COT 5310 Theory of Automata and Formal Languages Janua1y 23, 2024

Problem Set 1

Due January 30, 2024

1. (10 pts) Assume that a *computable function* is formalized as a Turing machine that can be represented as a finite length string from a finite alphabet. Thus, any string over this alphabet is simply defined to be a Turing machine. Is the set of all Turing machines countable? If so, give a listing (1-1 map with the integers) of all Turing machines. Note, such a listing is called an "effective enumeration." If not, explain why the set of all Turing machines is not countable.

2. (10 pts) Using induction, show that for every $x \in \{a, b\}^*$ such that x begins with a and ends with b, x contains the substring ab. Hint: use *string length* on which to do the induction.

3. (10 pts) Prove that the equation $(p/q)^2 = 3$ has no solution for $p, q \in N$.

4. (10 pts) Let $f(x) = 1$ if x is odd and $f(x) = 0$ if x is even. Write a program in $\mathscr{S}$ that computes f.

5. (10 pts) Let $P(x_1, x_2)$ be a predicate that is TRUE if $x_1 \leq x_2$, and FALSE otherwise. Explain whether or not the following program in $\mathscr{S}$ correctly implements this predicate. If it does not correctly implement the predicate, fix the program so it does.

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[A]  X₁ ← X₁ – 1
     X₂ ← X₂ – 1
     If  X₁ ≠ 0 GOTO B
     Y ← Y + 1
     GOTO E
[B]  If X₂ ≠ 0 GOTO A
     GOTO E
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6. The following problem will not be graded but will be covered in the solution set. However, it is important and useful to try to do the problem.
Prove Cantor's original result: for any nonempty set (whether finite or infinite), the cardinality of S (|S|) is strictly less than that of its power set 2^S. To do this first show that $|S| \leq |2^S|$ by defining a one-to-one (but not necessarily onto) map g from S to its power set. Next assume that there is a one-to-one and onto function f from S to its power set. Show that this assumption leads to a contradiction by defining a new subset R of S that cannot possibly be the image of the map f (similar to a diagonalization argument). *Hint*: Let $R = \{x \in S : x \notin f(x)\}$.