Analysis of Algorithms

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(Lecture 8: Reductions)

Welcome to COP 4531

Based on Kevin Wayne's slides

Algorithm Design Patterns and Anti-Patterns

- Algorithm design patterns. Ex.
  - Greed. $O(n^2)$ Dijkstra's SSSP (dense)
  - Divide-and-conquer. $O(n \log n)$ FFT.
  - Dynamic programming. $O(n^2)$ edit distance.
  - Duality. $O(n^3)$ bipartite matching.
  - Reductions
  - Approximation.
  - Randomization.
- Algorithm design anti-patterns.
  - NP-completeness $O(n^k)$ algorithm unlikely.
  - PSPACE-completeness. $O(n^k)$ certification algorithm unlikely.
  - Undecidability. No algorithm possible.

8.1 Polynomial-Time Reductions

Classify Problems According to Computational Requirements

- Q. Which problems will we be able to solve in practice?
- A working definition. [Cobham 1964, Edmonds 1965, Rabin 1966] Those with polynomial-time algorithms.

Yes	Probably no
Shortest path	Longest path
Matching	3D-matching
Min cut	Max cut
2-SAT	3-SAT
Planar 4-color	Planar 3-color
Bipartite vertex cover	Vertex cover
Primality testing	Factoring

Classify Problems

- Desiderata. Classify problems according to those that can be solved in polynomial-time and those that cannot.
- Provably requires exponential-time.
  - Given a Turing machine, does it halt in at most k steps?
  - Given a board position in an n-by-n generalization of chess, can black guarantee a win?
- Frustrating news. Huge number of fundamental problems have defied classification for decades.
- This chapter. Show that these fundamental problems are "computationally equivalent" and appear to be different manifestations of one really hard problem.

Polynomial-Time Reduction

- Desiderata'. Suppose we could solve X in polynomial-time. What else could we solve in polynomial time?
- Reduction. Problem X polynomial reduces to problem Y if arbitrary instances of problem X can be solved using:
  - Polynomial number of standard computational steps, plus
  - Polynomial number of calls to oracle that solves problem Y.

  computational model supplemented by special piece of hardware that solves instances of Y in a single step
- Notation. $X \leq_P Y$.
- Remark:
  - We pay for time to write down instances sent to black box $\Rightarrow$ instances of Y must be of polynomial size.

Polynomial-Time Reduction

- Purpose. Classify problems according to relative difficulty.
- Design algorithms. If $X \le_P Y$ and Y can be solved in polynomial-time, then X can also be solved in polynomial time.
- Establish intractability. If $X \le_P Y$ and X cannot be solved in polynomial-time, then Y cannot be solved in polynomial time.
- Establish equivalence. If $X \le_P Y$ and $Y \le_P X$, we use notation $X \equiv_P Y$.

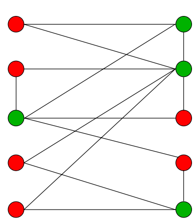
  up to cost of reduction

Reduction By Simple Equivalence

Basic reduction strategies.
Reduction by simple equivalence.
Reduction from special case to general case.
Reduction by encoding with gadgets.

Independent Set

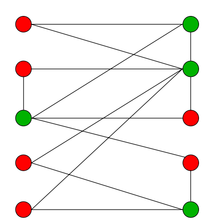
- INDEPENDENT SET: Given a graph $G = (V, E)$ and an integer k, is there a subset of vertices $S \subseteq V$ such that $|S| \ge k$, and for each edge at most one of its endpoints is in S?
- Ex. Is there an independent set of size ≥ 6? Yes.
- Ex. Is there an independent set of size ≥ 7? No.



independent set

Vertex Cover

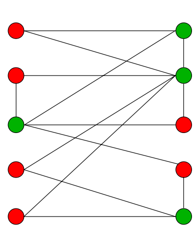
- VERTEX COVER: Given a graph $G = (V, E)$ and an integer k, is there a subset of vertices $S \subseteq V$ such that $|S| \le k$, and for each edge, at least one of its endpoints is in S?
- Ex. Is there a vertex cover of size ≤ 4? Yes.
- Ex. Is there a vertex cover of size ≤ 3? No.



vertex cover

Vertex Cover and Independent Set

- Claim. VERTEX-COVER $\equiv_P$ INDEPENDENT-SET.
- Pf. We show S is an independent set iff $V - S$ is a vertex cover.



independent set

vertex cover

Vertex Cover and Independent Set

- Claim. VERTEX-COVER $\equiv_P$ INDEPENDENT-SET.
- Pf. We show S is an independent set iff $V - S$ is a vertex cover.
- $\Rightarrow$
  - Let S be any independent set.
  - Consider an arbitrary edge (u, v).
  - S independent $\Rightarrow u \notin S$ or $v \notin S$ $\Rightarrow$ $u \in V - S$ or $v \in V - S$.
  - Thus, $V - S$ covers (u, v).
- $\Leftarrow$
  - Let $V - S$ be any vertex cover.
  - Consider two nodes $u \in S$ and $v \in S$.
  - Observe that $(u, v) \notin E$ since $V - S$ is a vertex cover.
  - Thus, no two nodes in S are joined by an edge $\Rightarrow$ S independent set. ▪

Reduction from Special Case to General Case

Basic reduction strategies.
Reduction by simple equivalence.
Reduction from special case to general case.
Reduction by encoding with gadgets.

Set Cover

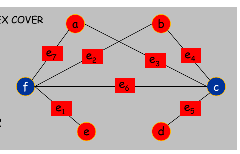
- SET COVER: Given a set U of elements, a collection $S_1, S_2, \ldots, S_m$ of subsets of U, and an integer k, does there exist a collection of $\leq$ k of these sets whose union is equal to U?
- Sample application.
  - m available pieces of software.
  - Set U of n capabilities that we would like our system to have.
  - The ith piece of software provides the set $S_i \subseteq U$ of capabilities.
  - Goal: achieve all n capabilities using fewest pieces of software.
- Ex:

U = { 1, 2, 3, 4, 5, 6, 7 }
k = 2
$S_1 = \{3, 7\}$ $S_4 = \{2, 4\}$
$S_2 = \{3, 4, 5, 6\}$ $S_5 = \{5\}$
$S_3 = \{1\}$ $S_6 = \{1, 2, 6, 7\}$

Vertex Cover Reduces to Set Cover

- Claim. VERTEX-COVER $\leq_P$ SET-COVER.
- Pf. Given a VERTEX-COVER instance G = (V, E), k, we construct a set cover instance whose size equals the size of the vertex cover instance.
- Construction.
  - Create SET-COVER instance:
    - k = k, U = E, $S_v = \{e \in E : e \text{ incident to } v\}$
  - Set-cover of size $\leq$ k iff vertex cover of size $\leq$ k. ▪

VERTEX COVER



k = 2

SET COVER

U = { 1, 2, 3, 4, 5, 6, 7 }
k = 2
$S_a = \{3, 7\}$ $S_b = \{2, 4\}$
$S_c = \{3, 4, 5, 6\}$ $S_d = \{5\}$
$S_e = \{1\}$ $S_f = \{1, 2, 6, 7\}$

8.2 Reductions via "Gadgets"

Basic reduction strategies.
Reduction by simple equivalence.
Reduction from special case to general case.
Reduction via "gadgets."

Satisfiability

- Literal: A Boolean variable or its negation. x_i or $\overline{x_i}$
- Clause: A disjunction of literals. $C_j = x_1 \vee \overline{x_2} \vee x_3$
- Conjunctive normal form: A propositional formula Φ that is the conjunction of clauses. $\Phi = C_1 \wedge C_2 \wedge C_3 \wedge C_4$
- SAT: Given CNF formula Φ, does it have a satisfying truth assignment?
- 3-SAT: SAT where each clause contains exactly 3 literals.

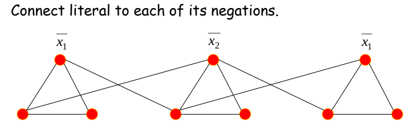
each corresponds to a different variable

Ex: $(\overline{x_1} \vee x_2 \vee x_3) \wedge (x_1 \vee \overline{x_2} \vee x_3) \wedge (x_2 \vee x_3) \wedge (\overline{x_1} \vee \overline{x_2} \vee \overline{x_3})$
Yes: x_1 = true, x_2 = true x_3 = false.

3 Satisfiability Reduces to Independent Set

- Claim. 3-SAT $\leq_P$ INDEPENDENT-SET.
- Pf. Given an instance Φ of 3-SAT, we construct an instance (G, k) of INDEPENDENT-SET that has an independent set of size k iff Φ is satisfiable.
- Construction.
  - G contains 3 vertices for each clause, one for each literal.
  - Connect 3 literals in a clause in a triangle.
  - Connect literal to each of its negations.

G

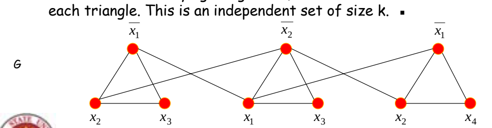


k = 3

$$\Phi = (\overline{x_1} \vee x_2 \vee x_3) \wedge (x_1 \vee \overline{x_2} \vee x_3) \wedge (\overline{x_1} \vee x_2 \vee x_4)$$

3 Satisfiability Reduces to Independent Set

- Claim. G contains independent set of size $k = |\Phi|$ iff Φ is satisfiable.
- Pf. $\Rightarrow$ Let S be independent set of size k.
  - S must contain exactly one vertex in each triangle.
  - Set these literals to true. ← and any other variables in a consistent way
  - Truth assignment is consistent and all clauses are satisfied.
- Pf $\Leftarrow$ Given satisfying assignment, select one true literal from each triangle. This is an independent set of size k. ▪



k = 3 $\quad \Phi = (\bar{x}_1 \vee x_2 \vee x_3) \wedge (x_1 \vee \bar{x}_2 \vee x_3) \wedge (\bar{x}_1 \vee x_2 \vee x_4)$

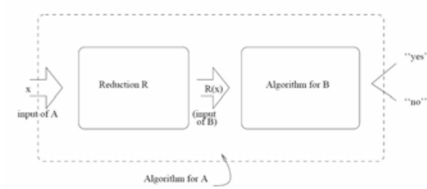
Review

- Basic reduction strategies.
  - Simple equivalence: INDEPENDENT-SET $\equiv_P$ VERTEX-COVER.
  - Special case to general case: VERTEX-COVER $\leq_P$ SET-COVER.
  - Encoding with gadgets: 3-SAT $\leq_P$ INDEPENDENT-SET.
- Transitivity. If $X \leq_P Y$ and $Y \leq_P Z$, then $X \leq_P Z$.
- Pf idea. Compose the two algorithms.
- Ex: 3-SAT $\leq_P$ INDEPENDENT-SET $\leq_P$ VERTEX-COVER $\leq_P$ SET-COVER.

Self-Reducibility

- Decision problem. Does there exist a vertex cover of size $\leq k$?
- Search problem. Find vertex cover of minimum cardinality.
- Self-reducibility. Search problem $\leq_P$ decision version.
  - Applies to all (NP-complete) problems in this chapter.
  - Justifies our focus on decision problems.
- Ex: to find min cardinality vertex cover.
  - (Binary) search for cardinality k^* of min vertex cover.
  - Find a vertex v such that $G - \{v\}$ has a vertex cover of size $\leq k^* - 1$.
    - any vertex in any min vertex cover will have this property
  - Include v in the vertex cover.
  - Recursively find a min vertex cover in $G - \{v\}$.

delete v and all incident edges



Decision Problems

- Decision problem.
  - X is a set of strings.
  - Instance: string s.
  - Algorithm A solves problem X: $A(s)$ = yes iff $s \in X$.
- Polynomial time. Algorithm A runs in poly-time if for every string s, $A(s)$ terminates in at most $p(|s|)$ "steps", where $p(\cdot)$ is some polynomial.

length of s

- PRIMES: $X = \{ 2, 3, 5, 7, 11, 13, 17, 23, 29, 31, 37, \ldots \}$
- Algorithm. [Agrawal-Kayal-Saxena, 2002] $p(|s|) = |s|^8$.

Definition of P

- P. Decision problems for which there is a poly-time algorithm.

Problem	Description	Algorithm	Yes	No
MULTIPLE	Is x a multiple of y?	Grade school division	51, 17	51, 16
RELPRIME	Are x and y relatively prime?	Euclid (300 BCE)	34, 39	34, 51
PRIMES	Is x prime?	AKS (2002)	53	51
EDIT-DISTANCE	Is the edit distance between x and y less than 5?	Dynamic programming	niether neither	acgggt ttttta
LSOLVE	Is there a vector x that satisfies Ax = b?	Gauss-Edmonds elimination	$\begin{bmatrix} 0 & 1 & 1 \\ 2 & 4 & -2 \\ 0 & 3 & 15 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 36 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

NP

- Certification algorithm intuition.
  - Certifier views things from "managerial" viewpoint.
  - Certifier doesn't determine whether $s \in X$ on its own; rather, it checks a proposed proof t that $s \in X$.
- Def. Algorithm $C(s, t)$ is a certifier for problem X if for every string s, $s \in X$ iff there exists a string t such that $C(s, t)$ = yes.

  "certificate" or "witness"
- NP. Decision problems for which there exists a poly-time certifier.

  $C(s, t)$ is a poly-time algorithm and $|t| \le p(|s|)$ for some polynomial $p(\cdot)$.
- Remark. NP stands for nondeterministic polynomial-time.

Certifiers and Certificates: Composite

- COMPOSITES. Given an integer s, is s composite?
- Certificate. A nontrivial factor t of s. Note that such a certificate exists iff s is composite. Moreover $|t| \le |s|$.
- Certifier.

```
boolean C(s, t) {
   if (t ≤ 1 or t ≥ s)
      return false
   else if (s is a multiple of t)
      return true
   else
      return false
}
```

- Instance. $s = 437{,}669$.
- Certificate. $t = 541$ or 809. $437{,}669 = 541 \times 809$
- Conclusion. COMPOSITES is in NP.

Certifiers and Certificates: 3-Satisfiability

- SAT. Given a CNF formula Φ, is there a satisfying assignment?
- Certificate. An assignment of truth values to the n boolean variables.
- Certifier. Check that each clause in Φ has at least one true literal.

$$\left(\overline{x_1} \vee x_2 \vee x_3\right) \wedge \left(x_1 \vee \overline{x_2} \vee x_3\right) \wedge \left(x_1 \vee x_2 \vee x_4\right) \wedge \left(\overline{x_1} \vee \overline{x_3} \vee \overline{x_4}\right)$$

instance s

- Ex.

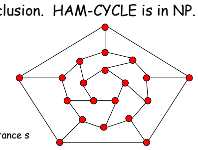
$$x_1 = 1,\ x_2 = 1,\ x_3 = 0,\ x_4 = 1$$

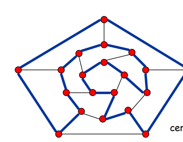
certificate t

- Conclusion. SAT is in NP.

Certifiers and Certificates: Hamiltonian Cycle

- HAM-CYCLE. Given an undirected graph $G = (V, E)$, does there exist a simple cycle C that visits every node?
- Certificate. A permutation of the n nodes.
- Certifier. Check that the permutation contains each node in V exactly once, and that there is an edge between each pair of adjacent nodes in the permutation.
- Conclusion. HAM-CYCLE is in NP.


instance s


certificate t

P, NP, EXP

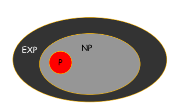
- P. Decision problems for which there is a poly-time algorithm.
- EXP. Decision problems for which there is an exponential-time algorithm.
- NP. Decision problems for which there is a poly-time certifier.

- Claim. $P \subseteq NP$.
- Pf. Consider any problem X in P.
  - By definition, there exists a poly-time algorithm $A(s)$ that solves X.
  - Certificate: $t = \varepsilon$, certifier $C(s, t) = A(s)$. ▪

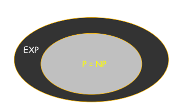
- Claim. $NP \subseteq EXP$.
- Pf. Consider any problem X in NP.
  - By definition, there exists a poly-time certifier $C(s, t)$ for X.
  - To solve input s, run $C(s, t)$ on all strings t with $|t| \le p(|s|)$.
  - Return yes, if $C(s, t)$ returns yes for any of these. ▪

The Main Question: P Versus NP

- Does P = NP? [Cook 1971, Edmonds, Levin, Yablonski, Gödel]
  - Is the decision problem as easy as the certification problem?
  - Clay $1 million prize.



If P ≠ NP



If P = NP

would break RSA cryptography
(and potentially collapse economy)

- If yes: Efficient algorithms for 3-COLOR, TSP, FACTOR, SAT, …
- If no: No efficient algorithms possible for 3-COLOR, TSP, SAT, …

Consensus opinion on P = NP? Probably no.