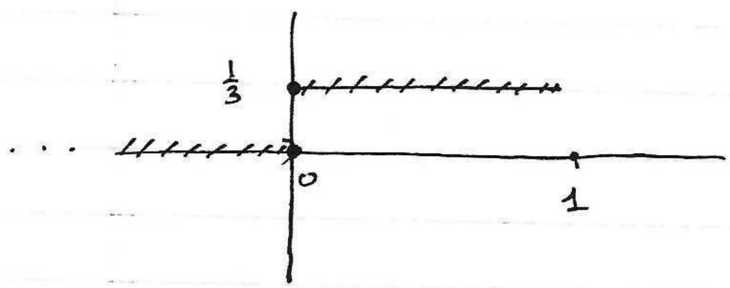
(3) $F(x)$ is right continuous

i.e. $\lim\limits_{\substack{h>0 \\ h\to 0}} F(x_0+h) = F(x_0)$

Some c.d.f's

Example Tossing a coin $H = 0$ with prob $(H) = 1/3$

$T = 1$ with prob $(T) = 2/3$

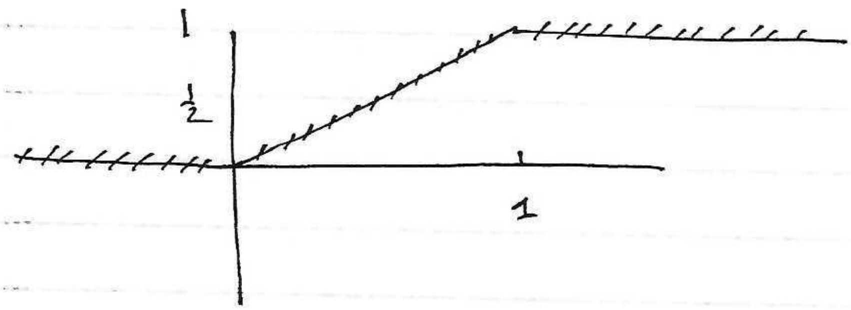


cdf is

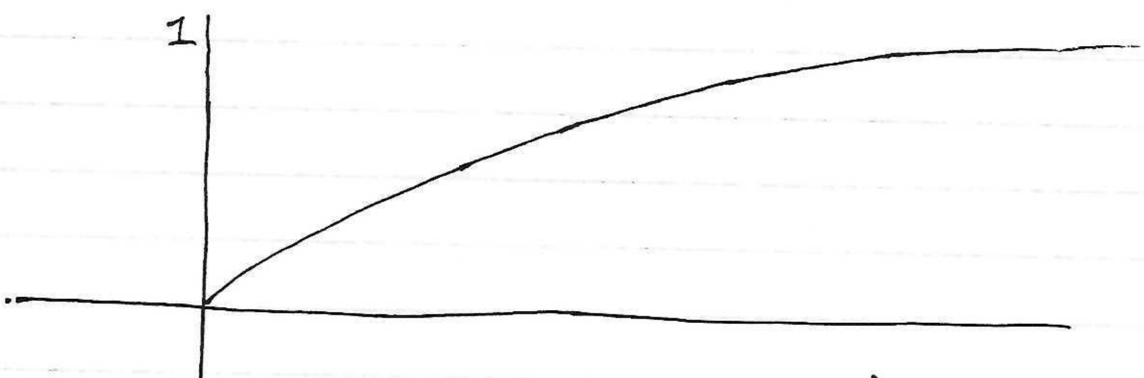
Example of a discrete r.v. with a finite or countable number of values x_i with

$$P(X = x_i) > 0 \quad \text{and} \quad \sum_i P(X = x_i) = 1$$

Example Uniform distribution on interval $[0, 1]$



Example exponential c.d.f.



$$F_X(x) = 1 - e^{-\lambda x}$$

Discrete Random Variables

A distribution function that jumps at a finite number of points in a finite interval is said to be a discrete random variable.

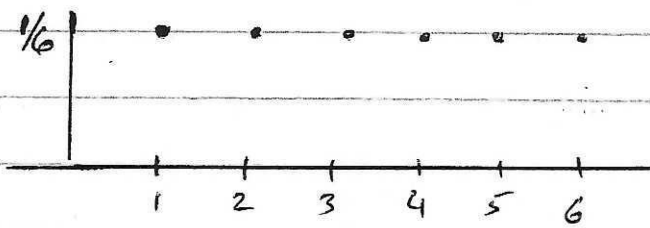
Characterized by $P[X = x_i] = p_i$

s.t. $\sum_i p_i = 1$

The probability density function f for a discrete r.v. is:

$$f_X(x_i) = P[X = x_i]$$

Example: dice



The Poisson Distribution

$$P(Z = k) = \frac{e^{-\lambda} \lambda^k}{k!}$$

$k = 0, 1, 2, 3, \ldots$

λ is a parameter

Let $\lambda = 2$, then:

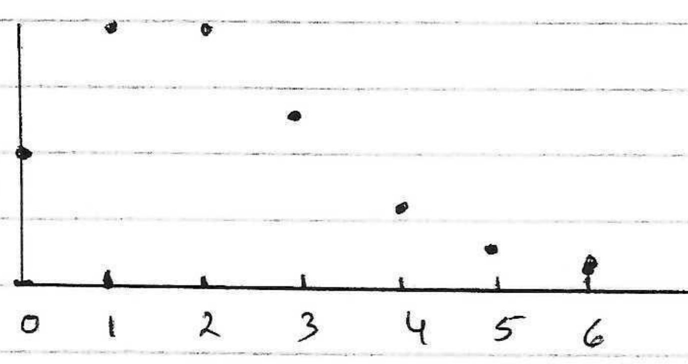
$P(Z = 0) = e^{-\lambda} = e^{-2}$

$P(Z = 1) = \frac{e^{-2} \cdot 2}{1} = 2e^{-2}$

$P(Z = 2) = e^{-2} \cdot 4/2 = 2e^{-2}$

$P(Z = 3) = 4/3 \, e^{-2}$

$P(Z = 4) = \ldots$



Note $\quad \sum_{k=0}^{\infty} \frac{e^{-\lambda} \lambda^k}{k!} = e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} = e^{-\lambda} \cdot e^{\lambda} = 1$

Continuous random variables

"no jumps"

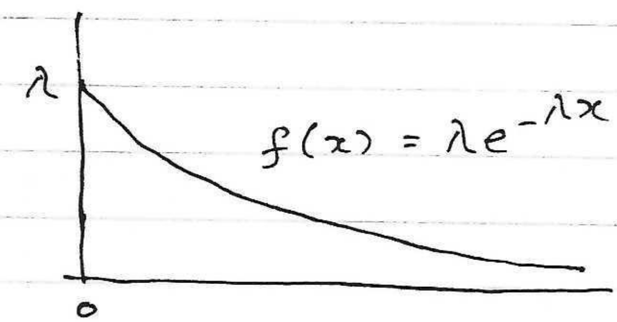
Suppose F is a continuous cdf and differentiable except at a finite # of points in any finite interval.

Then $F(x) = \int_{-\infty}^{x} f(t)\,dt$ and $F'(x) = f(x)$ at points where the derivative exists.

The function f is called the probability density function. (pdf)

Example. $F_X(x) = 1 - e^{-\lambda x}$

Exponential $f_X(x) = \lambda e^{-\lambda x}$

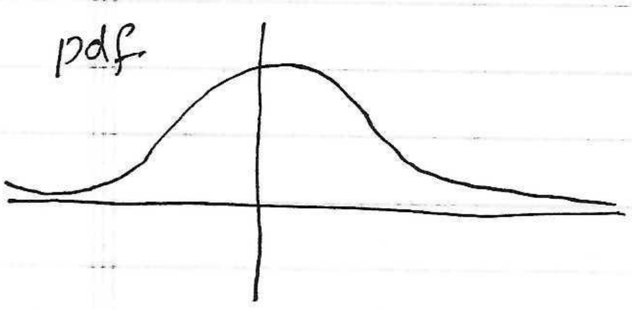


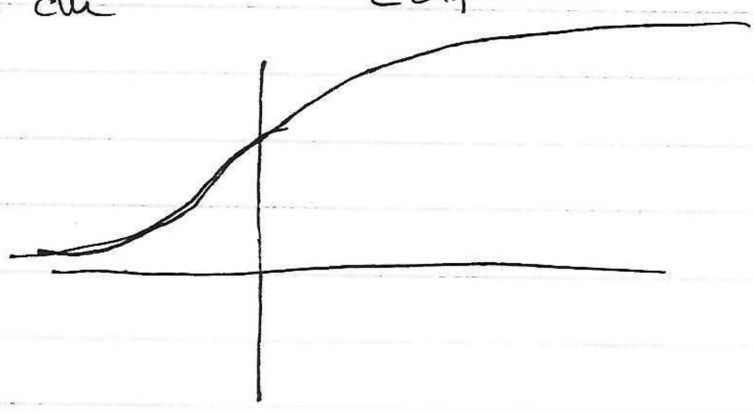
Example $f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$

Normal

$F(x) = \int_{-\infty}^{x} \frac{e^{-u^2/2}}{\sqrt{2\pi}}\,du$ cdf

pdf





Discrete	Continuous
$f(x_i) = P(X=x_i)$	$f(x)dx \doteq P(x < X < x+dx)$
$F(x) = \sum_{x_i \le x} f(x_i)$	$F(x) = \int_{-\infty}^{x} f(u)du$
$P(A) = \sum_{x_i \in A} f(x_i)$	$P(A) = \int_A f(u)du$
$\sum_i f(x_i) = 1$	$\int_{-\infty}^{\infty} f(x)dx = 1$
$f(x_i) = F(x_i) - F(x_{i-1})$	$f(x) = F'(x)$

Moments of R.V.

Expected value

discrete case. Let $p_i = P(X=x_i) = f(x)$ for r.v. X

$$\therefore E(X) = \sum_i x_i p_i \quad \text{weighted sum}$$

$$E(x) = \sum_i x_i f(x_i)$$

Continuous case $E(X) = \int_{-\infty}^{\infty} x f(x) dx$

Functions of a Random Variable

Let X be a r.v. ; then Y is also a r.v.

$Y = g(X)$

Examples $\quad X^2 + X - 1 \;;\quad X^5 \quad (X-\mu)^2$

What is the distribution of Y?

Need $P(Y \le y)$

$$\Omega \xrightarrow{X} R \xrightarrow{g} R$$
$$\omega \longmapsto X(\omega) \longrightarrow g(X(\omega))$$

$$P(Y \le y) = P^{\Omega}(\omega : g(X(\omega)) \le y)$$
$$= P^{R}(x : g(x) \le y)$$

Example

X is r.v.

-1	0	1
$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$

$Y = X^2$ $\qquad P[X \le -1] = 1/3 \;;\; P[X \le 0] = 2/3$

$P[X \le 1] = 1$

$$\text{Prob}[Y \le y] = P[x : x^2 \le y]$$

$\therefore \; \text{Prob}[Y \le 0] = \text{Prob}[X^2 \le 0] = \frac{1}{3}$

$\text{Prob}[Y \le 1] = \text{Prob}[X^2 \le 1] = 1$

or $\; \text{Prob}[Y = 0] = 1/3 \qquad \text{Prob}[Y = 1] = 2/3.$

Expected Values

Let $Y = g(X)$.

$$E(Y) = E(g(X)) = \int_{-\infty}^{\infty} g(x)\, dF_X\, dx$$

$$\text{or} \quad \int_{-\infty}^{\infty} g(x) f(x)\, dx.$$

N^{th} moment $E(X^n) = \int_{-\infty}^{\infty} x^n f(x)\, dx \qquad \sum_i x_i^n p_i$ (discrete)

1st moment or expectation $E(X) = \int x f(x)\, dx$ or $\sum_i x_i p_i$ where $p_i = \text{prob}(x_i) = f(x_i)$

$= \mu_X$

n^{th} central moment $E[(X-\mu)^n] = \int (X-\mu)^n f(x)\, dx$

$E[(X-\mu)^2] = \int (X-\mu)^2 f(x)\, dx$ (Variance or 2nd central moment)

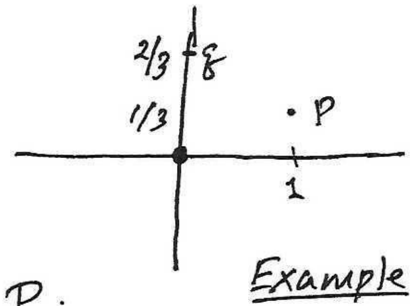
$= \sigma^2$

Note: $\sigma^2 = E(X^2) - \mu^2$.

Bernoulli random variable

$f(0) = P(X=0) = q = 1-p$

$f(1) = P(X=1) = p$



"Event" happens with probability p.

" does not happen " " q. (complement happens)

$E(X) = 0 \cdot q + 1 \cdot p = p$

$E(X^2) = \sum x_i^2 f(x_i) = 0^2 \cdot q + 1^2 \cdot p = p$

$Var(X) = \sigma^2 = E(X^2) - [E(X)]^2 = p - p^2 = pq$

Binomial Distribution

Sequence of independent Bernoulli experiments $X_1, \ldots, X_n$.

Define a new random variable S_n that is:

$$S_n = X_1 + X_2 + \cdots + X_n = \sum_{i=1}^{n} X_i$$

Values of S_n are $0, \ldots, n$. Distribution of S_n is:

$$P(S_n = k) = \binom{n}{k} p^k q^{n-k} \qquad k = 0, 1, \ldots, n.$$

$$E(S_n) = \sum_{k=0}^{n} k \binom{n}{k} p^k q^{n-k} = np \sum_{k=1}^{n} \binom{n-1}{k-1} p^{k-1} q^{n-k} = np$$

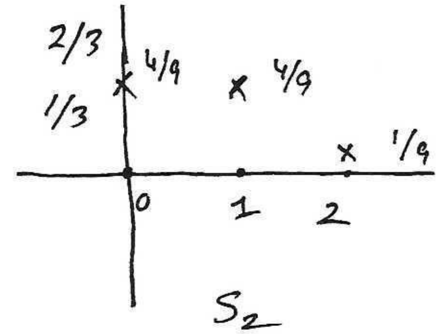
Example

$n = 2$, $p = 1/3$, $q = 2/3$

$P[S_2 = 0] = \binom{2}{0} q^2 = 4/9$

$P[S_2 = 1] = \binom{2}{1} pq = 4/9$

$P[S_2 = 2] = \binom{2}{2} p^2 = \frac{1}{9}$



Geometric Distribution

$$P(X = k) = q^k p \quad \text{for } k = 0, 1, 2, \ldots$$

k failures followed by a success.

q is probability of a failure
p is probability of a success $\qquad q = 1 - p$

Distribution until the first success.

Check that the sum $= 1$.

$$\sum_{k=0}^{\infty} q^k p = p + pq + pq^2 + pq^3 \ldots$$

$$= p \sum_{k=0}^{\infty} q^k = p \cdot \frac{1}{1-q} = \frac{p}{1-q} = 1$$

Mean # of trials before ~~until~~ the first success.

$$\sum_{k=0}^{\infty} k q^k p = pq \sum_{k=1}^{\infty} k q^{k-1} \quad ..$$

$$= pq \sum_{k=0}^{\infty} \frac{\partial}{\partial q} (q^k)$$

$$= pq \frac{\partial}{\partial q} \sum_{k=0}^{\infty} q^k = pq \frac{\partial}{\partial q} \left(\frac{1}{1-q}\right)$$

$$= \frac{pq}{(1-q)^2} = \frac{q}{p}. \qquad \{\text{Variance is } q/p^2\}$$

The number of trials to get first success is $1 + q/p = 1/p$.

Joint Probability Distributions

(Bivariate - 2 random variables.) (Multivariate ?)

X & Y random variables ; $X: \Omega \to R$, $Y: \Omega \to R$

Define the joint distribution $X,Y : \Omega \to R^2$ by

$$F_{X,Y}(x,y) = P[\{\omega : X(\omega) \le x \text{ and } Y(\omega) \le y\}] = P[X \le x, Y \le y]$$

Example

Let $F(x,y) = (1-e^{-x})(1-e^{-y}) \quad x > 0, y > 0$

$= 0$ otherwise.

Note: this is bivariate exponential distribution.

The density function is $f(xy) = e^{-x}e^{-y} = e^{-(x+y)}$

Properties of the joint c.d.f.

$F(x,y) \ge 0$

$F(x,-\infty) = F(-\infty,y) = 0$

$F(x,\infty)$ and $F(\infty,y)$ are cumulative distribution functio

$F_X(x)$ $F_Y(y)$ (also called the marginal dist. functs.

$$F(x+h, y+k) - F(x+h,y) - F(x,y+k) + F(x,y) \ge 0$$

If X & Y are independent $F_{X,Y}(x,y) = F_X(x) \cdot F_Y(y)$

For continuous random variables we have:

$$F_{X,Y}(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f_{X,Y}(u,v)\, dv\, du$$

$$F_X(x) = \int_{-\infty}^{x}\int_{-\infty}^{\infty} f(u,v)\, dv\, du \qquad \text{marginal for } X$$

$$F_Y(y) = \int_{-\infty}^{y}\int_{-\infty}^{\infty} f(u,v)\, du\, dv \qquad \text{marginal for } Y.$$

The discrete case is analagous.

Conditional Probability Distributions

Example discrete case:

Let X and Y be discrete r.v. and Let $P[X=x] = f(x) > 0$

$$P[Y \leq y \mid X=x] = \frac{P[Y \leq y, X=x]}{P[X=x]} = \frac{\sum_{v \leq y} f_{X,Y}(x,v)}{f_X(x)}$$

Thus, we can define:

$F_{X|Y}$, $f_{X|Y}$ ... etc.

The continuous case is more complex but similar

Some Random Variables
Associated with Queuing Theory

Arrival time of n^{th} customer : $\tilde{\tau}_n$

Interarrival time between n^{th} and $(n-1)^{th}$ customer : $\tilde{t}_n$

Service time for n^{th} customer : $\tilde{x}_n$

Waiting time in queue for n^{th} customer : $\tilde{w}_n$

System time (waiting + service) for n^{th} customer : $\tilde{s}_n$

If interarrival times have the same distribution for all customers we can use $\tilde{I}$ as the r.v.

Exponential interarrival time

Suppose $\tilde{I}$ is distributed such that

$$P(\tilde{I} > t) = e^{-\lambda t}$$

Then,

$$F_{\tilde{I}}(t) = P(\tilde{I} \le t) = 1 - e^{-\lambda t} \qquad \text{(c.d.f. for the exponential)}$$

$$f_{\tilde{I}}(t) = \lambda e^{-\lambda t}$$

$$E(\tilde{I}) \quad \int_0^\infty t \lambda e^{-\lambda t} dt = \frac{1}{\lambda}.$$

$\therefore$ $\frac{1}{\lambda}$ is the expected interarrival time.

and λ is the arrival rate.

Let $\tilde{X}$ be the service time distribution for customers and suppose this is also exponentially distributed.

$$P[\tilde{X} \le t] = 1 - e^{-\mu t}$$

$\therefore \frac{1}{\mu}$ is the expected service time and μ is the service rate.

$$Var(\tilde{X}) = \frac{1}{\mu^2}$$

<u>Memory-less Property of the Exponential Distribution</u>

(a) Suppose that an arrival occured. We know that the time to the next arrival, if exponentially distributed, is $P(\tilde{I} \le t) = 1 - e^{-\lambda t}$ for parameter λ.

(b) Now, suppose some time t_0 elapses since the arrival. What is the conditional distribution of the remaining time to the next arrival?

$$P(\tilde{I} \le t + t_0 \mid \tilde{I} > t_0) = \frac{P(t_0 < \tilde{I} \le t + t_0)}{P(\tilde{I} > t_0)}$$

$$= \frac{P[\tilde{I} \le t + t_0] - P[\tilde{I} \le t_0]}{1 - P[\tilde{I} \le t_0]}$$

$$= \frac{1 - e^{-\lambda(t+t_0)} - (1 - e^{-\lambda t_0})}{e^{-\lambda t_0}} = 1 - e^{-\lambda t}$$

But this is exactly the same distribution, the exponential.

$\therefore$ the distribution of the remaining time to the next arrival is exactly the same as the (unconditional) distribution of the interarrival time. (for the exponential)

<u>Example</u>

An office has two phones. Bob and Carol are on the phone and Alice wants to use the phone. Alice can use the phone as soon as either Bob or Carol finishes.

Suppose the holding time of each call is exponentially distributed with parameter μ:

$$P(\tilde{I} \le t) = 1 - e^{-\mu t}.$$

What is the probability that Alice completes the call before Carol?

Random Variables in Queuing Theory Cont.

A <u>stochastic process</u> $\{ X(t), t \in T \}$ is an indexed family of random variables.

Often interested in $P[X(t_1) \le x_1, \dots X(t_n) \le x_n]$
Variable t is often viewed as time and $X(t)$ is the state at time t.

Many ideas associated with stochastic processes:
Markov process, markov chain, ~~Station~~ stationary process, independent increments, etc.

<u>Counting Process</u>

A stochastic process is a <u>counting process</u> $\{ \tilde{N}(t), t \ge 0 \}$ if it expresses the number of events that have occurred by time t. (inclusive).
That is:

(a) $\tilde{N}(t)$ is integer valued
(b) $\tilde{N}(t)$ is non-negative
(c) $\tilde{N}(t)$ is non-decreasing
(d) For $s < t$, $\tilde{N}(t) - \tilde{N}(s)$ is the number of events that ~~occurred~~ occured in the interval $(s, t]$.

The Poisson Process

A stochastic process $\{\tilde{A}(t), t \geq 0\}$ is a Poisson Process with rate λ, $\lambda > 0$, if:

1. $\tilde{A}(t)$ is a counting process with $\tilde{A}(0) = 0$. $\tilde{A}(t)$ counts the number of arrivals through time t.
2. $\tilde{A}(t)$ has independent increments. That is, the numbers of events occurring in disjoint intervals are independent.
3. The number of arrivals in an interval of length τ is Poisson with parameter $\lambda\tau$. That is:

$$P[A(t+\tau) - A(t) = n] = \frac{e^{-\lambda\tau}(\lambda\tau)^n}{n!} \qquad n = 0, 1, 2, \ldots$$

for all t.

An equivalent definition is:

1'. $\tilde{A}(t)$ is a counting process with $\tilde{A}(0) = 0$.

2'. $\tilde{A}(t)$ has stationary and independent increments. (If the distribution of the number of events in an interval depend only on the length of the interval then it is stationary.)

3'. $P[\tilde{A}(h) = 1] = \lambda h + o(h)$

4'. $P[\tilde{A}(h) \geq 2] = o(h)$.

$o(h)$ is "little oh" of h.

A function f is $o(h)$ if: $\lim_{h \to 0} \frac{f(h)}{h} = 0$

Poisson process. Number of arrivals in an interval of time t is: $A(t)$ with:

$$P[A(t) = n] = \frac{e^{-\lambda t}(\lambda t)^n}{n!}$$

λ is the rate or expected number of arrivals in unit time.

$$E[A(t)] = \sum_{n=0}^{\infty} \frac{n\, e^{-\lambda t}(\lambda t)^n}{n!} = e^{-\lambda t}\lambda t \sum_{n=1}^{\infty} \frac{(\lambda t)^{n-1}}{(n-1)!}$$

$$= \lambda t$$

$$E[A^2(t) - A(t)] = \sum_{n=0}^{\infty} \frac{n(n-1)\, e^{-\lambda t}(\lambda t)^n}{n!} = (\lambda t)^2$$

$$Var[A(t)] = E\left[\left[A(t) - E[A(t)]\right]^2\right] = \lambda t$$

What is the distribution of the Interarrival time I for this Poisson process?

$$P[I > t] = P[\text{no arrivals in time } t]$$
$$= e^{-\lambda t}$$

$\therefore P[I \leq t] = 1 - e^{-\lambda t}$ which is an exponential interarrival time with expected value $1/\lambda$.

Moment Generating Functions

The moment generating function of a random variable X is

$$\psi(t) = E(e^{tX}) = \int_{-\infty}^{\infty} e^{tx}\, dF(x) = \int_{-\infty}^{\infty} e^{tx} f(x)\, dx$$

$$\psi(t) = \int_{-\infty}^{\infty} \sum_{k=0}^{\infty} \frac{(tx)^k}{k!}\, dF(x) = \sum_{k=0}^{\infty} \frac{t^k}{k!} \int_{-\infty}^{\infty} x^k\, dF(x)$$

$$= \sum_{k=0}^{\infty} E(X^k) \frac{t^k}{k!}$$

Let $a_k = E(X^k)$

$$\therefore \quad \psi(t) = \sum_{k=0}^{\infty} a_k \frac{t^k}{k!}$$

which is the Maclaurin power series expansion of $\psi(t)$.

Note that $\psi'(t) = \sum_{k=1}^{\infty} a_k \frac{t^{k-1}}{(k-1)!} = a_1 + a_2 t + \frac{a_3 t^2}{2!} + \cdots$

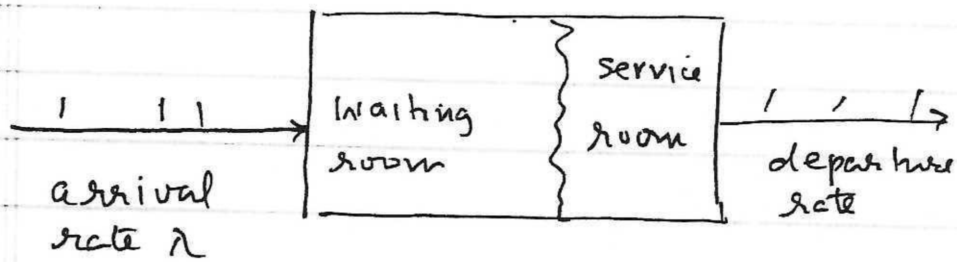
and $\psi'(0) = a_1 = E(X)$.

Similarly $\psi''(0) = a_2 = E(X^2)$

and $\psi^{(k)}(0) = a_k = E(X^k)$

Little's Theorem

$$N = \lambda T$$



Let system be empty at time 0.
Let $A(t)$ be number of arrivals in time $[0, t]$
Let $D(t)$ " " " departures in time $[0, t]$

$\therefore$ If $N(t)$ is the number of customers in the "system" at time t
we have $N(t) = A(t) - D(t)$

Assume λ is the arrival rate (long-term)

$$\lambda = \lim_{t \to \infty} \frac{A(t)}{t} \quad \text{(customers / second)}$$

Let T be the average delay per customer that is the average time in the system.
Let N = average # of customers in the system

Consider the equilibrium situation where the long term departure rate is the same as the arrival rate. This holds for most queuing systems independent of arrival & service distribution.
Little's theorem says that $N = \lambda T$.

What is the actual distribution of customers in the system? (ie probability distribution)

Need to make stronger assumptions on the arrival process & service process in order to answer this.

Assumptions

1. Customers arrive according to a Poisson process with rate λ
2. Customers have a service time that is exponential with mean $\frac{1}{\mu}$.
3. One server

$\therefore$ Interarrival process is exponential with mean $\frac{1}{\lambda}$
& Service rate is μ.

This type of queuing system is said to be:

M / M / 1 $\longrightarrow$ # of servers.
↳ departure process.
↙ inter arrival process

M means exponential
D means deterministic
G means general

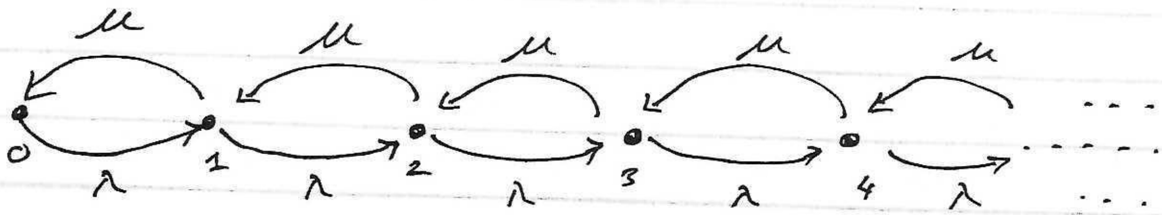
M / G / 1
M / D / 1
M / M / k

M/M/1.

Balance Equations Approach

Consider the distribution of the number of customers in the system and let

$$\text{Prob}[N = i] = p_i \text{ in steady state.}$$



Balance equations for steady state condition
λ is rate at which customers are added
μ is rate at which customers are serviced

Consider a small time δ. Probability of an arrival (1) in time δ is $\lambda\delta + o(\delta)$

$$\text{Prob}[\text{system in state } n \text{ and transitions to } n+1 \text{ in time } \delta] = p_n \lambda \delta + o(\delta)$$

$$\text{Prob}[\text{system in state } n+1 \text{ and transitions to } n \text{ in time } \delta] = p_{n+1} \mu \delta + o(\delta)$$

In equilibrium we can equate these two and get $p_n \lambda \delta + o(\delta) = p_{n+1} \mu \delta + o(\delta)$

Now divide by δ and take limit as $\delta \to 0$ we get

$$\lambda p_n = \mu p_{n+1}$$

$$\therefore \quad p_{n+1} = \frac{\lambda}{\mu} p_n = \rho p_n$$

$$\therefore \quad p_n = p_0 \rho^n \quad \leftarrow \text{geometric distribution}$$

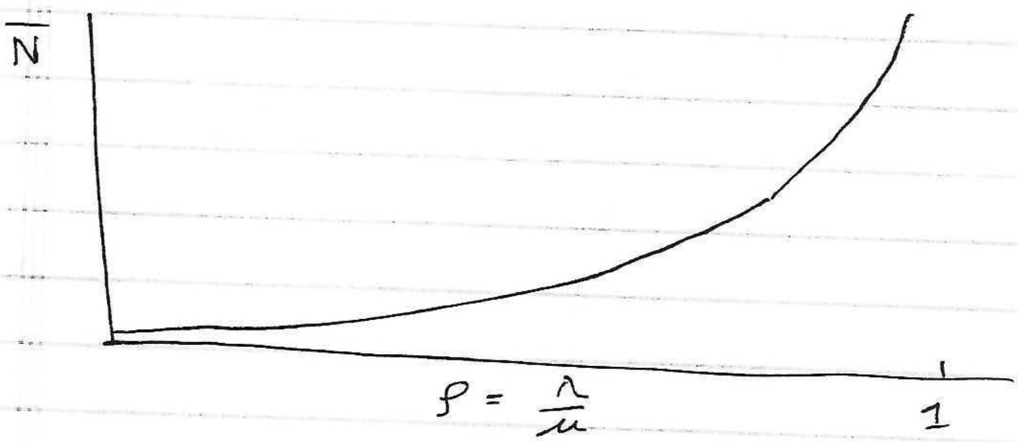
Note: $\sum p_i = 1 \quad \therefore \quad \sum_{n=0}^{\infty} p_0 \rho^n = 1$

$\therefore \quad p_0 \left(\frac{1}{1-\rho}\right) = 1 \quad \Rightarrow \quad p_0 = 1-\rho.$

$\therefore \quad p_n = (1-\rho)\rho^n$

$\bar{N}$ = average # in the system (mean of N)

$$= \frac{\rho}{1-\rho} = \frac{\lambda/\mu}{1-\lambda/\mu} = \frac{\lambda}{\mu-\lambda}$$



$$T = \frac{\bar{N}}{\lambda} = \frac{1}{\mu-\lambda}$$

$$W = T - \frac{1}{\mu} = \frac{\rho}{\mu-\lambda}$$

$$N_q = \lambda W = \frac{\rho^2}{1-\rho}$$

for M/M/1 queue.

$$N = \frac{\lambda}{\mu-\lambda}$$