Introduction to the Numerical Simulation of Stochastic Differential Equations with Examples

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Introduction

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Stochastic Differential Equations

Stoke's law for a particle in fluid

$$dv(t) = -\gamma\, v(t)\, dt$$

where

$$\begin{aligned} \gamma &= \frac{6\pi r}{m}\eta, \\ \eta &= \text{viscosity coefficient.} \end{aligned}$$

Langevin's eq. For very small particles bounced around by molecular movement,

$$dv(t) = -\gamma\, v(t)\, dt + \sigma\, dw(t),$$

$w(t)$ is a Brownian motion, $\gamma =$ Stoke's coefficient. $\sigma =$Diffusion coefficient.

1-D Brownian Motion

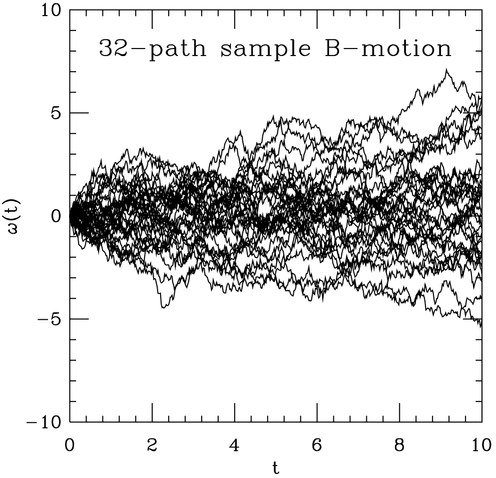

Figure: 1-D Brownian motion

2-D, or Complex Brownian Motion

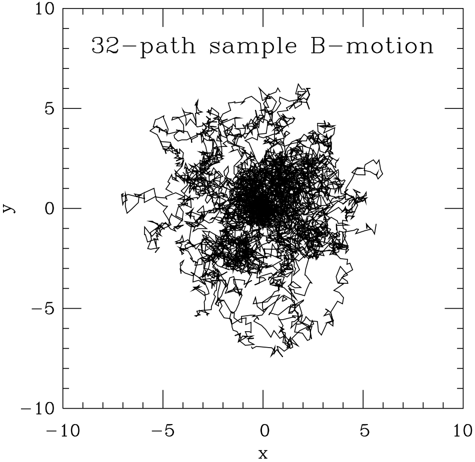

Figure: 2-D Brownian motion

Brownian Motion

$w(t) =$ Brownian motion. Einstein's relation gives diffusion coefficient

$$\sigma = \sqrt{\frac{2kT\gamma}{m}}.$$

and probability density function for Brownian motion satisfies heat equation:

$$\frac{\partial p(w,t)}{\partial t} = \frac{1}{2}\frac{\partial^2 p(w,t)}{\partial w^2}$$

Formal solution to LE is called an **Ornstein-Uhlenbeck** process

$$v(t) = v_0 e^{-\gamma t} + \sigma e^{-\gamma t}\int_0^t e^{\gamma s} dw(s)$$

A Simple Stochastic Differential Equation

What does $dw(t)$ mean?

$$w(t) = \Delta w_1 + \Delta w_2 + \cdots + \Delta w_n$$

each **increment** is independent, and

$$\boldsymbol{E}\{\Delta w_i \Delta w_j\} = \delta_{ij} \Delta t$$

or infinitesimal version

$$\begin{aligned} \boldsymbol{E} dw(t) &= 0 \\ \boldsymbol{E}\{dw(t)\, dw(s)\} &= \delta(t-s)\, dt\, ds \end{aligned}$$

The Langevin Equation

Solution to LE has properties

$$\begin{aligned} \boldsymbol{E}v(t) &= v_0 e^{-\gamma t} + \sigma e^{-\gamma t} \int_0^t e^{\gamma s} \boldsymbol{E} dw(s) \\ &= v_0 e^{-\gamma t} \end{aligned}$$

and

$$\begin{aligned} \boldsymbol{E}(v(t))^2 &= (v_0)^2 e^{-2\gamma t} + \sigma^2 e^{-2\gamma t} \frac{e^{2\gamma t} - 1}{2\gamma} \\ &\to \frac{\sigma^2}{2\gamma} \quad \text{as} \quad t \to \infty \end{aligned}$$

Something familiar about this?

$$\frac{m}{2} \boldsymbol{E}(v)^2 = \frac{m}{2} \frac{\sigma^2}{2\gamma} = \frac{1}{2} kT$$

Itô Calculus

Itô calculus for multi-dimensional version

$$dw(t)^2 \equiv dt \quad \text{or} \quad dw_i(t)dw_j(t) \equiv \delta_{ij}dt$$

In non-isotropic case, system

$$d\mathbf{z} = \mathbf{b}(\mathbf{z})\, dt + \sigma(\mathbf{z})\, d\mathbf{w}(t) \qquad \text{(SDE)}$$

is shorthand for

$$\mathbf{z}(t) = \mathbf{z}_0 + \int_0^t \mathbf{b}(\mathbf{z}_s)\, ds + \int_0^t \sigma(\mathbf{z}_s)\, d\mathbf{w}_s.$$

Itô rule for Stochastic integral:

$$\boldsymbol{E}\{\int_0^t \sigma(\mathbf{z}_s)d\mathbf{w}_s\} = 0,$$

and

$$\boldsymbol{E}\{\int_0^t \sigma(\mathbf{z}_s)d\mathbf{w}_s\}^2 = \int_0^t \sigma\sigma^T(\mathbf{z}_s)ds.$$

These integrals are **martingales.**

A Standing Martingale

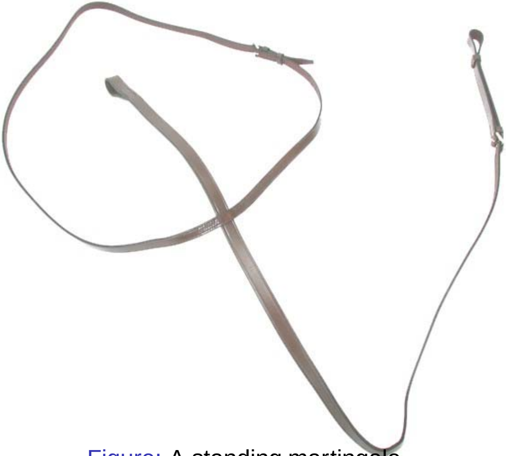

Figure: A standing martingale

Numerical Solution of SDEs

Simulation? First,

$$\boldsymbol{E} f(z(t)) \approx \frac{1}{N} \sum_{i=1}^{N} f(z^{[i]}(t))$$

for sample of N paths $z(t)$. Paths $\{z^{[1]}, z^{[2]}, ..., z^{[N]}\}$ integrated by some rule, e.g. Euler Two criteria two versions of solution $\tilde{z}(t), z(t)$ are equivalent ($\tilde{z}(t) \equiv z(t)$) for $0 \leq t \leq T$, **strong** criteria:

$$P(\sup_{0 \leq t \leq T} |\tilde{z}(t) - z(t)| > 0) = 0$$

weak: for **any** sufficiently smooth $f(x)$,

$$|\boldsymbol{E} f(\tilde{z}(T)) - \boldsymbol{E} f(z(T))| = 0$$

Weak Solutions

Example: weak simulation ($m \geq 0$):

$$dx = -x|x|^{m-1}dt + dw(t)$$

has solution whose distribution law satisfies Kolmogorov equation

$$\frac{\partial p(x,t)}{\partial t} = \frac{\partial}{\partial x}\left(\frac{1}{2}\frac{\partial}{\partial x} + x|x|^{m-1}\right)p(x,t) \to 0$$

when $t \to \infty$. That is, $x(t)$ becomes stationary. $p(x, t \to \infty)$, properly normalized, is

$$p(x,\infty) = N_m e^{-\frac{2}{m+1}|x|^{m+1}}.$$

Two examples

$$\begin{aligned} p(x,\infty) &= e^{-2|x|} && \text{for } m = 0 \\ p(x,\infty) &= \frac{1}{\sqrt{\pi}}e^{-|x|^2} && \text{for } m = 1 \end{aligned}$$

Strong Solutions

Example: a **strong** test,

$$dx = -\lambda x dt + \mu x dw(t)$$

having formal solution

$$x(t) = x_0 \exp\left(-(\lambda + \frac{\mu^2}{2})t + \mu w(t)\right). \quad (1)$$

Notice $x(t) \to 0$ as $t \to \infty$. Many authors (Mitsui et al, Higham, ...) have studied stability regions, λ, μ, for asymptotic stability $x(t_n) \to 0$, when

$$t_n = h_1 + h_2 + \ldots + h_n \to \infty$$

may have varying stepsizes. Cases

$$t = T_1 = n \cdot h,$$

and $h \to h/2^m = h'$,

$$t = T_m = n2^m \cdot h'$$

Strong Solutions

allow pathwise comparisons when

$$
\begin{aligned}
t_n = T_1 = T_m \quad & = n \cdot h \\
\Delta w(T_m + h') \quad & = \sqrt{h'}\,\xi_1 \\
\Delta w(T_m + h' + h') \quad & = \sqrt{h'}\,\xi_1 + \sqrt{h'}\,\xi_2 \\
& \ldots \\
\Delta w(T_m + h) \quad & = \textstyle\sum_{k=1}^{m} \sqrt{h'}\,\xi_k \\
\Delta w(T_1 + h) \quad & = \Delta w(T_m + h)
\end{aligned}
$$

Here, one follows the pathwise convergence as m is changed. See Kloeden and Platen, chapt. 9, p. 309. One compares "exact" solution, equation (1), with simulation values at points $T_1 = T_m$.

Strong Solutions

Numerical criteria similar: discrete times $t_k = kh$, $h =$ step size,
$T = Mh$, and
$z_k =$ numerical approx.,
strong order β:

$$(\mathbf{E} \max_{0 \le k \le M} |z_k - z(t_k)|^2)^{1/2} \le K_1 h^\beta$$

weak order β: for $f(z) \in C^{2\beta}$,

$$|\mathbf{E} f(z_M) - \mathbf{E} f(z(T))| \le K_2 h^\beta$$

Examples

Example methods:
Euler-Maruyama

$$z_{k+1} = z_k + b(z_k)h + \sigma(z_k)\Delta W$$

is **strong** order $\beta = 1/2$, **weak** order 1.
Milstein

$$\begin{aligned} z_{k+1} =& z_k + b(z_k)h + \sigma(z_k)\Delta W \\ &+ \frac{1}{2}\sigma(z_k)\sigma'(z_k)(\Delta W^2 - h) \end{aligned}$$

is **strong** order $\beta = 1$, **weak** order 1

Higher-Order Methods

Higher order weak methods require modeling

$$I_{ij} = \int_0^h w_i dw_j \qquad I_{i0} = \int_0^h w_i(s) ds$$

$$I_{ijk} = \int_0^h w_i w_j dw_k \qquad I_{ii0} = \int_0^h w_i^2 ds$$

For example, for Runge-Kutta type methods

$$\begin{aligned} I_{ij} &\approx \frac{1}{2}\xi_i \xi_j + \frac{h}{2}\Xi_{ij}, \\ I_{i0} &\approx \frac{h}{2}\xi_i, \\ I_{ijk} &\approx \frac{h}{2}\delta_{ij}\xi_k \\ I_{ii0} &\approx \frac{h}{2}\xi_i^2 \end{aligned}$$

Ξ_{ij} is a model for $\int w_i dw_j - w_j dw_i$.

Examples

$\Delta w = \xi$ is approximately gaussian

$$\boldsymbol{E}\xi = 0, \boldsymbol{E}\xi^2 = h, \boldsymbol{E}\xi^3 = 0, \boldsymbol{E}\xi^4 = 3h^2.$$

Do N sample paths per time-step - one for each $z^{[i]}$. A simple Δw is

$$\begin{aligned} \xi &= \sqrt{3h} && \text{with probability } \tfrac{1}{6}, \\ &= -\sqrt{3h} && \text{with probability } \tfrac{1}{6}, \\ &= 0 && \text{with probability } \tfrac{2}{3}. \end{aligned}$$

Important facts about these bounded increments:

- they introduce Fourier spectra with wave vectors $= \mathbf{k}\sqrt{3h}$, where $\mathbf{k} \in \mathbb{Z}^d$.
- in $d > 1$ dimensions, $\Delta \mathbf{w}$ is not isotropic.

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Examples of Bounded Increments

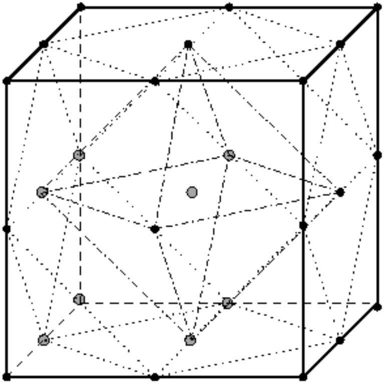

Figure: 3-D distribution of bounded increments

Some Applications

Some applications:

- Black-Scholes model for asset volatility
- Langevin dynamics
- shearing of light in inhomogeneous universes

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Black-Scholes

Black-Scholes model. Let $S =$ asset price, $r =$ interest rate. Without volatility,

$$dS = r\ S\ dt.$$

With **efficient market hypothesis**, fluctuations$(S) \propto S$:

$$dS = rS\ dt + \sigma S\ dw.$$

σ is called the **volatility**. Solution to SDE

$$S(t) = S_0\ e^{(r - \frac{1}{2}\sigma^2)t + \sigma w(t)}.$$

Langevin Dynamics

Langevin dynamics: we want some physical quantity

$$\mathbf{E}f = \int p(\mathbf{x})f(\mathbf{x})d^n x = \frac{\int e^{-S(\mathbf{x})}f(\mathbf{x})d^n x}{\int e^{-S(\mathbf{x})}d^n x}.$$

To find a covering distribution $q(\mathbf{x}), \alpha q(\mathbf{x}) \geq p(\mathbf{x})$, but $\alpha \geq 1$ is not large - difficult if n large.

Alternative is Langevin dynamics:

$$d\mathbf{x}(t) = -\frac{1}{2}\frac{\partial S}{\partial \mathbf{x}}dt + d\mathbf{w}(t),$$

and use

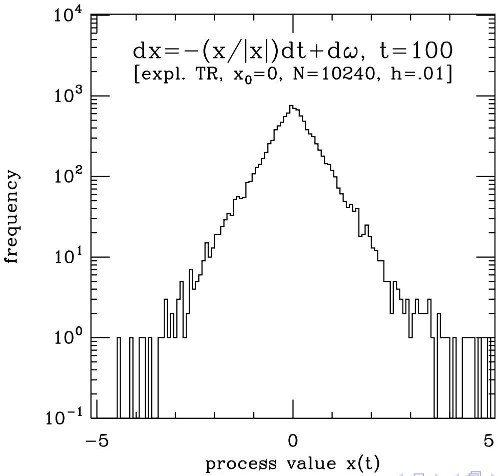
$$\mathbf{E}f = \lim_{T\to\infty}\frac{1}{T}\int_0^T f(\mathbf{x}(t))dt.$$

The following is sufficient for convergence: if $|\mathbf{x}|$ big,

$$\mathbf{x}\cdot\frac{\partial S}{\partial \mathbf{x}} > 1$$

A Simple Example

A simple example: $dx = -\text{sign}(x)dt + dw$, whose p.d.f as $t \to \infty$ is $p(x, \infty) = e^{-2|x|}$.



Stochastic Dyer-Roeder

Stochastic Dyer-Roeder: Sachs' equations for shear (σ), ray separation θ, in free space with scattered point-like particles:

$$\begin{aligned} \frac{d\sigma}{ds} + 2\theta\sigma &= \mathcal{F} \\ \frac{d\theta}{ds} + \theta^2 + |\sigma|^2 &= 0 \end{aligned}$$

σ is complex, $\mathcal{F}$ is the Weyl term, and s is an affine parameter - related to redshift z.

$$\theta = \frac{1}{2}\frac{d}{dz}ln(A)$$

where $A \propto D^2$ is the beam area, get two eqs.,

$$\begin{aligned} \frac{d\sigma}{ds} + 2\frac{1}{D}\frac{dD}{ds}\sigma &= \mathcal{F} \\ \frac{1}{D}\frac{d^2D}{ds^2} + |\sigma|^2 &= 0. \end{aligned}$$

Stochastic Dyer-Roeder

In Lagrangian coordinates (contract with redshift z), the Weyl term to 1^{st} order has derivatives of the gravitational potential $\Phi(x,y)$, with $z = x + i\,y$:

$$\mathcal{F} = \frac{1}{c^2}(1+z)^2 \frac{d^2\Phi}{dz^2}.$$

Light "sees" shearing forces orthogonal to congruence. Problem is essentially 2-D:

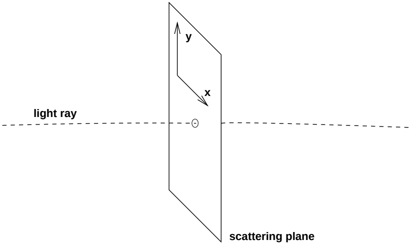


Figure: 2-D character of light scattering

Stochastic Dyer-Roeder

Correlation length is about 7 cells, i.e. ~ 7 Mpc at $z = 0$. Softened (2-3 cells) shears are normal in < 128 Mpc.

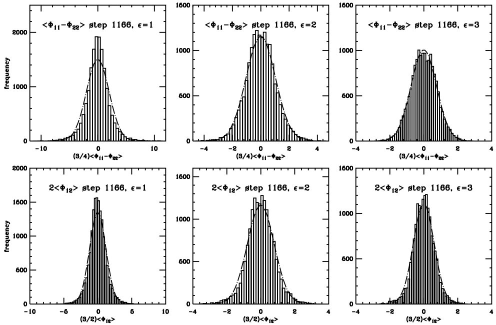


Figure: Shearing forces, from H. Couchman's code

Stochastic Dyer-Roeder

More useful form for 1st:

$$D^2\sigma = \int_0^s D^2(s')\mathcal{F}(s')ds'.$$

Expressing the affine parameter in terms of the redshift

$$s = \int_0^z \frac{d\xi}{(1+\xi)^3\sqrt{1+\Omega\xi})}$$

Yields a generalized Dyer-Roeder eq.

$$\begin{aligned}(1+z)(1+\Omega z)\frac{d^2D}{dz^2} \\ +(\frac{7}{2}\Omega z + \frac{\Omega}{2} + 3)\frac{dD}{dz} \\ +\frac{|\sigma(z)|^2}{(1+z)^5}D &= 0.\end{aligned}$$

Stochastic Dyer-Roeder

Shear can be well approximated by

$$\sigma(z) = \gamma \frac{3\Omega}{8\pi(D(z))^2} \times \int_0^z (D(\xi))^2(1+\xi)(1+\Omega\xi)^{-\frac{1}{2}} dw(\xi)$$

where $w(z)$ is a complex (2-D) B-motion. Constant $\gamma \approx 0.62$ was determined by N-body simulations.

Stochastic Dyer-Roeder

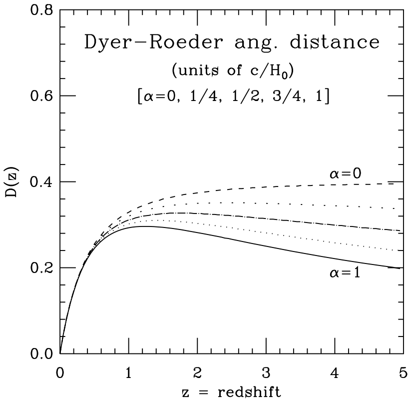


Figure: Shear free Dyer-Roeder $D(z)$

Stochastic Dyer-Roeder

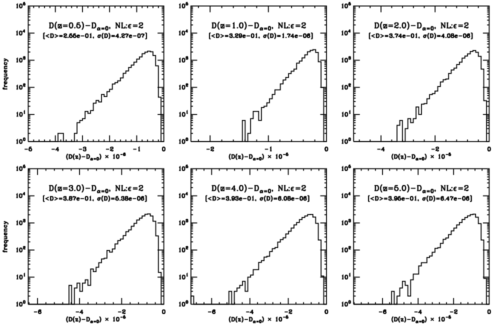


Figure: $D(z)$ histograms at $0 \leq z \leq 5$. Non-linear integration. Scales for the abscissas are: 10^{-6} for $z = 1/2$, 10^{-5} for $z = 1, 2, 3, 4, 5$.

Weak Simulations

Recall some basic rules of the Itô calculus

$$\begin{aligned} \boldsymbol{E}\, dw(t) &= 0 \\ \boldsymbol{E}\{dw(t)\, dw(s)\} &= \delta(t-s)\, dt\, ds \end{aligned}$$

Multi-dimensional version

$$dw_i(t)^2 \equiv dt \quad \text{or} \quad dw_i(t) dw_j(t) \equiv \delta_{ij} dt$$

Usual $\mathbf{z}(t) \in C^0$ process:

$$d\mathbf{z} = \mathbf{b}(\mathbf{z})\, dt + \sigma(\mathbf{z})\, d\mathbf{w}(t) \quad \text{(SDE)}$$

is shorthand for

$$\mathbf{z}(t) = \mathbf{z}_0 + \int_0^t \mathbf{b}(\mathbf{z}_s)\, ds + \int_0^t \sigma(\mathbf{z}_s)\, d\mathbf{w}_s.$$

Stochastic integral is non-anticipating. Important thing about Itô rule:

$$\boldsymbol{E}\{\int_0^t \sigma(\mathbf{z}_s) d\mathbf{w}_s\} = 0.$$

Weak Simulations

Taking the expression for $\mathbf{z}(t)$ for one step $t \to t+h$,

$$\mathbf{z}(t+h) = \mathbf{z}_t + \int_t^{t+h} \mathbf{b}_s\, ds + \int_t^{t+h} \sigma_s\, d\mathbf{w}_s,$$

and substituting $\mathbf{z}(s)$ from the right-hand side into the left side integrals, e. g.

$$\int_t^{t+h} \mathbf{b}(\mathbf{z}_s)\, ds =$$

$$\int_t^{t+h} \mathbf{b}(\mathbf{z}_t + \int_t^s \mathbf{b}_u du + \int_t^s \sigma_u\, d\mathbf{w}_u)\, ds.$$

Since $t \le u \le s \le t+h$ and

$$\int_t^s \sigma_u d\mathbf{w}(u) = O((s-t)^{1/2})$$

an expansion gives, including the $\int \sigma dw$ term, Picard-fashion, a stochastic Taylor series (due to Wolfgang Wagner)

Weak Simulations

Truncating Taylor series to $O(h)$ accuracy, we get Milstein's method (scalar case):

$$\begin{aligned} z(t+h) &= z(t) + hb(z(t)) + \sigma(z(t))\Delta\omega \\ &+ \frac{1}{2}\sigma'\sigma(\Delta\omega^2 - h) \end{aligned}$$

Again

$$\Delta\omega = \sqrt{h}\,\xi$$

where ξ =zero-centered, univariate normal:

$$\boldsymbol{E}\xi = 0, \quad \boldsymbol{E}\xi^2 = 1.$$

Notice that because $\boldsymbol{E}\Delta\omega^2 = h$, Milstein's term preserves the Martingale property

$$\boldsymbol{E}\frac{1}{2}\sigma_t'\sigma_t(\Delta\omega^2 - h) = 0.$$

Weak Simulations

It is not hard to modify this for vector case:

$$
\begin{aligned}
\mathbf{z}_{t+h} &= \mathbf{z}_t + h\mathbf{b}_t \\
&\quad +\sigma_t \Delta \mathbf{w} + \frac{1}{2}\sigma_t' \sigma_t \, \Xi
\end{aligned}
$$

Where matrix Ξ is a model

$$\Xi^{\epsilon\gamma} \approx \int_t^{t+h} \omega^\epsilon d\omega^\gamma$$

$$
\begin{aligned}
\Xi^{\epsilon\gamma} &= \frac{h}{2}(\xi_1^\epsilon \xi_1^\gamma - \tilde{\xi}^{\epsilon\gamma}) & \epsilon > \gamma \\
&= \frac{h}{2}(\xi_1^\epsilon \xi_1^\gamma + \tilde{\xi}^{\gamma\epsilon}) & \epsilon < \gamma \\
&= \frac{h}{2}((\xi_1^\epsilon)^2 - 1) & \epsilon = \gamma
\end{aligned}
$$

Additional variables $\tilde{\xi}^{\gamma\epsilon}$ are also zero-centered, univariate normals but independent of the ξ's in $\Delta\omega^\alpha = \sqrt{h}\,\xi^\alpha$.

Higher-Order Schemes

Here is a second order accurate method. Writing $\mathbf{b} = \mathbf{A} + \mathbf{B}$,

$$
\begin{aligned}
z^{\alpha}_{t+h} &= z^{\alpha}_{t} \\
&+ \frac{h}{2}(A^{\alpha}(z_{t+h}) + B^{\alpha}(z_t + \sigma_t \xi_1 + (A_t + B_t)h) \\
&+ A^{\alpha}(z_t) + B^{\alpha}(z_t)) \\
&+ \frac{1}{2}\{\sigma^{\alpha\beta}(z_t + \sqrt{\frac{1}{2}}\sigma_t \xi_0 + \frac{h}{2}(A_t + B_t)) \\
&\qquad + \sigma^{\alpha\beta}(z_t - \sqrt{\frac{1}{2}}\sigma_t \xi_0 + \frac{h}{2}(A_t + B_t))\}\xi^{\beta}_1 \\
&+ (\partial_\beta \sigma^{\alpha\delta}_t)\, \sigma^{\beta\epsilon}_t\, \Xi^{\epsilon\delta}.
\end{aligned}
$$

The first $\mathbf{A}(z_{t+h})$ is implicit.

An Example

Let's take a simple case, $M > 0$ (stable matrix),

$$d\mathbf{z} = -M\mathbf{z}dt + d\mathbf{w}$$

and write $M = A + B$, where $\mathbf{I} + hA$ is easy to invert. The semi-implicit algorithm is

$$(\mathbf{I} + hA)\,\mathbf{z}_{t+h} = (\mathbf{I} - hB)\,\mathbf{z}_t + \Delta\mathbf{w}$$

or

$$\mathbf{z}_{t+h} = (\mathbf{I} + hA)^{-1}\left((\mathbf{I} - hB)\,\mathbf{z}_t + \Delta\mathbf{w}\right)$$

In particular case $A = B = \frac{1}{2}M$,

$$\mathbf{z}_{t+h} = (\mathbf{I} + \frac{h}{2}M)^{-1}((\mathbf{I} - \frac{h}{2}M)\mathbf{z}_t + \Delta\mathbf{w}).$$

Stability of procedure will depend on L_2 norm

$$||(\mathbf{I} + \frac{h}{2}M)^{-1}(\mathbf{I} - \frac{h}{2}M)|| < 1.$$

An Example

Even in scalar case, when h is large enough ($h > 2/M$), $|1 - hM| > 1$, but

$$|(1 - hM/2)/(1 + hM/2)| \leq 1$$

for all $h > 0$.

Two dimensional case when scales of e.v.'s are very different:

$$\begin{bmatrix} dx \\ dy \end{bmatrix} = -\frac{1}{2} \begin{bmatrix} \lambda_1 + \lambda_2, & \lambda_1 - \lambda_2 \\ \lambda_1 - \lambda_2, & \lambda_1 + \lambda_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} dt + \begin{bmatrix} dw_1(t) \\ dw_2(t) \end{bmatrix}.$$

which converges for $\forall\ \lambda_i > 0$, but if $\lambda_1 \gg \lambda_2$, the stepsize $h < 2/\lambda_1$ - too small to be useful. This is **stiffness**, just like in the ODE case.

Another Example

For real, SPD matrix M:

$$d\mathbf{z} = -M\mathbf{z}dt + d\mathbf{w}.$$

The solution is formally

$$\mathbf{z}(t) = e^{-Mt}\mathbf{z}(0) + \int_0^t e^{M(s-t)} d\mathbf{w}(s).$$

Large t corr. matrix approximates $\frac{1}{2}M^{-1}$:

$$\boldsymbol{E} z_i(\infty) z_j(\infty) = \frac{1}{2}[M^{-1}]_{ij}.$$

For big M, actual computational method is

$$\boldsymbol{E} z_i(\infty) z_j(\infty) \approx \frac{1}{T}\int_0^T z_i(t) z_j(t) dt$$

as T gets big, from the ergodic theorem.

Non-Symmetric Case

Non-symmetric case:

$$\begin{aligned} d\mathbf{X} &= -M\mathbf{X}dt + d\mathbf{w}, \\ d\mathbf{Y} &= -M^T\mathbf{Y}dt + d\mathbf{w}, \end{aligned}$$

initial conditions $\mathbf{X}(0) = \mathbf{Y}(0) = \mathbf{0}$. Same $n-$D $\mathbf{w}$ for both $\mathbf{X}(t), \mathbf{Y}(t)$. From formal solutions, extract X, Y covariance

$$\boldsymbol{E}X(t)Y^T(t) \to \frac{1}{2}M^{-1}$$

as $t \to \infty$.

Again, splitting $M = A + B$, a stabilized and cheap procedure for each $\mathbf{X}(t), \mathbf{Y}(t)$ is

$$\mathbf{z}_{t+h} = (\mathbf{I} + hA)^{-1}(\mathbf{I} - hB)\,\mathbf{z}_t + \Delta\mathbf{w}$$

where in the diffusion term, we ignore the $O(h^{3/2})$ contribution. Examples: $A = \mathrm{diag}(M)$, or $A = \mathrm{tridiag}(M)$

A Test Problem

Test problem: $M = U^T Tr U$, where Tr = upper triangular, $diag(Tr) = (1, \ldots, N)$, $[Tr]_{i,j} \in \mathcal{N}(0,1)$, $j > i$. Random orthogonal U by Pete Stewart's procedure: $S = diag(sign(u_1))$

$$U = S U_0 U_1 \ldots U_{N-2}$$

where

$$U_k = \begin{pmatrix} I_k & \\ & H_{N-k} \end{pmatrix}$$

H_j = Householder transforms,

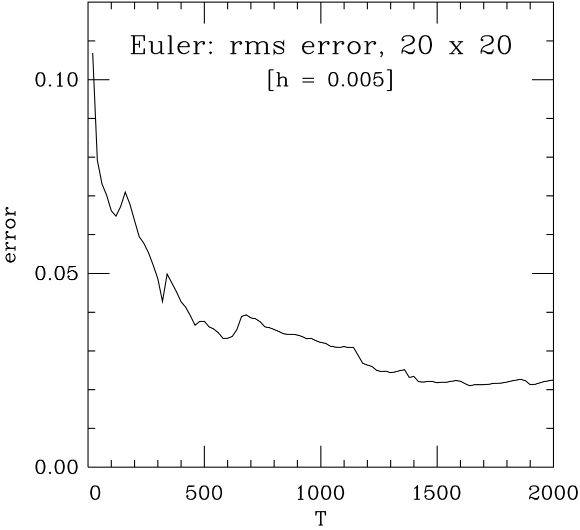
$$H_j = I_j - 2\frac{\mathbf{u}\mathbf{u}^T}{||\mathbf{u}||^2}$$

with $j-$length vectors $\mathbf{u}$

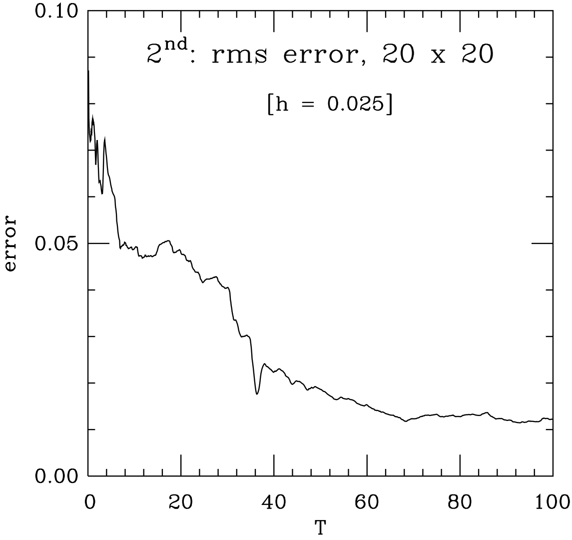
$$\mathbf{u} = \mathbf{x} - ||\mathbf{x}||\mathbf{e}_1,$$

each $x_i \in \mathcal{N}(0,1)$, $i = 1, \ldots, j$. Also, $cond(M) \sim N$.

Convergence of the Euler Method



Convergence of a Second-Order Method



More Examples

More general problems? Some has been done. Talay, Tubaro, and Bally's Euler estimates

$$|\boldsymbol{E}f(z(T)) - \boldsymbol{E}f(z_n(T))| \le h\frac{K(T)||f||_\infty}{T^q}$$

$h = T/n =$ time step, $q > 0$ constant, and $K(T)$ is *non-decreasing*. Optimal choice of T is unclear. Example of Langevin dynamics,

$$d\mathbf{z}(t) = -\mathbf{b}(\mathbf{z})dt + d\mathbf{w}(t), \tag{2}$$

want $\mathbf{z}$ to converge to stationary. For large $|\mathbf{z}(t)|$,

$$\boldsymbol{E}|\mathbf{z} + \Delta\mathbf{z}|^2 \le \boldsymbol{E}|\mathbf{z}|^2.$$

From eq. (2),

$$2\mathbf{z} \cdot \mathbf{b}(\mathbf{z}) \ge 1$$

Discretization errors $O(h)$ for Euler, $O(h^2)$ for 2nd order RK.

A Final Example

A final example model problem, where $m \in \mathbb{Z}^+$

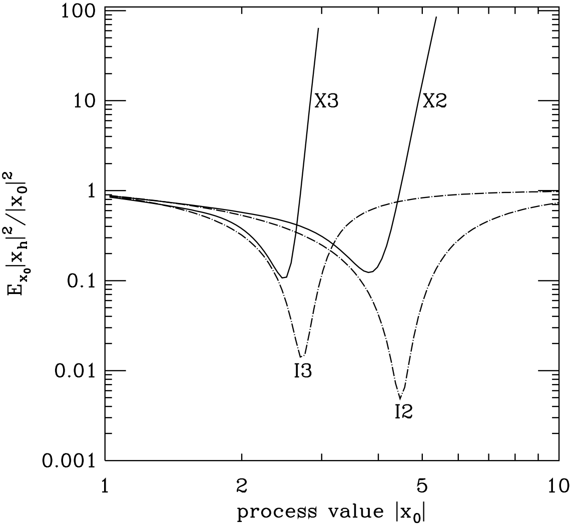
$$dx = -x|x|^{m-1}dt + dw(t)$$

Two procedures: $\Delta\omega = \sqrt{h}\,\xi$,

$$\begin{aligned} x_h &= \mathrm{XTR}(x_0, \xi) \\ &= x_0 - \frac{h}{2}(x_{euler}|x_{euler}|^{m-1} + x_0|x_0|^{m-1}) \\ &\quad +\Delta\omega \end{aligned}$$

$$\begin{aligned} x_h &= \mathrm{ITR}(x_0, \xi) \\ &= x_0 - \frac{h}{2}(x_h|x_h|^{m-1} + x_0|x_0|^{m-1}) \\ &\quad +\Delta\omega \end{aligned}$$

A Final Example



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