Chapter II: Brownian Motion (Wiener Measure)

Let $\Omega$ be cont. functions $\beta(t)$ so $\Omega[0,1]$ and $\beta(0) = 0$. 

Ask $P\{\int \beta_{n} d\beta \leq x^3\} = ?$

Consider $(\Omega, (B_{t})_{t \geq 0})$ some idea of what we have been talking about.

Let $X_{1}, X_{2}, \ldots$ be i.i.d. with $\mu = 0$ and $\sigma^2 = 1$ and C.L.T.

$$\lim_{n \to \infty} P\left\{ \frac{S_{n}}{\sqrt{n}} \leq x \right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{x}{\sqrt{2}}} e^{-\frac{u^2}{2}} du \quad \text{in 1944}$$

Erdos-Kac proved $\lim_{n \to \infty} P\left\{\max_{1 \leq i \leq n} \frac{S_{i}}{\sqrt{i}} \leq x \right\} = \phi(x)$

and under the same hypotheses on the r.v.'s they proved

$$\lim_{n \to \infty} P\left\{ \frac{S_{1}^2 + S_{2}^2 + \ldots + S_{n}^2}{n} \leq x \right\} = \sigma_{2}(x)$$

$$\lim_{n \to \infty} P\left\{ \frac{S_{1} + S_{2} + \ldots + S_{n}}{\sqrt{n}} \leq x \right\} = \sigma_{3}(x)$$

Let $N_{n}$ be the # of partial sums out of $S_{1}, S_{2}, \ldots, S_{n}$ which are strictly positive then

$$\lim_{n \to \infty} P\left\{ \frac{1}{n} \leq x \right\} = \begin{cases} 0 & x < 0 \\ \frac{3}{2} \arcsin \sqrt{x} & 0 \leq x \leq 1 \\ 1 & x > 1 \end{cases}$$
Let the $X_i$'s be i.i.d. with $\mu = 0, \sigma^2 = 1$. For each $n \in \mathbb{N}$, define:

$$X_n(t) = \begin{cases} \frac{S_i}{\sqrt{n}} & t = 0 \\ \frac{i}{\sqrt{n}} \frac{i-1}{n} \leq t \leq \frac{i}{n} & i = 1, 2, \ldots, n \end{cases}$$

Let $F: \mathbb{R}$ (space of Riemann integrable functions on $[0,1]$) $\rightarrow \mathbb{R}$ with weak hypotheses. Then, for a sequence $\{F_n\}$ of functions in $F$:

**Theorem:** \( \lim_{n \to \infty} P \{ F_n(\mu) < \alpha \} = P_w \{ F(\beta) < \alpha \} \)

Connected with the Wiener measure.

**Examples:**
1. $F[\beta] = \int_0^\infty \beta^2 \, ds$ by theorem:
   \[
   \lim_{n \to \infty} P \left( \sum_{i=1}^{n} \frac{S_i^2}{n} \leq \alpha \right) = P_w \left\{ \int_0^1 \beta^2 \, ds \leq \alpha \right\}
   \]
2. $F[\beta] = \beta(1)$ so:
   \[
   \lim_{n \to \infty} P \left( \frac{\sum S_i}{\sqrt{n}} \leq \alpha \right) = P_w \left\{ \beta(1) \leq \alpha \right\}
   \]
3. Let $F[\beta] = \int_0^1 \left( 1 + \text{signum}(s) \right) \frac{\beta}{2} \, ds$ with \text{signum}$\beta = \begin{cases} 1 & \beta > 0 \\ 0 & \beta = 0 \\ -1 & \beta < 0 \end{cases}$
   \[
   \lim_{n \to \infty} P \left( \frac{\sum S_i}{\sqrt{n}} \leq \alpha \right) = P_w \left\{ \int_0^1 \left( 1 + \text{signum}(s) \right) \frac{\beta}{2} \, ds \leq \alpha \right\}
   \]

**Formal:** For any integer $n$ and any choice of $a_1 < a_2 < \cdots < a_n$, Lebesgue measurable set in $\mathbb{R}$, we define an "interval" $[a_1, a_2, \ldots, a_n]$ on $[0,1]$ as follows...
\[ I = I(\tau_1, \tau_2, ..., \tau_n, E) = \{ \beta(\cdot) \in \mathbb{C} \cap [0,1] : (\beta(t_1), \beta(t_2), ..., \beta(t_n)) \in E \} \]

Let \( \mathcal{E} \) be the class of intervals, i.e., for all \( n \), all choices of \( \tau_1, \tau_2, ..., \tau_n \) and all Lebesgue measurable sets \( E \subset \mathbb{R}^n \). We see that \( \mathcal{E} \) is an algebra of sets in \( [0,1] \). Let \( \mathcal{B} \) be the smallest \( \sigma \)-algebra generated by \( \mathcal{E} \), this is the class of Wiener measurable sets.

Given an interval \( I \) we define \( m(I) = \int_I \frac{1}{\sqrt{(2\pi)^n \det A}} e^{-\frac{1}{2} \sum_{i,j=1}^n A_{ij} u_i u_j} \, du_1 \cdots du_n \).

Let \( A = \min(\tau_1, \tau_2) \) be an \( n \times n \) matrix with \( A_{ij} = \frac{1}{2} e \left( -\sum_{k=1}^n (c_{ik} - c_{kj})^2 \right) \), where \( c_{ij} \) are the components of \( \tau_i \) and \( \tau_j \) in a Brownian Motion.

Consider \( \beta \) a component of a Brownian Motion.

\[ P\left( a \leq \beta(\tau_1) \leq b, \beta(\tau_2) \leq \eta \right) = \frac{1}{2 \eta^2 \sqrt{\pi}}, \int_a^b e^{-\frac{u^2}{\eta^2}} \, du \quad \text{assuming Gaussian} \]

\[ P\left( a \leq \beta(\tau_1) \leq b, \beta(\tau_2) \leq \eta \right) = \frac{1}{\sqrt{\eta^2 + \sigma^2}}, \int_a^b e^{-\frac{u^2}{\eta^2}} \, du \]
Theorem: If $I = \bigcup I_j$, where $I_j \cap I_k = \emptyset$ for $i \neq k$ and $i, k \in \mathbb{K}$, and $I \in \mathcal{I}$, then $m(I) = \sum_{j=1}^{\infty} m(I_j)$.

Let $n = \text{dimension}$, $E$ be the defining points, and $F$ the defining set.

Let $I_1, I_2$ be intervals

$I_1 = (n_1, 2n_1, 3n_1, \ldots, n_2, 2n_2, 3n_2, E^{(n_2)})$

$I_2 = (n_0, 2n_0, 3n_0, \ldots, n_1, 2n_1, 3n_1, E^{(n_1)})$

Suppose $n_2 > n_1$.

Suppose $I_1 = \{ \beta(c) \in C_0[0,1] \mid \beta(c) \leq 2 \}$ also we can write

$I_1 = \{ \beta(c) \in C_0[0,1] \mid \beta(c) \leq 2 \}$

by trivial restrictions. So we make $I_1 \cup I_2$ such that they have the same dimension, some defining points and so $\mathcal{E}$ is algebra since $C_0[0,1] \subseteq \mathcal{E}$ and $I, UI \subseteq \mathcal{E}$ commuting.

Sets on $L^2$ topology are measurable, if originally measurable sets are uniform.

It will also discover that almost all paths are non-diffracting.

Every point and a path will satisfy a Hölder (lip-$\alpha$) condition for order $\alpha$ where $2 \leq \alpha \leq 1/\beta(c) - \beta(c+1)$.
Let \( E \subset \mathbb{R}^n \), \( 0 < \tau_1, \tau_2, \ldots, \tau_n < 1 \),
\[
I = \{ \beta \in C([0,1]) \mid \beta(\tau_1), \ldots, \beta(\tau_n) \in E \qquad \text{measurable set in } \mathbb{R}^n \}
\]
define \( m(I) = \frac{1}{\tau_1 \tau_2 \cdots \tau_n} \int \cdots \int \sum_{\beta} \frac{\prod u_i}{\tau_i} \prod \frac{(\beta(u_i) - \beta(u_{i-1}))^2}{\tau_i} \, du_1 \cdots du_n
\]
where \( \beta(x,y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-y)^2}{2}} \), notice this solves
\[
\psi_0 = \frac{\partial}{\partial y} \psi_y, \quad \psi(0,0) = 5(y-x)
\]
It is obvious that this finitely additive
\[ I = I, U I, \quad I, \cap I = \emptyset \text{ since integrals are additive set function.} \]

**Theorem 1:** Let \( a > 0, \quad 0 < y < \frac{1}{2} \), and define
\[ A_{a,y} = \{ \beta \in C([0,1]) \mid \beta(\tau_1), \ldots, \beta(\tau_n) \in a C_{\tau_1, \ldots, \tau_n} \}
\]
For any interval \( I \subset C([0,1]) \Rightarrow I \cap A_{a,y} = \emptyset \)
\[ m(I) < K a^{\frac{n-2}{3}} \quad \text{K is indep. of a.} \]
But \( A_{a,y} \) is a compact set in \( C([0,1]) \) and eventually one can prove that almost all \( \beta \in C([0,1]) \) satisfy some Hölder cond.

**Theorem 2:** \( m \) is countably additive on \( \mathcal{I} \), i.e. if \( I_i \in \mathcal{I} \) which are disjoint \( I = \bigcup_{i=1}^{\infty} I_i \) then
\[ m(I) = \sum_{i=1}^{\infty} m(I_i) \]

\[ I = \bigcup_{n=1}^{\infty} I_n \in \mathcal{I} \]
Proof. Let \( J_n = I - \bigcup_I J_i \). Clearly \( J_n \supset J_{n-1} \) and
\[
\lim_{n \to \infty} J_n = \bigcap_{n=1}^\infty I_n = \emptyset.
\]
From finite additivity we have
\[
m(I) = \sum_{j=1}^n m(I_j) + m(J_n)
\]
so it suffices to show that
\[
\lim_{n \to \infty} m(J_n) = 0.
\]
Let \( x_1, x_2, \ldots, x_m \) and \( E_n \) be a defining set.

Lebesgue measurable set \( A \subseteq \mathbb{R}^2 \) be the definition of \( J_n \).

Clearly \( \exists E_n^* \) closed in \( \mathbb{R}^2 \) \( E_n^* \subset E_n \) the interval
\( K_n \) whose defining points are also \( x_1, x_2, \ldots, x_m \) but whose defining set is \( E_n^* \) with
\[
m(E_n - E_n^*) < \frac{\varepsilon}{2^{n+1}}.
\]

Let \( L_n = \bigcap_{j=1}^n K_j \) then \( L_n \subset K_n \subset J_n \) and
\[
m(J_n) = m(J_n - L_n) + m(L_n)
\]
since
\[
J_n - L_n = J_n - \bigcap_{j=1}^n K_j \subseteq \bigcup_{j=1}^n (J_i - K_j)
\]

\[
\therefore m(J_n - L_n) \leq \sum_{j=1}^n m(J_i - K_j) \leq \sum_{j=1}^n \frac{\varepsilon}{2^{n+1}} < \frac{\varepsilon}{2}
\]

So for all \( n > N \), \( m(J_n) < m(L_n) + \frac{\varepsilon}{2} \). Want to show this is small.

I. From Theorem 1, for any \( \delta \in (0, \frac{\varepsilon}{2}) \) there is a large
\( N \) such that \( m(J_n) < \frac{\varepsilon}{2} \). Hence it will suffice for
such an \( \delta \) to find an \( N_0 \) so large that \( n > N_0 \) implies
\[
m(I) < \frac{\varepsilon}{2}
\]

\( \lim_{n \to \infty} m(A_n) = 0 \).
Let \( M_n = L_n \cap A_n \), since \( M_n \) is decreasing by \( L_n \) decreasing and
\[
\lim_{n \to \infty} M_n = \lim_{n \to \infty} L_n = \emptyset. \text{ We must that } \exists N_0 \ni \forall n > N_0, M_n = \emptyset. \\
Assume the contrary, i.e., \( \exists \) a sequence \( \beta_k \in [0,1] \) s.t. \( \beta_k \in M_k \), \( k = 1, 2, 3, \ldots \).

For any \( n_k, \beta_k \in M_{n_k}, k > n_k \), consider this infinite set of functions belonging to \( M_{n_k} = L_{n_k} \cap A_{n_k} \). Since each of these functions belong to \( A_{n_k} \), they constitute a uniformly bounded equicontinuous family, by Ascoli's theorem. A subsequence converging to a function \( \beta_0 \in A_{n_k} \) more over since each \( \beta_k \) with \( n_k > n_{k-1} \) is in \( L_{n_k} \), which is an interval whose defining set is closed. Therefore we have \( \beta_0 \in L_{n_k} \), \( \beta_0 \in M_{n_k} \) for \( n_k \) arbitrary so \( \beta_0 \in M_{n_k} \), \( n_k \) which contradicts \( \lim_{n \to \infty} M_n = \emptyset \). Which proves the

**Theorem (Theorem 2).**

**Lemma 1:** For any positive integer \( n \), let \( 0 = \xi_0 < \xi_1 < \xi_2 < \cdots < \xi_{2^n} = 1 \)

where \( \xi_j = \frac{2^n}{2^n} j = 0, 1, 2, 3, \ldots, 2^n \). Let \( \beta \in C_0, 1 \) and linear between \( \xi_2 \) and \( \xi_2 \), \( i = 1, 2, \ldots, 2^n \). Suppose \( \exists a > 0 \) and \( 0 < \gamma < \frac{1}{2} \) \( \forall k = 1, 2, \ldots, n \) and all \( i = 1, 2, 3, \ldots, 2^n \).

\[
|\beta(\xi_{2^i}) - \beta(\xi_{2^{i+1}})| \leq \left( \frac{2^n}{24} \right) a / (2^n k - 2^{i+1} k) \frac{1}{2^n} 
\]
Then $4 \alpha, \epsilon \in [0, 1], |\beta(x) - \beta(y)| \leq \alpha |x - y|$. A picture:

\[
\begin{array}{c}
\begin{array}{c}
\gamma_1 \gamma_2 \gamma_1 \gamma_2 \gamma_1 \gamma_2 \gamma_1 \gamma_2 \\
\frac{1}{4} \frac{1}{2} \frac{1}{4} \frac{1}{2}
\end{array}
\end{array}
\]

Differences over gaps of $1, 2, 4, \ldots, 2^n$ are required to prove this.

**Lemma 2.** Let $0 < \epsilon, \delta < 1$ and $I = \{ \beta \in [0, 1]: |\beta(x) - \beta(y)| \geq \alpha |x - y| \}$. Then

\[ m(I) \leq c \frac{\epsilon^2}{(2\alpha \cdot \epsilon)^{1-\delta}}. \]

**Proof:** Note that $I$ is an interval if we define $E \subset \mathbb{R}^3$

\[ E = \{ (u_1, u_2) \in \mathbb{R}^2 | u_1 - u_2 > a |x_1 - x_2| \} \]

Hence

\[ I = \{ \beta \in [0, 1]: (\beta(x_1), \beta(x_2)) \in E \}. \]

But

\[ m(I) = \frac{1}{\sqrt{2\pi} \epsilon} \int_{-\infty}^{\infty} e^{-\frac{u_1^2}{2\epsilon^2}} \int_{-\infty}^{\infty} e^{-\frac{(u_1 - u_2)^2}{2(1-\epsilon)^2}} \, du_1 \, du_2. \]

\[ = \frac{1}{\sqrt{2\pi} \epsilon} \int_{-\infty}^{\infty} e^{-\frac{u_1^2}{2\epsilon^2}} \left( \int_{-\infty}^{\infty} e^{-\frac{u_1 - u_2}{2(1-\epsilon)^2}} \, du_2 \right) \, du_1. \]

\[ = \frac{1}{\sqrt{2\pi} \epsilon} \int_{-\infty}^{\infty} e^{-\frac{(u_1 + b)^2}{2(1-\epsilon)^2}} \, du_1, \quad \text{with} \quad b = \frac{a}{1-\epsilon}. \]

\[ \leq \frac{1}{\sqrt{2\pi} \epsilon} \int_{-\infty}^{\infty} e^{-\frac{u_1^2}{2(1-\epsilon)^2}} \, du_1 = c. \]

**Theorem 1.** Let $x_1, x_2, \ldots, x_s$ and $E^* \subset \mathbb{R}^3$ be the definition of $I$, i.e.,

\[ I = \{ \beta \in [0, 1]: (\beta(x_1), \beta(x_2), \ldots, \beta(x_s)) \in E^* \}. \]
Choose \( n \) so large that \( \frac{1}{2^{n-1}} \leq \min \left( \tau_1, \tau_2 - \tau_1, \ldots, \tau_s - \tau_{s-1} \right) \) and if \( \tau_s \neq 1 \), let \( \frac{1}{2^{n-1}} < \lambda(\tau_s) \). For such \( n \), let
\[
\varepsilon_j = \frac{1}{2^j} \quad \text{for} \quad j = 0, 2, 4, 6, \ldots, 2^n.
\]
Clearly between any two successive \( \varepsilon_j \)'s, \( j = 0, 2, 4, \ldots, 2^n \), there is at most one \( \varepsilon \) pair.

If \( \varepsilon \) falls between \( \tau_1 \) and \( \tau_2 \), define \( \varepsilon_{j+1} = \frac{\varepsilon + \varepsilon_{j+2}}{2} \).

Thus \( S = \{ \varepsilon_0, \varepsilon_1, \ldots, \varepsilon_2^n \} \supset (\tau_1, \tau_2, \ldots, \tau_s) \) and by imposing trivial restrictions at appropriate points, I can be defined in terms of \( S \) and a Lebesgue measurable set in \( \mathbb{R}^2 \). For any \( \beta \in I \), define
\[
\beta^* = \begin{cases} 
\beta(\varepsilon_j) \quad &\text{for} \quad \varepsilon_j \in S \\
\text{linear in between the points of} \quad S &
\end{cases}
\]
Thus \( \beta^* \in I \), \( \beta^* \in A_{a_1} \). Hence from Lemma 1
\[
\exists \, \delta = 0, 1, 2, \ldots, 2^n + 1 \quad \text{such that} \quad |\beta^*(\varepsilon_{k+1}) - \beta^*(\varepsilon_{k+1})| > \left( \frac{1}{2^n} \right)^{a_1} \delta_{k+1} - \frac{1}{2^n} \delta_{k+1},
\]
by Lemma 1.

Since \( \beta^*(\varepsilon) = \beta(\varepsilon) \) at all \( \varepsilon \in S \), the last inequality also holds for \( \beta \) also. Let \( J_{kl} = \{ \beta \in (0, 1) \mid |\beta^*(\varepsilon_{k+1}) - \beta^*(\varepsilon_{k+1})| > \left( \frac{1}{2^n} \right)^{a_1} \delta_{k+1} - \frac{1}{2^n} \delta_{k+1} \} \).

We have just shown that \( \bigcup_{k=0}^n \bigcup_{l=1}^{2^n} J_{kl} \) is finite by additivity:
\[
m(I) \leq \sum_{k=0}^{2^n} \sum_{l=1}^{2^n} m(J_{kl})
\]
but
letting $b = \left( \frac{1 - 2^{-y}}{2} \right)^a$ we have from lemma 2

$$m(I) \leq \sum_{k=0}^{n} \sum_{x=1}^{2^{-y}} e^{-\frac{b^2}{2 \left( (2^{y-1}) \cdot (2^{x-1}) \right)^{1-2y}}}.$$

For $k = 0$

$$\frac{2^{y-1}}{2} \leq \frac{x}{2^n}.$$

For $k = 1$

$$\frac{2^{y-1}}{2} \leq \frac{x}{2^n}.$$

So

$$m(I) \leq \frac{2^{y-1}}{2} e^{-\frac{b^2}{2 \left( 2^{y-1} \right)^{1-2y}}} \sum_{k=0}^{n} \sum_{x=1}^{2^{y-1}/2} e^{-\frac{b^2}{2 \left( 2^{y-1} \right)^{1-2y}}}$$

Let $\phi(u) = u^2 e^{-\frac{b^2}{2} u^2}$, $u > 0$, its maximum is at

$$u = \left( \frac{4}{b^2} \right)^{1/4}$$

with maximum $\left( \frac{b^2}{4} \right)^{1/4}$

So replacing with the maximum

$$m(I) \leq 3 \sum_{k=0}^{n} 2^{y-1} e^{-\frac{b^2}{2 \left( 2^{y-1} \right)^{1-2y}}}$$

We get the next theorem on xerox next time. The lemma

We can consider $E\{F\} = E\{F(\beta, \cdot)\} = \int F(\beta, \cdot) \, d\mu$. Suppose $F: C[0, 1] \to \mathbb{R}$ and it is measurable, i.e.

$$\{ \beta \in C[0, 1] \mid F(\beta) \leq x \}$$

is measurable, i.e.
Consider \( x \in [0, t] \) changes only the first term.

But \( P\{ \beta(0) = x, \beta(t) = A \} = \frac{1}{\sqrt{2\pi t}} \int_A \exp\left(-\frac{y^2}{2t}\right) dy \) but

\( \varphi(t, x, y) \) satisfies \( u_t = \frac{1}{2} u_{xx} \)

\( u(x, 0) = \delta(x) \), i.e. \( \lim_{t \to 0} \int \varphi(t, x, y) g(y) dy = g(x) \),

\( \varphi(t, x, y) \) is the fundamental solution of the heat equation. Now

\[
E\{ \beta(t) \} ; E\{ g(\beta(t), \beta(t), \ldots, \beta(t)) \}
\]

\[
= \frac{1}{2\pi t} \int_{-\infty}^{\infty} e^{-\frac{u^2}{2t}} du = \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} \sqrt{2\pi t} \varphi(t, x, y) \delta(y) dy = \frac{\delta(x - \beta(t))}{\sqrt{2\pi t}}
\]

But \( P\{ \beta(x) = x \} = m (I = \{ \beta \in C([0, t]) | \beta(0) = x \}) = \frac{1}{2\pi t} \int_{-\infty}^{\infty} e^{-\frac{u^2}{2t}} du \).

Let us use \( \delta \) function notation. Let \( g \) be Borel measurable, then

\[
E\{ g(\beta(t)) \} = \frac{1}{2\pi t} \int_{-\infty}^{\infty} g(u) e^{-\frac{u^2}{2t}} du \hspace{1cm} \text{let} \hspace{1cm} g(u) = \delta(u - x)
\]

\[
E\{ \delta(\beta(t) - x) \} = \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} \delta(u - x) e^{-\frac{u^2}{2t}} du = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}
\]

Consider \( u_t = \frac{1}{2} u_{xx} \) so \( u(x, t) = E\{ \delta(\beta(t) - x) \} = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} \)

\( u(x, 0) = \delta(x) \)

Consider \( u_t = \frac{1}{2} u_{xx} - V(x) u \hspace{1cm} u(x, 0) = \delta(x) \hspace{1cm} V(x) \geq 0 \) continuous
But we can write \( u(x,t) = E \left\{ e^{-\int_0^t V(\beta(s)) \, ds} \delta(\beta(s) - x) \right\}. \)

This is the Feynman–Kac formula.

Example: \( u_t = \frac{1}{2} u_{xx} - \frac{k^2}{2} u \quad u(x,0) = \delta(x), \quad V = \frac{k^2}{2} \)

\[ u(x,t) = E \left\{ e^{-\frac{1}{2} \int_0^t \beta(s)^2 \, ds} \delta(\beta(s) - x) \right\} \]
Let $X_1, X_2, \ldots$ be independent with means $\mu_k$ and variance $\sigma_k^2$.
Let $S_n = \sum_{i=1}^{n} X_i$. A necessary and sufficient condition that
\[
\lim_{n \to \infty} P \left[ \frac{S_n - E(S_n)}{\sigma(S_n)} \leq x \right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2} \, du 
\]
and that the r.v. $\frac{X_k - \mu_k}{\sigma(S_n)}$ be infinitesimal is that $\varepsilon > 0$
the Lindeberg condition:
\[
\lim_{n \to \infty} \frac{1}{\sigma(S_n)} \sum_{k=1}^{n} \int_{|x| > \varepsilon \sigma(S_n)} x^2 \, dF_k(x + \mu_k) = 0.
\]

What does the Lindeberg Condition mean? Consider
\[
P \left[ \max_{1 \leq k \leq n} |X_k - \mu_k| > \varepsilon \sigma(S_n) \right] \leq \sum_{k=1}^{n} P \left[ |X_k - \mu_k| > \varepsilon \sigma(S_n) \right] = \sum_{k=1}^{n} \int_{|x| > \varepsilon \sigma(S_n)} x^2 \, dF_k(x)
\]
\[
= \frac{1}{e \sigma(S_n)^2} \sum_{k=1}^{n} \int_{|x| > \varepsilon \sigma(S_n)} (x - \mu_k)^2 \, dF_k(x) = \frac{1}{e \sigma(S_n)^2} \sum_{k=1}^{n} \int_{|x| > \varepsilon \sigma(S_n)} x^2 \, dF_k(x - \mu_k).
\]

This is much stronger than infinitesimal, the variables are really uniformly really small.

Theorem: Let $X_k$ be a sequence of finite sequences of independent
random variables with means $\mu_k$ and variances $\sigma_k^2$. A necessary condition
that $S_n = \sum_{i=1}^{n} X_i$ be $\text{N}(\mu, \sigma^2)$, where $\text{N}$ is an appropriate normal
distribution, is that $\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \int_{|x| > \varepsilon \sigma(S_n)} x^2 \, dF_k(x + \mu_k) = 0$
Proof: Sufficiency: Consider \( \sup_{1 \leq k \leq n} \sum_{1 \leq j \leq k} x_j^2 dF_k(x_k + \mu_k) \leq \varepsilon \), so they are infinitesimal. Moreover, \( \sigma^2(S_n) \to 1 \) by 1) and 2) \( \sigma^2(S_n) \) is uniformly bounded and conditions 1 and 2 of Batsy's theorem are satisfied. Therefore, by Batsy's theorem, looking at the accompanying laws, a sufficient condition for convergence of the d.f.'s is that

\[
K_n(x) = \sum_{k=1}^{n} \int \frac{x^2}{1 + x^2} dF_k(x_k + \mu_k) = 0, \quad x > 0
\]

and \( K_n(+\infty) \to K(+\infty) \) since \( \sigma^2(S_n) \to 1 \). Since we can choose \( A_n = \zeta \mu_k \), we get \( \kappa_n \to \kappa \) with \( \kappa_n = \sum_{k=1}^{n} \mu_k - \Lambda_n \).

Necessity: Trivial.

Corollary: Suppose \( \sigma^2(S_n) = 1 \) then conditions 1 and 2 reduce to

\[
\lim_{n \to \infty} \sum_{k=1}^{n} \int x^2 dF_k(x_k + \mu_k) = 0
\]

and moreover if \( \mu_k = 0 \), then this last one is

\[
\lim_{n \to \infty} \sum_{k=1}^{n} \int x^2 dF_k(x_k - \mu_k) = 0
\]

Theorem follows from this corollary let \( X = \frac{X_k - \mu_k}{\sigma(S_n)} \) and \( S_n = X_{1k} + \ldots + X_{nk} \) and \( \sigma(S_n) = \frac{S_n}{\sigma(S_n)} = \frac{X_k - \mu_k}{\sigma(S_n)} \) and

\[
F_{X_k}(x) = P(X_k < x) = P \left( \frac{X_k - \mu_k}{\sigma(S_n)} < x \right) = F_k(x \sigma(S_n) + \mu_k)
\]

the condition to be checked is

\[
\lim_{n \to \infty} \sum_{k=1}^{n} \int x^2 dF_k(x \sigma(S_n) + \mu_k) = 0 \quad \text{which is}
\]
\[
\lim_{n \to \infty} \frac{1}{\delta(n)} \sum_{k=1}^{n} x_k^2 dF_n(x + \mu_k) = 0
\]

**Theorem:** Let \( \{X_n\} \) be a seq. of i.i.d. random variables, with means \( \mu_k \) and variance \( \sigma_k^2 \). A key condition that the diff. of \( S_n = X_1 + \ldots + X_n = A_n \) (where the \( A_n \)'s are properly chosen) converge to the Poisson \( P(\lambda) = \sum_{m=0}^{\infty} \frac{e^{-\lambda} \lambda^m}{m!} \) and that \( \sigma_k(n) \to \sigma_k \) is that

\[
\forall \delta > 0 \quad \lim_{n \to \infty} \frac{1}{\delta(n)} \sum_{k=1}^{n} x_k^2 dF_n(x + \mu_k) = 0 \quad \text{and}
\]

\[
\lim_{n \to \infty} \frac{1}{\delta(n)} \sum_{k=1}^{n} x_k^2 dF_n(x + \mu_k) = 0 \quad \text{so that} \quad A_n = \sum_{k=1}^{n} \mu_k - \lambda + o(1) \quad \text{as} \ n \to \infty.
\]

We will talk about \( (\mathbb{N}, \mathcal{B}, P) \sum X_n(w) \) converge a.e. for certain cases.

**Brownian Motion:** \( (\mathbb{R}, \mathcal{B}, P) \sum_{t \in [0,1]} \mathbf{1} \text{ the open sets} \)

Let \( F: \Omega \to \mathbb{R} \in \mathcal{B}, F[\beta] = \omega \beta \) is measurable \( F \).

Let \( \omega \in (\Omega, \mathcal{B}, P) \) and \( X(\omega) \) an r.v. \( P(\mathbb{E}X = x^2 = F(x)) \) and \( g(x) \) is a Borel measurable function in \( \mathbb{R} \). Then

\[
E_g^\mathbb{E}(X) = \int_{-\infty}^{\infty} g(x) F(x) dx
\]

and if \( X_1, \ldots, X_n \) are any r.v.'s on \( (\Omega, \mathcal{B}, P) \) whose joint density \( f(x_1, \ldots, x_n) \) is \( f(x_1, \ldots, x_n) \) and \( g(x_1, \ldots, x_n) \) is a Borel measurable function in \( \mathbb{R}^n \),

\[
E g(X_1, \ldots, X_n) = \int_{\mathbb{R}^n} g(x_1, x_2, \ldots, x_n) f(x_1, x_2, \ldots, x_n) dx_1 \ldots dx_n
\]
But \[ P\{ \beta(x) \leq x \} = \frac{\pi}{2} \int_{-\infty}^{x} e^{-u^2} du \quad \text{and similarly} \]
\[ P\{ \beta(x_1) \leq x_1, \ldots, \beta(x_n) \leq x_n \} = \frac{1}{\sqrt{(2\pi)^n}} \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_n} e^{-\sum_{i=1}^{n} u_i^2} du_1 \cdots du_n. \]

Hence \[ E \left[ g(\beta(x_1), \beta(x_2), \ldots, \beta(x_n)) \right] = \frac{1}{(2\pi)^{n/2}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{-\sum_{i=1}^{n} u_i^2} g(u_1, u_2, \ldots, u_n) du_1 \cdots du_n \]
and \[ A_{ij} = (\min(\beta(x_i), \beta(x_j))). \]

Suppose \[ F[\beta] = \int_0^t \beta^2(s) ds \] so \[ E \left[ \int_0^t \beta^2(s) ds \right] = \int_0^t E[\beta^2] ds. \]

So \[ E \left[ \int_0^t \beta^2(s) ds \right] \] do a little classical analysis. Consider the integral \( e^{\int_0^t \beta^2(s) ds} \) and let \( \rho_0, \rho_1, \rho_2, \ldots \) such solutions and so \( \int_0^t \rho^2(s) ds = u(t) \) for \( u(t) \) such solutions are normalized. 

So \[ \rho \int_0^t \rho^2(s) ds + \rho \int_0^t u(s) ds = u(t) \] differentiate w.r.t. \( t \) with \( \rho u(t) - \rho u'(t) + \rho \int_0^t u(s) ds = u'(t) \) and then again 
\[ \rho u(t) - \rho u'(t) + \rho \int_0^t u(s) ds = u''(t) \] with \( u(0) = 0 \) and \( u'(0) = 0 \).

So \( \min(c_1, c_2) = \sum_{k=0}^{\infty} \frac{u(kc_1) - u(kc_2)}{c_1 - c_2} \) the Spectral Theorem.
Let \( x_1, x_2, \ldots \) be independent random variables, each \( N(0, 1) \). Consider
\[
\sum_{k=0}^{\infty} \frac{a_k(w) u_k(x)}{\sqrt{p_k}} = \beta(x)
\]
will prove that this converges for a.e. \( u \).

This is a Fourier series which has random coefficients.

\[
E\{\beta^2(x)\} = E\left[ \sum_{k=0}^{\infty} \frac{a_k(w) u_k(x)}{\sqrt{p_k}} \right]^2 = E\left[ \sum_{k=0}^{\infty} a_k(w) u_k(x) \right]^2
\]

because \( N(0, 1) \)

exchange summation + integration:
\[
E\left\{ \frac{1}{p_k} \right\} = \min (c_k, \tau) = \tau
\]

This proves \( \tau \) because is \( N(0, 1) \).

\[
E\{\beta(x) \beta(s)\} = \min (c_x, c_s)
\]

replace with Fourier Series

\[
E \left\{ e^{-\frac{1}{2} \int_0^t \beta(s) ds} \right\} = E \left\{ e^{-\frac{1}{2} \int_0^t \sum_{k=0}^{\infty} a_k(w) u_k(s) ds} \right\}
\]

\[
= E \left\{ e^{-\frac{1}{2} \int_0^t \sum_{k=0}^{\infty} \frac{a_k(w) u_k(s)}{\sqrt{p_k}} ds} \right\} = \prod_{k=0}^{\infty} E \left\{ e^{-\frac{1}{2} \int_0^t \frac{a_k(w) u_k(s)}{\sqrt{p_k}} ds} \right\}
\]

\[
= \prod_{k=0}^{\infty} e^{-\frac{1}{2} \int_0^t \frac{a_k(w) u_k(s)}{\sqrt{p_k}} ds} = e^{\frac{1}{2} \int_0^t s ds} = e^{t^2/2}
\]

\[
E \left\{ e^{-\frac{1}{2} \int_0^t \beta(s) ds} \right\} = E \left\{ e^{-\frac{1}{2} \sum_{k=0}^{\infty} a_k(w) u_k(s)} \right\} \quad \text{and}
\]

\[
\prod_{k=0}^{\infty} \left( 1 - \frac{1}{\tau^{2k+1}} \right) = \frac{1}{\prod_{k=0}^{\infty} \left( 1 + \frac{2k+1}{\tau^{2k+1}} \right)} = \frac{\tau^{2k+1}}{\tau^{2k+1} + 1}
\]

by induction on \( k \).
Let \( T = 1 \quad \mathbb{E} \{ e^{-\frac{1}{4} \int_0^T \beta(s)^2 \, ds} \} = \frac{1}{\sqrt{\cos(\theta)}} \) how does this behave as \( t \to \infty \)

And \( \lim_{t \to \infty} t \log \mathbb{E} \{ e^{-\frac{1}{4} \int_0^T \beta(s)^2 \, ds} \} = -\frac{1}{2} \)

**Theorem:** If \( V(y) \to \infty \) as \( |y| \to \infty \) then

\[
\lim_{t \to \infty} t \log \mathbb{E} \{ e^{-\frac{1}{4} \int_0^T V(\beta(s)) \, ds} \} = -\lambda, \quad \text{where} \; \lambda \; \text{is the lowest eigenvalue of} \; -\frac{1}{2} \frac{d^2}{dy^2} + V(y), \text{the Schröedinger equation. This is small potatoes.}
\]

Feynman - Kac : \( \{ u(t, x) = \frac{1}{2} u_{xx} - V(x) u \} \) \( V \) measurable and bounded below

\[
u(x, y) = \mathbb{E}_x \left\{ e^{-\int_0^T V(\beta(s)) \, ds} u_0(\beta(T)) \right\}
\]

Let's play with it

**Special Cases:** \( V \equiv 0 \rightarrow \mathbb{E}_x \{ u_0(\beta(T)) \} = \frac{1}{2\pi t} \int_{-\infty}^{\infty} u_0(cy) e^{-\frac{(x-y)^2}{2t}} \, dy
\]

2) \( V(x) = \frac{x^2}{2} \) so presumably

\[
u(x, y) = \mathbb{E}_x \left\{ e^{-\int_0^T (\beta(s)^2 + x^2) \, ds} \right\}
\]

\[
u(x, y) = e^{-\frac{x^2}{2}} \mathbb{E}_x \left\{ e^{-\int_0^T \beta(s)^2 \, ds} u_0(cy) \right\}
\]

\[
u(x, y) = e^{-\frac{x^2}{2}} \mathbb{E}_x \left\{ e^{-\int_0^T \beta(s)^2 \, ds} \right\} \mathbb{E}_x \left\{ e^{-\frac{1}{4} \int_0^T \beta(s)^2 \, ds} u_0(cy) \right\}
\]

\[
u(x, y) = e^{-\frac{x^2}{2}} \mathbb{E}_x \left\{ e^{-\int_0^T \beta(s)^2 \, ds} \right\} \mathbb{E}_x \left\{ e^{-\frac{1}{4} \int_0^T \beta(s)^2 \, ds} \right\} \mathbb{E}_x \left\{ e^{-\frac{1}{4} \int_0^T \beta(s)^2 \, ds} u_0(cy) \right\}
\]
\[ E \left\{ e^{-\frac{x^2}{2}} \int_0^\infty e^{-\frac{t^2}{2}} \sum_{n=1}^{\infty} \frac{\sin \lambda_n s}{\lambda_n} \sin \lambda_n \beta(x) \, ds \right\} \]

\[ = e^{-\frac{x^2}{2}} \int \frac{1}{\sqrt{2 \pi}} e^{-\frac{x^2}{2}} \int_0^\infty e^{-\frac{s^2}{2}} \sum_{n=1}^{\infty} \frac{\sin \lambda_n s}{\lambda_n} \sin \lambda_n \beta(x) \, ds \, ds \, ds \]

\[ = e^{-\frac{x^2}{2}} \frac{1}{\sqrt{\cosh(\beta)}} e^{-\frac{x^2}{2}} \int_0^\infty \sum_{n=1}^{\infty} \frac{u_n(\beta)}{\beta - \beta_n} \sin \lambda_n \beta(x) \, ds \, ds \]

Where \( R(s, \tau; -\beta) = \min(s, \tau) + \frac{1}{\beta^2} \int_0^s \min(s, \tau) R(s, \tau; -\beta) \, ds \)

Does \[ -\sum_{n=1}^{\infty} \frac{u_n(\beta)}{\beta - \beta_n} + \sum_{n=1}^{\infty} \frac{u_n(\beta)}{\beta + \beta_n} = \frac{1}{\beta^2} \sum_{n=1}^{\infty} \frac{u_n(\beta)}{\beta - \beta_n} \sum_{n=1}^{\infty} \frac{u_n(\beta)}{\beta + \beta_n} \]

\[ R(s, \tau; -\beta) = \begin{cases} 
-\cosh \frac{\beta(\eta - s)}{\beta} \sinh \frac{\beta s}{\beta} & 0 \leq s \leq \tau \\
\cosh \frac{\beta(\eta - s)}{\beta} \sinh \frac{\beta s}{\beta} & s \geq \tau 
\end{cases} \]

So \( u(x, t) = \frac{1}{\sqrt{\cosh t}} e^{-\frac{x^2}{2} \left( t + \int_0^t R(s, \tau; -1) \, ds \right)} = \frac{1}{\sqrt{\cosh t}} e^{-\frac{x^2 + t \tanh t}{t}} \)

Exercise: \( v(x) = \frac{x^2}{2} \); \( u_0 = x \), hint

\( u(x, t) = E_x \left\{ e^{-\frac{t}{2} \beta^2 \sin \beta \cos \beta} \right\} \)

Calculate \( E_x \left\{ e^{-\frac{t}{2} \beta^2 \sin \beta \cos \beta} \right\} \)
Solving for \( \psi(x) \) where:

1. \( \frac{1}{2} \psi''(x) - (s + 2) \psi(x) \psi = 0 \quad \psi(x) = \begin{cases} 1 & x > \alpha \\ 0 & x \leq \alpha \end{cases} \)

2. \( \psi \rightarrow 0 \) as \( x \rightarrow \pm \infty \)

3. \( \psi \) is continuous

4. \( \psi \) is continuous at \( x = 0 \)

5. \( \psi'(0^-) - \psi'(0^+) = 2 \)

---

Case 1: \( x > 0 \)

for \( x > \alpha \)

\[
\frac{1}{2} \psi''(x) - (s + 2) \psi = 0
\]

\[
\psi'(x) - 2(s + 1) \psi = 0
\]

\[
\lambda_2 = \pm \sqrt{2(s+2)} \quad \text{so} \quad \psi(x) = c_1 e^{-\sqrt{2(s+2)}x}
\]

for \( 0 < x < \alpha \)

\[
\frac{1}{2} \psi''(x) - (s) \psi = 0
\]

\[
\psi(x) = c_2 e^{-\sqrt{2s}x} + c_3 e^{\sqrt{2s}x}
\]

for \( x < 0 \)

\[
\frac{1}{2} \psi''(x) - s \psi = 0
\]

\[
\psi(x) = c_4 e^{-\sqrt{2s}x}
\]

3. \( \Rightarrow \) \( c_4 = c_2 + c_3 \quad \text{at} \quad x = 0 \)

5. \( \Rightarrow \) \( -\sqrt{2s} c_4 = -\sqrt{2s} c_2 + \sqrt{2s} c_3 + 2 \)

\[
\begin{align*}
\lambda_1 &= \pm \sqrt{2(\alpha+2)} \\
\lambda_2 &= \sqrt{2(s+2)} \\
\end{align*}
\]

\[
\begin{align*}
\psi(x) &= c_1 e^{-\sqrt{2(\alpha+2)}x} + c_2 e^{-\sqrt{2(s+2)}x} + c_3 e^{\sqrt{2s}x} \\
\lambda_3 &= -\sqrt{2s} \\
\end{align*}
\]

\[
\begin{align*}
\Rightarrow \quad c_1 e^{-\sqrt{2(\alpha+2)}x} &= c_2 e^{-\sqrt{2(s+2)}x} + c_3 e^{\sqrt{2s}x} \\
\end{align*}
\]
4. \quad c_1 \left( -\sqrt{2(5+1)} \right) e^{\frac{-\sqrt{2(5+1)}}{\sqrt{2}} x} = -\sqrt{2} c_2 e^{\frac{-\sqrt{2} x}{\sqrt{2}}} + \sqrt{2} c_3 e^{\frac{-\sqrt{2} x}{\sqrt{2}}} e^{\sqrt{2}x} x

\begin{align*}
\Rightarrow \quad c_1 \left( -\sqrt{2(5+1)} \right) e^{\frac{-\sqrt{2(5+1)}}{\sqrt{2}} x} &= \sqrt{2} \left( c_2 e^{\frac{-\sqrt{2} x}{\sqrt{2}}} - c_3 e^{\frac{-\sqrt{2} x}{\sqrt{2}}} e^{\sqrt{2}x} x \right) \\
\Rightarrow \quad \frac{\sqrt{2(5+1)}}{\sqrt{2}} c_1 e^{\frac{-\sqrt{2(5+1)}}{\sqrt{2}} x} &= c_2 e^{\frac{-\sqrt{2} x}{\sqrt{2}}} - c_3 e^{\frac{-\sqrt{2} x}{\sqrt{2}}} e^{\sqrt{2}x} x \\
\Rightarrow \quad c_1 e^{\frac{-\sqrt{2(5+1)}}{\sqrt{2}} x} &= -c_3 e^{\frac{-\sqrt{2} x}{\sqrt{2}}} e^{\sqrt{2}x} x + c_2 e^{\frac{-\sqrt{2} x}{\sqrt{2}}}
\end{align*}

\begin{align*}
2c_3 e^{\sqrt{2}x} x &= c_1 \left( 1 - \sqrt{\frac{5+1}{5}} \right) e^{\frac{-\sqrt{2(5+1)}}{\sqrt{2}} x} \\
2c_3 &= c_1 \left( 1 - \sqrt{\frac{5+1}{5}} \right) e^{\frac{-\sqrt{2(5+1)}}{\sqrt{2}} x}
\end{align*}

\begin{align*}
c_3 &= c_1 \frac{1}{2} \left( 1 - \sqrt{\frac{5+1}{5}} \right) e^{\frac{-\sqrt{2(5+1)}}{\sqrt{2}} x}
\end{align*}

\begin{align*}
\Rightarrow \quad c_1 &= \frac{2}{1 - \sqrt{\frac{5+1}{5}}} \left( \frac{\sqrt{2(5+1)}}{\sqrt{2}} + \sqrt{2} \right) e^{\frac{-\sqrt{2(5+1)}}{\sqrt{2}} x} \\
\Rightarrow \quad c_1 &= \frac{2}{1 - \sqrt{\frac{5+1}{5}}} \left( \frac{\sqrt{2(5+1)}}{\sqrt{2}} + \sqrt{2} \right) \frac{\sqrt{2(5+1)}}{\sqrt{2}} e^{\frac{-\sqrt{2(5+1)}}{\sqrt{2}} x} e^{\sqrt{2}x} x \\
\Rightarrow \quad c_1 &= \frac{2}{1 - \sqrt{\frac{5+1}{5}}} \frac{\sqrt{2(5+1)}}{\sqrt{2}} \frac{\sqrt{2(5+1)}}{\sqrt{2}} e^{\frac{-\sqrt{2(5+1)}}{\sqrt{2}} x} e^{\sqrt{2}x} x
\end{align*}

\begin{align*}
\Rightarrow \quad c_1 &= \frac{2}{1 - \sqrt{\frac{5+1}{5}}} \frac{\sqrt{2(5+1)}}{\sqrt{2}} \frac{\sqrt{2(5+1)}}{\sqrt{2}} e^{\frac{-\sqrt{2(5+1)}}{\sqrt{2}} x} e^{\sqrt{2}x} x
\end{align*}

\begin{align*}
\Rightarrow \quad c_2 &= \frac{\sqrt{2}}{\sqrt{5} - \sqrt{5+1}} e^{\frac{-\sqrt{2(5+1)}}{\sqrt{2}} x} e^{\sqrt{2}x} x
\end{align*}

\begin{align*}
\Rightarrow \quad c_4 &= \frac{-1}{\sqrt{2}} \left( \frac{1}{\sqrt{5} - \sqrt{5+1}} e^{\frac{2\sqrt{2}x}{\sqrt{2}}} x \right)
\end{align*}

\begin{align*}
-1 \left( \frac{1}{\sqrt{2}} + \frac{\sqrt{2}}{\sqrt{5} - \sqrt{5+1}} e^{\frac{2\sqrt{2}x}{\sqrt{2}}} x \right)
\end{align*}
2. Strong Limit Theorem

Strong Law of Large Numbers: Bernoulli. Let \( X_1, X_2, \ldots \) be i.i.d. Bernoulli r.v.s.

\[
S_n = X_1 + \cdots + X_n, \quad P(E X) = \mathbb{E} X = p, \quad P(E X_i = 0) = \mathbb{P}(1 - p)
\]

\[\varepsilon > 0, \lim_{n \to \infty} P(E(\frac{S_n}{n} - P) \geq \varepsilon) = 0, \quad \text{convergence in measure.}\]

**Theorem:** Let \( X_1, X_2, \ldots \) be i.i.d. r.v.s with mean \( \mu \) and variance \( \sigma^2 \), \( S_n = \frac{1}{n} \sum X_i \), then \( \varepsilon > 0, \lim_{n \to \infty} P(E(\frac{S_n}{\sqrt{\frac{n}{\sigma^2}}} - \mu) \geq \varepsilon) = 0 \). (Note: \( \mu \) and \( \sigma \) are the same.)

**Proof:** Reminder of Chebyshev inequality.

If \( X \) has mean \( \mu \) and variance \( \sigma^2 \), then \( P(E|X - \mu| \geq \varepsilon) \leq \frac{\sigma^2}{\varepsilon^2} \)

\[
\sigma^2 = \int (X - \mu)^2 f(X) dX \geq \varepsilon^2 \int f(X) dX = \varepsilon^2 P(E|X - \mu| \geq \varepsilon).
\]

So, \( P(E(\frac{S_n}{\sqrt{n}} - \mu) \geq \varepsilon) \leq \frac{\sigma^2}{\varepsilon^2} \to 0 \) as \( n \to \infty \). Very trivially.

Steinhaus (1922) first people to state probability.

Borel (1909), \( a, a, \ldots \) — binary expansion for \( x \in (0, 1) \)

\[
P(E(l_n) = \frac{1}{2}) = 1.
\]

Consider \((\Omega, B, P)\)

**Theorem:** Kolmogorov's Inequality. Let \( X_1, X_2, \ldots X_n \) be i.i.d. r.v.s with \( \mu = 0 \) and variances \( \sigma^2, \leq \mathbb{E} X_i^2 \). Let \( S_n = \frac{1}{\sqrt{n}} \sum X_i \).

\[
P(E \max (|X_1|, \ldots, |X_n|) \geq \varepsilon) \leq \frac{\sigma^2}{\varepsilon^2}
\]

Remark: Chebyshev's inequality applied to \( S_n \).
\[ P \left( \left| S_n \right| \geq \varepsilon \right) = \sum_{k=1}^{n} \frac{e_k}{\varepsilon^2} \]

**Proof:** Let \( E = \bigcup_{k=1}^{n} (S_k, S_k, \ldots, S_n) \geq \varepsilon \) (case 1 in \( A \))

\[ E_k = \bigcup_{k=1}^{n} (S_k, S_k, \ldots, S_n) \geq \varepsilon \quad \text{and} \quad 1S_k \geq \varepsilon \quad \text{if} \quad k \in k_k^3 \]

note \( E = E_1 \cup E_2 \cup \cdots \cup E_n \). \( E_i \cap E_j = \emptyset \quad \forall \quad i \neq j \). Therefore

\[ P(E) = \sum_{k=1}^{n} P(E_k) \quad \text{consider} \quad \sum_{k=1}^{n} e_k^2 = \sum_{k=1}^{n} \int X_k^2 \, dP = \int \sum_{k=1}^{n} X_k^2 \, dP = \int \left( \sum_{k=1}^{n} X_k^2 \right) \, dP = \int S_n^2 \, dP = \sum_{k=1}^{n} \int S_k^2 \, dP \]

\[ = \sum_{k=1}^{n} \int \left[ S_k - (S_k - S_n) \right]^2 \, dP = \sum_{k=1}^{n} \left[ \int S_k^2 \, dP + P(E_k) \sum_{j=k+1}^{n} e_j^2 \right] \]

\[ \geq \sum_{k=1}^{n} \int S_k^2 \, dP \quad \text{on} \quad E_k \quad 1S_k \geq \varepsilon \]

\[ \geq \varepsilon^2 \sum_{k=1}^{n} P(E_k) = \varepsilon^2 P(E) \quad Q.E.D. \]

**Theorem:** Let \( X_1, X_2, \ldots \) be a sequence of independent random variables with mean 0 and variances \( \sigma_k^2 \). If \( \sum_{k=1}^{n} \sigma_k^2 < \infty \), then

\[ \sum_{k=1}^{n} X_k(w) \] converges a.e. in \((\Omega, \mathcal{F}, P)\).

**Proof:** Let \( S_n(w) = \sum_{j=1}^{n} X_j(w) \) and let \( a_m(w) = \sup_{k \geq m} |S_m(w) - S_k(w)| \)

\[ a(w) = \inf_{m \geq 1} a_m(w). \]

The m.a.s. condition that the series converge at a.e. \( w \) is

\[ a(w) < \infty. \]
Let $\varepsilon > 0$ be given. For any integers $m,n$ we have from Kolmogorov inequality:

$$P\left( \max_{1 \leq k \leq n} |S_{m+k}(w) - S_m(w)| \geq \varepsilon \right) \leq \frac{\sum_{k=m+1}^{n} \sigma_k^2}{\varepsilon^2}$$

from which,

$$P\{a_m(w) \geq \varepsilon\} \leq \frac{\sum_{k=m+1}^{n} \sigma_k^2}{\varepsilon^2} < \infty \quad \text{by the definition of inf}$$

for $m \to \infty$.

So for $\varepsilon > 0$,

$$P\{a_m(w) \geq \varepsilon\} = 0 \quad \text{as } m \to \infty, \quad P\{a_m(w) = \varepsilon\} = 1.$$ 

**Application:** Recall that for independent Gaussian variables $u_k(z)$,

$$\sum_{k=0}^{\infty} u_k(z) \text{ is a series for all } z$$

mean zero and variance $\sigma_k^2$. From theorems just proved this series converges for all $z$ if $\sum_{k=0}^{\infty} \sigma_k^2 < \infty$.

So it in $\varepsilon > 0$ and the covariance is $\min(c, \varepsilon)$, so it is another form of Brownian motion.

**Lemma 1:** If $\{y_n\}$ is a sequence of real numbers $\Rightarrow \lim_{n \to \infty} y_n = x$ then $\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} y_i = x$.

**Lemma 2:** If $\{y_n\}$ is a sequence of real numbers $\Rightarrow \sum_{n=1}^{\infty} \frac{y_n}{n} < \infty$ then $\lim_{n \to \infty} \frac{y_1 + y_2 + \cdots + y_n}{n} = 0$. 

prob
Theorem: (1st form of Strong Law of Large Numbers) Let \( X_1, X_2, \ldots \) be a sequence of r.v.s with mean zero and variance \( \sigma_k^2 \neq \sum_{k=1}^\infty \frac{\sigma_k^2}{k^2} \) converges. Then \( P \{ \lim_{n \to \infty} \frac{S_n}{n} = 0 \} = 1 \).

Remark: The r.v.s have two moments and if they were identically distributed the hypothesis is automatic and we have trivially
\[ P \{ \lim_{n \to \infty} \frac{S_n}{n} = \mu \} = 1. \]

Proof: Let \( Y_k = \frac{X_k}{n} \). Then the \( Y \)s are independent with mean zero and variance \( \frac{\sigma_k^2}{k^2} \). Therefore by theorem on infinite series
\[ \sum_{k=1}^\infty \frac{Y_k}{k} \text{ converges for almost all } \omega \text{ i.e. } \sum_{k=1}^\infty \frac{X_k}{k} \text{ converges.} \]

i.e. Then by Lemma 2 \( \frac{S_n}{n} \to 0 \) a.e. Q.E.D.

Application: Let \( X_1, X_2, \ldots \) be r.v.s having common d.f. \( F(x) \) and \( N(x) = \# \text{ of } X_j \text{'s out of } X_1, X_2, \ldots \text{ such that } x \leq X_j \). Then \( Y \)

\[ P \{ \lim_{n \to \infty} \frac{N(x)}{n} = F(x) \} = 1. \] This is the empirical distribution function, so this keeps statisticians in business.

Proof: Let \( Y_i = \begin{cases} 1 & \text{if } x_i \leq x \\ 0 & \text{if } x_i > x \end{cases} \) for a given \( x \) fixed. Note that
\( Y_1, Y_2, \ldots \) are independent r.v.s with mean \( F(x) \) and (second moment) \( F(x) \). So by the remark \( S_n = \sum_{i=1}^n Y_i = N(x) \) so \( P \{ \lim_{n \to \infty} \frac{S_n}{n} = F(x) \} = 1. \)
We will prove: **Theorem:** Let \( X_1, X_2, \ldots \) be i.i.d. r.v.s with mean \( \mu \) and variance one. Let \( N_n = \# \) of partial sums \( S_j \) out of \( S_1, S_2, \ldots, S_n \) which are \( \geq 0 \). Then

\[
\lim_{n \to \infty} P\{ N_n \leq \alpha n \} = 0 \quad \alpha < 0
\]

\[
\sum_{\alpha < 1} = \left\{
\begin{array}{ll}
\frac{\pi}{2} \arcsin \sqrt{\alpha} & 0 \leq \alpha \leq 1 \\
1 & \alpha \geq 1 
\end{array}
\right. 
\text{(the arcsin law)}.
\]

**Proof:** (Using the Feynman - Kac formula). Let \( x^{(n)}(\alpha) = \left\{ \begin{array}{ll}
\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2} t^2} \cos(t \alpha) dt & \alpha < 1 \\
\frac{1}{\alpha} \text{sgn} \alpha & \alpha \geq 1
\end{array} \right. \)

This is a random step function. The invariance principle states that for a large class of functionals \( F_t \), \( F \in \mathbb{F} \), B.M.,

\[
\lim_{n \to \infty} P\{ F[X^{(n)}(\cdot)] \leq \alpha \} = P\{ F[X(\cdot)] \leq \alpha \}
\]

For example, let \( F[\beta] = \int_0^1 (1 + \text{sgn} \beta(s)) ds = \text{sgn} x = \text{sgn} \beta \geq 0 \)

Then \( \lim_{n \to \infty} P\{ N_n \leq \alpha n \} = P\{ \int_0^1 (1 + \text{sgn} \beta(s)) ds \leq \alpha \}
\]

Then \( \lim_{n \to \infty} P\{ N \leq \alpha n \} = P\{ \int_0^1 (1 + \text{sgn} \beta(s)) ds \leq \alpha \}
\]

Use the Lebesgue measure of the B.M. path that is positive. It will suffice to show

\[
P\left\{ \int_0^1 \frac{1 + \text{sgn} \beta(s)}{2} ds \leq \alpha \right\} = \sum \{ \alpha \}
\]

Let \( \sigma_\alpha(t) = P\{ \int_0^t \frac{1 + \text{sgn} \beta(s)}{2} ds \leq \alpha \} \) consider for \( \alpha > 0 \)

\[
E\left\{ e^{-\int_0^T \frac{1 + \text{sgn} \beta(s)}{2} ds} \right\} = \int_0^T e^{-\int_0^t ds} \sigma_\alpha(t)
\]

Let \( u(x, \beta(s)) = E\left\{ e^{-\int_0^T \frac{1 + \text{sgn} \beta(s)}{2} ds} \right\} \) therefore
$3. \int_0^\infty u(x,t;\lambda) \, dt = \int_0^\infty e^{-2\mu t} \, d\sigma(x,t).$ So now we try to find $u$, from the Feynman-Kac formula

$$u(x,0;\lambda) = \frac{1}{\sqrt{4\pi \lambda}} e^{-\frac{x^2}{4\lambda}}.$$ $u_t = \frac{1}{2} u_{xx} - \lambda V(x) u, \quad V(x) = \begin{cases} 3 \quad & x > 0 \\ 0 \quad & x \leq 0 \end{cases}$

This is equivalent to the following integral equation

$$u(x,t;\lambda) = \frac{1}{\sqrt{4\pi \lambda t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4\lambda t}} \left( \int_{-\infty}^{\infty} \frac{e^{-\frac{(y-s)^2}{2\lambda t}}}{\sqrt{2\pi \lambda t-s}} \, ds \right) \, dy.$$

Apply $\frac{2}{3} \frac{\partial}{\partial x} - \frac{1}{2} \frac{\partial^2}{\partial x^2}$ on both sides, heuristically a convolution.

$$\frac{\partial u}{\partial t} - \frac{1}{2} \frac{\partial^2 u}{\partial x^2} = 0 - \lambda V(x) u(x,t;\lambda).$$ Let

$$\psi(x,s;\lambda) = \int_0^\infty e^{-st} u(x,t;\lambda) \, dt \quad s > 0.$$ And so do the integral

$$\psi(x,s;\lambda) = \frac{1}{\sqrt{s}} e^{-\frac{x^2}{4s}} - \lambda \int_0^\infty V(y) \psi(y,s;\lambda) \, dy.$$

But this is equivalent to an ODE.

$$\frac{1}{2} \psi'' - (s - \lambda V(x)) \psi = 0 \quad \psi \to 0, \quad \text{as} \quad x \to \pm \infty$$

$\psi'(0-) - \psi'(0+) = 2$. If you solve this and $\psi$ is continuous at $x=0$.

$$\psi(x,s;\lambda) = \begin{cases} \frac{1}{\sqrt{2}} \left( \frac{\sqrt{s} + \sqrt{5}}{\sqrt{5} + \sqrt{s}} \right) e^{-\sqrt{5}x} x \quad & x \geq 0 \\ \frac{1}{\sqrt{2}} \left( \frac{\sqrt{s} + \sqrt{5}}{\sqrt{5} + \sqrt{s}} \right) e^{\sqrt{2}x} x \quad & x < 0 \end{cases}$$
Therefore \[ \int_{-\infty}^{0} \mathcal{H}(s, x; \lambda) \, dx = \frac{1}{\sqrt{s(s+\lambda)}} \] from (2).

By Fubini,
\[
\int_{-\infty}^{0} \int_{0}^{\infty} u(x, t; \lambda) \, dx \, dt = \int_{0}^{\infty} \int_{-\infty}^{0} \mathcal{H}(s, x; \lambda) \, dx \, ds = \int_{0}^{\infty} e^{-st} \int_{0}^{\infty} e^{-\lambda x} \, dx \, ds.
\]

Inverse Laplace T,
\[
\frac{1}{\sqrt{s(s+\lambda)}} = \int_{0}^{\infty} \mathcal{H}(s, x; \lambda) \, dx = \int_{0}^{\infty} e^{-st} \int_{0}^{\infty} e^{-\lambda x} \, dx \, ds.
\]

Inverse Laplace T,
\[
F(t) = e^{-\frac{2t}{\lambda}} \int_{0}^{\infty} \mathcal{H}(s, x; \lambda) \, dx \to \int_{0}^{\infty} e^{-x t} \, dx = \frac{1}{t},
\]

So \( F(t) = e^{-\frac{2t}{\lambda}} \frac{1}{t} = \Gamma(\frac{1}{2}, -\frac{2}{\lambda}t) \).

So \( \mathcal{H}(s, x; \lambda) \) is a Bessel Function \( J_{\frac{1}{2}}(\sqrt{s(s+\lambda)}) \).

So \( \mathcal{H}(s, x; \lambda) = \begin{cases} 0 & x \leq 0 \\ \frac{1}{\sqrt{\pi}} \arcsin \sqrt{s} & 0 < x < \lambda \\ 1 & x \geq \lambda \end{cases} \)

This theorem was first proved this way by Kac etc. This is a combinatorial result, actually.

Problem: \( P \{ \max_{0 \leq y \leq 1} \beta(s) \leq x \} \)

\[ = \lim_{n \to \infty} P \{ \max \left( \frac{S_1, S_2, \ldots, S_n}{n} \right) \leq x \} = H(x) \text{, Let.} \]

(\( \beta \) is continuous so if \( \max_{0 \leq y \leq 1} \beta(s) \leq x \) then \( V_n(\beta(s)) = 1 \) on a set of positive measure)

\[ V_n(x) = \left\{ \begin{array}{ll} 0 & x < \alpha \\ 1 & x \geq \alpha \end{array} \right. \text{ Let } \alpha > 0 \text{ and consider } \]

\[ \lim_{n \to \infty} E \left[ e^{-\lambda \int_{0}^{x} V_n(\beta(s)) \, ds} \right] = H(\alpha) \text{ because for } \beta(s) > \alpha, \lambda > 0 \]

This is calculable by Fuyman – Kac

\[ H(\alpha) = \frac{1}{\pi} \int_{0}^{\frac{\sqrt{\alpha}}{\lambda}} e^{-u^2} \, du. \]
Thm. 1: If \( X_1, X_2, \ldots \) are i.i.d. with mean 0 and variances \( \sigma^2 < \infty \), then \( \sum_{n=1}^{\infty} X_n \) converges a.e.

Thm. 2: If \( X_1, X_2, \ldots \) be i.i.d. for which \( \mathbb{E} X_i \leq c \), and \( \mathbb{E} X_i^2 < \infty \), then if \( \sum_{n=1}^{\infty} \mathbb{E} X_n \) converges to a set of positive measure then \( \sum_{n=1}^{\infty} \sigma^2 < \infty \). (Also \( \mathbb{E} X_i = 0 \))

\[ \mathbb{P} \{ \{ X_n \} \text{ is an essentially uniformly bounded sequence of i.i.d.'s} \} \] then \( \sum_{i=1}^{\infty} X_i \) either converges or diverges a.e., the \( X_i \)'s have mean zero compared to later with the zero-one law.

\[ \text{Proof: Let } S_0 = 0, S_n = \sum_{k=1}^{n} X_k. \text{ Let } G = \mathcal{O} \text{ where the } \mathbb{E} \text{ is converges, i.e. for } \delta \text{ lim } S_n \text{ exists and } \mathbb{P}(G) > 0. \]

By Egorov's theorem \( \exists K > 0 \text{ s.t. } \mathbb{E} = \{ \omega : |S_n(\omega)| \leq K \} \). Let \( E_n = \{ \omega : |S_n(\omega)| \leq K \} \) have positive measure, i.e. \( \mathbb{P}(E_k) > 0 \). Let \( E_n = \bigcap_{n=0}^{\infty} E_n \).

\[ \{ E_n \} \text{ is a decreasing sequence so } \lim_{n \to \infty} E_n = E. \text{ Consider } \]

\[ \int_{E_n} S_{n+1}^2 dP - \int_{E_{n-1}} S_{n+1}^2 dP = \int_{E_{n-1}} S_n^2 dP - \int_{E_{n-1}} S_{n+1}^2 dP - \int_{E_{n-1}} S_n^2 dP \]

\[ = \int_{E_{n-1}} (X_n + S_{n-1})^2 dP - \int_{E_{n-1}} S_n^2 dP - \int_{E_{n-1}} S_{n-1}^2 dP \]

\[ \text{Prob.} \]

\[ \text{Thm. 1: If } X_1, X_2, \ldots \text{ are i.i.d. with mean 0 and variances } \sigma^2 < \infty \text{ then } \sum_{n=1}^{\infty} X_n \text{ converges a.e.} \]

\[ \text{Thm. 2: If } X_1, X_2, \ldots \text{ be i.i.d. for which } \mathbb{E} X_i \leq c \text{, and } \mathbb{E} X_i^2 < \infty \text{ then if } \sum_{n=1}^{\infty} \mathbb{E} X_n \text{ converges to a set of positive measure then } \sum_{n=1}^{\infty} \sigma^2 < \infty \text{ (Also } \mathbb{E} X_i = 0 \) \]

\[ \mathbb{P} \{ \{ X_n \} \text{ is an essentially uniformly bounded sequence of i.i.d.'s} \} \] then \( \sum_{i=1}^{\infty} X_i \) either converges or diverges a.e., the \( X_i \)'s have mean zero compared to later with the zero-one law.

\[ \text{Proof: Let } S_0 = 0, S_n = \sum_{k=1}^{n} X_k. \text{ Let } G = \mathcal{O} \text{ where the } \mathbb{E} \text{ is converges, i.e. for } \delta \text{ lim } S_n \text{ exists and } \mathbb{P}(G) > 0. \]

By Egorov's theorem \( \exists K > 0 \text{ s.t. } \mathbb{E} = \{ \omega : |S_n(\omega)| \leq K \} \). Let \( E_n = \{ \omega : |S_n(\omega)| \leq K \} \) have positive measure, i.e. \( \mathbb{P}(E_k) > 0 \). Let \( E_n = \bigcap_{n=0}^{\infty} E_n \).

\[ \{ E_n \} \text{ is a decreasing sequence so } \lim_{n \to \infty} E_n = E. \text{ Consider } \]

\[ \int_{E_n} S_{n+1}^2 dP - \int_{E_{n-1}} S_{n+1}^2 dP = \int_{E_{n-1}} S_n^2 dP - \int_{E_{n-1}} S_{n+1}^2 dP - \int_{E_{n-1}} S_n^2 dP \]

\[ = \int_{E_{n-1}} (X_n + S_{n-1})^2 dP - \int_{E_{n-1}} S_n^2 dP - \int_{E_{n-1}} S_{n-1}^2 dP \]
\[ \sum S_n^2 dP = \int E_{n-1}^E \sum S_n^2 dP = \sigma_n^2 P(E_{n-1}) - \int (X_n + S_{n-1})^2 dP \]

on \( E_{n-1} \), \(|S_{n-1}(\omega)| \leq K \) and \(|X_n| \leq c\ a.s.\)

\[ |X_n + S_{n-1}| \leq c + K \ a.s. \ \text{on} \ E_{n-1} \]

\[ \geq \sigma_n^2 P(E_{n-1}) - (c+K)^2 P(E_{n-1} - E_n) \]
\[ \geq \sigma_n^2 P(E) - (c+K)^2 P(E_{n-1} - E_n) \]

So, summing from \( n=1 \) to \( N \) because of the telescopic series

\[ \sum S_n^2 dP \geq P(E) \sum_{n=1}^{N} \sigma_n^2 - (c+K)^2 \]. But also

\[ \int E_{n-1}^E \sum S_n^2 dP \leq K^2 P(E_n) \leq K^2 \]. So

\[ \sum_{n=1}^{N} \sigma_n^2 \leq \frac{K^2 + (c+K)^2}{P(E)} \] so the series converges. Q.E.D.

**Theorem 3:** Let \( \{X_n\} \) be i.i.d. taking values for which \( \exists \gamma > 0 \) and \( |X_n| \leq c \ a.s. \ n=1,2, \ldots \) then \( \sum X_n \) converges a.e. if both series

\[ \sum_{n=1}^{\infty} \mu_n \ \text{and} \ \sum_{n=1}^{\infty} \sigma_n^2 \] converge where \( \mu_n \) are the mean and \( \sigma_n^2 \) are the variances of \( X_n \).

**Proof:** If the series converge and \( |X_n| \leq c \ \forall n=1,2,3, \ldots \) then \( X = X_1 + X_2 + \cdots \) converges almost everywhere.
\[ \sum_{n=1}^{\infty} \sigma_n^2(Y_n) = \sum_{n=1}^{\infty} \sigma_n^2 \leq \infty \text{ we have by theorem 1 that } \sum_{n=1}^{\infty} Y_n \text{ converges a.e. but } \sum_{n=1}^{\infty} \mu_n \text{ converges so } \sum_{n=1}^{\infty} X_n \text{ converges a.e.} \]

Otherwise: Consider \(\mathbb{R} \times \mathbb{R}\) with measure \(\mathbb{P} \times \mathbb{P}\) let

\[ h_n(x_1, x_2) = X_n(x_1) - X_n(x_2) \quad n = 1, 2, 3, \ldots \]

from our hypothesis \(\sum_{n=1}^{\infty} h_n(x_1, x_2)\) converges a.e. on \(\mathbb{R} \times \mathbb{R}\), moreover

\[ \int_{\mathbb{R} \times \mathbb{R}} h_n(x_1, x_2) \, d\mathbb{P} \, d\mathbb{P} = \int_{\mathbb{R}} X_n \, d\mathbb{P} - \int_{\mathbb{R}} X_n \, d\mathbb{P} = 0 \]

also \(\|h_n\|_2 \leq 2\epsilon\) a.e. on \(\mathbb{R} \times \mathbb{R}\). By theorem 2 the series

\[ \sum_{n=1}^{\infty} \sigma_n^2(h_n) \]

converges.

\[ \int_{\mathbb{R} \times \mathbb{R}} h_n^2 \, d\mathbb{P} \, d\mathbb{P} = \int_{\mathbb{R}} X_n^2 \, d\mathbb{P} - 2 \int_{\mathbb{R}} X_n \, d\mathbb{P} \int_{\mathbb{R}} X_m \, d\mathbb{P} + \int_{\mathbb{R}} X_m \, d\mathbb{P} \int_{\mathbb{R}} X_n \, d\mathbb{P} \]

\[ = \sigma_n^2 + \sigma_m^2 - 2\mu_n \sigma_m + \sigma_n^2 + \sigma_m^2 = 2\sigma_n^2 \]

therefore \(\sum_{n=1}^{\infty} \sigma_n^2(h_n) = 2 \sum_{n=1}^{\infty} \sigma_n^2 \leq \infty\). Let \(Y_n = X_n - \mu_n\). Then

\[ \sigma_n^2(Y_n) = \sigma_n^2 : \sum_{n=1}^{\infty} \sigma_n^2(Y_n) < \infty \] this implies that

\[ \sum_{n=1}^{\infty} Y_n \text{ converges a.e. but then } \sum_{n=1}^{\infty} \mu_n < \infty. \]

since \(X_n \text{ and } \mu_n\) converge a.e. and \(Y_n = X_n - \mu_n\).

Hence \(\sum_{n=1}^{\infty} X_n\) converges a.e. from theorem 3. Let \(\epsilon > 0\) be any constant, a necessary and sufficient condition that \(\sum_{n=1}^{\infty} X_n\) converges a.e. is that the three series converge.
\[
\sum_{n=1}^{\infty} P\left( \left| X_n(w) \right| > c \right) = \sum_{n=1}^{\infty} \int_{|x| > c} dF_n(x)
\]

2) \[
\sum_{n=1}^{\infty} \int_{|x_n(w)| \leq c} x_n dF_n(x) = \sum_{n=1}^{\infty} \int_{|x| \leq c} x dF_n(x) \quad \text{truncated mean}
\]

3) \[
\sum_{n=1}^{\infty} \left( \int_{|x_n(w)| \leq c} x_n^2 dP - \left( \int_{|x_n(w)| \leq c} x_n dP \right)^2 \right) \quad \text{truncated variance}
\]

\[
= \sum_{n=1}^{\infty} \left[ \int_{|x| \leq c} x^2 dF_n(x) - \left( \int_{|x| \leq c} x dF_n(x) \right)^2 \right]
\]

**Proof:** Let \( Y_n = \begin{cases} X_n & \text{if } |X_n| \leq c \\ C & \text{if } |X_n| > c \end{cases} \)

\( Z_n = \begin{cases} X_n & \text{if } |X_n| \leq c \\ -C & \text{if } |X_n| > c \end{cases} \)

It is obvious that \( \sum X_n, \sum Y_n, \sum Z_n \) converge at exactly the same points.

From Theorem 3 a u.a.s. for \( \sum Y_n \) converges e.e. if:

\( \sum E(Y_n) < \infty \quad \sum \sigma^2(Y_n) < \infty \quad \text{i.e.} \)

\( E(Y_n) = \int_{|x| \leq c} x_n dP + c P\{ |X_n(w)| > c \} \)

\( \sum_{n=1}^{\infty} \int_{|x| \leq c} x_n dP + c P\{ |X_n(w)| > c \} < \infty \quad \text{and} \)

\( \sum_{n=1}^{\infty} \left( \int_{|x| \leq c} x_n^2 dP - \left( \int_{|x| \leq c} x_n dP \right)^2 \right) + c^2 P\{ |X_n| \leq c \} P\{ |X_n| > c \} \)

\( + 2c P\{ |X_n| > c \} \int_{|x| \leq c} x dP \)
So we have four series consider the \( \sum_{k=1}^{\infty} Z_k \) by theorem
3. adding and subtracting we get this because theorem 3 is.

If you have i.i.d.\( \sum_{k=1}^{\infty} X_k \) converges or diverges a.s. Some fun
with problems. Let \( \Omega = [0,1] \) be Lebesgue measurable sets
on \([0,1]\) and \( \mathbb{P} \) is Lebesgue measure. So \( \omega \in [0,1] \).

Let \( X_\omega(\omega) =\begin{cases} 1 & \text{if } \omega \text{ is odd } \left( \frac{\omega - 1}{2} \leq \omega < \frac{\omega}{2} \right) \\ -1 & \text{if } \omega \text{ is even } \end{cases} \)

\[
X_\omega(\omega) = \begin{cases} 1 & \text{if } \frac{\omega}{2} \leq \omega < 1 \\ -1 & \text{if } \frac{1}{2} \leq \omega < 1 
\end{cases}
\]

These are Rademacher functions. Show that \( X_1, X_2, X_3, \cdots \) is a
sequence of independent r.v.'s. Consider \( \mathcal{E}q\# 3 \) a bounded sequen-
sequence of real numbers and consider

\[
\sum_{k=1}^{\infty} a_k X_k(\omega) \quad \text{if } Y_n = a_n X_n(\omega) \quad \mathbb{E}(Y_n) = 0
\]

\[ \sigma^2(Y_n) = a_n^2 \quad \text{is from Theorem 3 a n.s. condit-}
\]

\[
\sum_{n=1}^{\infty} \text{converges if and only if } \sum_{n=1}^{\infty} a_n^2 \text{ converges.}
\]

So when does \( \sum_{n=1}^{\infty} a_n X_n(\omega) \) converge? When \( \sum_{n=1}^{\infty} a_n^2 < \infty \).

Random coin flips determine this. But this is done by the Rademacher functions.
Brownian motion Heuristics: (A new formalism to play with)

with \( \beta \in \mathbb{R} \), B.M. \( 0 \leq s \leq t \), \( \beta(s) = 0 \).

\[
E \{ F[\beta] \} \quad \text{if} \quad f(\beta, \beta_0, \ldots, \beta_n) \quad \text{is a Borel measurable function on } IR^n \quad \text{and if} \quad 0 \leq t_1 \leq \ldots \leq t_n = t \quad \text{then}
\]

\[
E \{ f(\beta(t_1), \ldots, \beta(t_n)) \} = \frac{1}{\sqrt{\det (2\pi \Sigma)}} \int_{IR^n} f(\beta(t_1), \ldots, \beta(t_n)) e^{-\frac{1}{2} (\beta(t_1) - \mu_1)^T \Sigma^{-1} (\beta(t_1) - \mu_1)} \, d\beta(t_1, \ldots, t_n)
\]

For every path \( \beta(s) \) on \([0, t]\) evaluate at \( \frac{d}{ds} \), \( j = 0, 1, 2, \ldots, n \) and linearize in between. For nice functionals we expect

\[
E \{ F[\beta] \} = \lim_{n \to \infty} \sqrt{\det (2\pi \Sigma)} \int_{IR^n} f(\beta(t_1), \ldots, \beta(t_n)) e^{-\frac{1}{2} (\beta(t_1) - \mu_1)^T \Sigma^{-1} (\beta(t_1) - \mu_1)} \, d\beta(t_1, \ldots, t_n)
\]

Von Neumann has proved that there is no Haar measure in function space, translational invariance that is. So suppose a flat integral, element in function space.

\[
E \{ F[\beta] \} = \int F[\beta] e^{\frac{1}{2} \int_0^t \beta' \beta' \, ds} \, d\beta
\]

Consider for \( \varepsilon > 0 \) \( E \{ e^{\frac{1}{\varepsilon} \int_0^t \beta' \beta' \, ds} \} = \frac{1}{\varepsilon^{\frac{1}{2}}} \int_0^\infty e^{-\frac{x^2}{2\varepsilon}} \, dx\)

\[
\lim_{\varepsilon \to 0} \varepsilon \log E \{ e^{\frac{1}{\varepsilon} \int_0^t \beta' \beta' \, ds} \} = \frac{1}{6}. \quad \text{Consider}
\]
\[ E \{ e^{\frac{t}{\varepsilon} F[\varepsilon \beta]} \}, \text{ how does this behave as } \varepsilon \to 0? \]

\[ n = \int e^{\frac{t}{\varepsilon} F[\varepsilon \beta]} e^{-\frac{\varepsilon}{2} \int_0^t \beta(s) \, ds} \, d\beta \quad \text{Let } \sqrt{\varepsilon} \beta = w - \frac{1}{2} \int_0^t [w'(s)]^2 \, ds \]

\[ = \int e^{\frac{t}{\varepsilon} \left[ F[w] - \frac{1}{2} \int_0^t [w'(s)]^2 \, ds \right]} \, d\beta \quad \text{by Laplace} \]

\[ \sim \int e^{\frac{t}{\varepsilon} \sup_{w \in C_0^+} \left[ F[w] - \frac{1}{2} \int_0^t [w'(s)]^2 \, ds \right]} \, d\beta \]

**Conjecture**

\[ \lim_{\varepsilon \to 0} \varepsilon \log E \{ e^{\frac{t}{\varepsilon} F[\varepsilon \beta]} \} = \sup_{w \in C_0^+} \left( F[w] - \frac{1}{2} \int_0^t [w'(s)]^2 \, ds \right) \quad \text{and} \]

Three cleverly chosen examples should be done.

1. \( F[\beta] = \int_0^t \beta(s) \, ds \)

\[ E \{ e^{\frac{t}{\varepsilon} F[\varepsilon \beta]} \} = E \{ e^{\frac{t}{\varepsilon} \int_0^t \beta(s) \, ds} \} \quad \text{from conjecture} \]

\[ \lim_{\varepsilon \to 0} \varepsilon \log E \{ e^{\frac{t}{\varepsilon} \int_0^t \beta(s) \, ds} \} = \sup_{w \in C_0^+} \left[ \int_0^t w(s) \, ds - \frac{1}{2} \int_0^t [w'(s)]^2 \, ds \right] \]

\[ = \frac{1}{6}, \]

By calculus of variations the Euler equation, \( t \)

\[ 1 + w''(s) = 0 \]
\[ w(0) = 0 \Rightarrow w(s) = -\frac{s^2}{2} + ts \]
\[ w'(0) = 0 \]
\[ w'(t) = 0 \]

So it works, wooper-doopee. This is action asymptotics.
\[ E\{F[\beta]\} \quad \gamma(t) \in C(0, \infty) \quad \text{and we want} \]

\[ E\{F[\beta + y]\} = \int F[\beta + y] e^{-\frac{1}{2}\int_0^t [\beta'(s)]^2 ds} \, d\beta \quad \text{let} \ \beta + y = w \]

\[ = \int F[w] e^{-\frac{1}{2}\int_0^t [\gamma'(s)]^2 ds} \, dw \]

\[ = e^{-\frac{1}{2}\int_0^t \gamma'(s)^2 ds} \int F[w] e^{-\frac{1}{2}\int_0^t [\gamma'(s)]^2 ds} \, dw \]

\[ = e^{-\frac{1}{2}\int_0^t \gamma'(s)^2 ds} \int F[\beta] e^{-\frac{1}{2}\int_0^t \gamma''(s)^2 ds} \, d\beta \]

A conjecture:

\[ E\{F[\beta + y]\} = e^{-\frac{1}{2}\int_0^t \gamma'(s)^2 ds} E\{F[\beta] e^{-\frac{1}{2}\int_0^t \gamma''(s)^2 ds}\} \]

where \( y \in B.V. \)
Lemma 1: The first Borel-Cantelli lemma: With $(\Omega, B, P)$,

If $\sum P(A_n) < \infty$ then $P(\limsup_{n \to \infty} A_n) = 0$ with $A_n \in B$ for all $n = 1, 2, \ldots$

$\limsup_{n \to \infty} A_n = \text{set of points that belong to infinitely many}
\liminf_{n \to \infty} A_n = \text{all but a finite number}

\liminf_{n \to \infty} A_n \subset \limsup_{n \to \infty} A_n \Rightarrow \text{the event} A_n \text{ occur infinitely often.}$

Proof: Let $A = \limsup_{n \to \infty} A_n$ then $A = \bigcup_{m} A_m$ for $m = 1, 2, 3, \ldots$

Therefore $P(A) = \sum_{m} P(A_m)$ for $m = 1, 2, 3, \ldots$ Q.E.D.

Lemma 2: The second Borel-Cantelli lemma: If the series

$\sum P(A_n)$ diverges and if the $(A_n)$ are

dependent then $P\left(\limsup_{n \to \infty} A_n\right) = 1$

Proof: Let $A = \limsup_{n \to \infty} A_n = \bigcup_{m} A_m$ then

$1 - P(A) = P(A^c) = P\left(\bigcup_{m} A_m^c\right)$

But $\bigcap_{m} A_m = (\bigcup_{m} A_m)^c$ is an increasing sequence of sets in $\mathcal{B}$.

So $1 - P(A) = \limsup_{m \to \infty} P(\bigcap_{m} A_m^c) \text{ (decreasing)}$

$= \lim \liminf_{m \to \infty} P(\bigcap_{m} A_m^c)$
by independence \[ \lim_{n \to \infty} \lim_{m \to \infty} \frac{N}{n} (1 - P(A_m)). \] It is an elementary exercise in advanced calculus that if \( 0 \leq a \leq 1 \) and \( \sum_{n=1}^{\infty} a_n = \infty \) then \( \lim_{n \to \infty} \prod_{j=1}^{n} (1 - a_j) = 0 \). So for each \( n \)

\[ \lim_{N \to \infty} \frac{N}{n} (1 - P(A_m)) = 0 \implies 1 - P(A) = 0 \text{ Q.E.D.} \]

Thus we have two Borel-Cantelli lemmas.

We have proved (1st form of SLLN): Let \( X_n \) be a sequence of r.v.'s with mean zero and finite variances \( \sigma_n^2 \). If

\[ \sum_{n=1}^{\infty} \frac{\sigma_n^2}{n} < \infty \]

the \( \lim_{n \to \infty} \frac{\sum_{n=1}^{n} X_n}{n} = 0 \) \( \implies 1 \). We will prove a much nicer theorem.

Theorem: (Kolmogorov law of large numbers): Let \( X_1, X_2, \ldots \) be iid.r.v. which are integrable (has a first moment). Let \( \mu \) be the mean of them. Then

\[ \lim_{n \to \infty} \frac{\sum_{n=1}^{n} X_n}{n} = \mu \]

(They are identically distributed). This is a sharp theorem.

Proof: Let \( F(x) \) be the common d.f. Define \( Y_n = \begin{cases} X_n, & \text{if } X_n \leq \mu \\ 0, & \text{otherwise} \end{cases} \)

\( (Y_n) \) is a sequence of random variables. Now the variance of \( Y_n \) can be calculated

\[ \sigma^2(Y_n) = E(Y_n^2) = \int \frac{X_n^2}{P = \int X^2 \, dF(x)} 
\]

\[ \quad \quad \quad = \sum_{n=1}^{\infty} \int \frac{X_n^2}{P = \int X^2 \, dF(x)} 
\]
\[ \sum_{n=1}^{\infty} \frac{\sigma^2(Y_n)}{u^2} \leq \sum_{n=1}^{\infty} \frac{1}{n^2} \left( \sum_{j=1}^{\infty} x_j^2 dF(x) \right) = \sum_{n=1}^{\infty} \frac{1}{n^2} J_{-1 \leq x \leq 1} \]

But \[ \sum_{n=1}^{\infty} \frac{1}{n^2} \leq \sum_{j=1}^{\infty} \frac{1}{j^2} = \frac{\pi^2}{6} \]

so \[ \sum_{n=1}^{\infty} \frac{\sigma^2(Y_n)}{u^2} \leq \frac{\pi^2}{6} \sum_{j=1}^{\infty} x_j^2 dF(x) \leq \frac{\pi^2}{6} \sum_{j=1}^{\infty} x_1 dF(x) \leq \frac{\pi^2}{6} \cdot 1 = \frac{\pi^2}{6} \]

Therefore by the first form of the strong law of large numbers

\[ P \left( \lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} Y_j - E(Y_1) \right) = 0 \]

Let \( A \) be the set \( \{ x : X_n(x) \neq Y_n(x) \} \) and \( \{ x : |X_n(x)| \geq n \} \). Let

\[ C_n = \{ x : X_n(x) \neq Y_n(x) \} \]

and the first Borel-Cantelli lemma if we can show \[ \sum_{n=1}^{\infty} P(C_n) < \infty \]

then we will know \( P(C) = 0 \).

\[ P(C_n) = \int_{1 \times J} dF(x) = \sum_{j=1}^{\infty} \int_{1 \times J} dF(x) \]

\[ \sum_{n=1}^{\infty} P(C_n) = \sum_{n=1}^{\infty} \sum_{j=n+1}^{\infty} \int_{1 \times J} dF(x) = \sum_{j=1}^{\infty} \left( J_{-1 \leq x \leq 1} \right) \int_{-1 \leq x \leq 1} dF(x) \]

\[ \leq \sum_{j=1}^{\infty} \int_{1 \times J} dF(x) \leq \int_{-\infty}^{\infty} 1 \times 1 dF(x) < \infty \] by hypothesis so
The series converges and so $P(C) = 0$. Let $B = C^c$. Then $P(B) = 1$. \( B = \{ x : \text{at most a finite number of indices} \} \)

Also $P(\bigcap \{ N \}) = 1$ for any $w \in \bigcap \{ N \}$ from (2) and the definition of $B$.

(2) \[ \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} |X_i(w) - E(Y_i(w))| = 0. \] Because $P(C) = 0$, we show now $\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} E(Y_i(w)) = 0$ (4) and that will finish the proof.

\[ E(Y_n) = \int_{\mathbb{R}_{+}} x \, dF(x) = \int_{-\infty}^{\infty} x \, I_{(0,1)}(x) \, dF(x) \]

therefore by the Lebesgue dominated convergence theorem

\[ \lim_{n \to \infty} E(Y_n) = \int_{-\infty}^{\infty} \lim_{n \to \infty} x \, I_{(0,1)}(x) \, dF(x) = \mu \]

\[ \lim_{n \to \infty} E(Y_n) - \mu = 0 \]

so by a little lemma:

\[ \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} [E(Y_i) - \mu]^2 = 0 \]

Exercise: Apply this theorem to the Rademacher functions and you get the famous theorem of Borel (1909).

Consider the interval $[0,1]$ and for each $x \in [0,1]$ in its binary expansion, $x = a_1 a_2 a_3 \ldots$ where $a_j = \begin{cases} 1 & \text{if } S_n \text{ is the excess in the first } n \text{ binary places of plus ones over minus ones we have } \lim \frac{S_n}{n} = 0 \end{cases}$. a.e. in $[0,1]$. Borel in 1909 “almost”
Hausdorff (1913) proved \( S_n = O(n^{1/2 + \varepsilon}) \) \( \forall \varepsilon > 0 \).

Hardy & Littlewood (1914) proved \( S_n = O(\sqrt{n \log n}) \).

Steinhaus (1922) \( \lim_{n \to \infty} \frac{S_n}{\sqrt{2n \log n}} = 1 \).

Khintchine (1923) \( S_n = O(\sqrt{n \log \log n}) \).

Khintchine (1924) \( \lim_{n \to \infty} \frac{S_n}{\sqrt{2n \log \log n}} = 1 \). The law of the iterated logarithm.

How precise is this statement?

If \( \lambda > 1 \) then for almost all \( x \in [0,1] \) \( \exists N \gg n > N \Rightarrow S_n \leq 2\sqrt{2n \log \log n} \)

and if \( \lambda \leq 1 \) the for almost all \( x \in [0,1] \) \( \exists \) infinitely many \( n \gg S_n \geq 2\sqrt{2n \log \log n} \).

Definition: We say \( \phi(n) \) belongs to the upper class if \( S_n \geq \sqrt{n} \phi(n) \) for only finitely many \( n \). We say \( \phi(n) \) belongs to the lower class if \( S_n \geq \sqrt{n} \phi(n) \) for infinitely many \( n \). In this language, Khintchine's law of the iterated logarithm

\[
\lambda \sqrt{2 \log \log t} \in \begin{cases} 
\text{upper class if } \lambda > 1 \\
\text{lower class if } \lambda < 1
\end{cases}
\]

Paul Lévy (1933) \( \sqrt{2 \log \log t} + \alpha \log \log \log t \in \begin{cases} 
\text{upper class if } \alpha \geq 3 \\
\text{lower class if } \alpha < 1
\end{cases}
\)

Kolmogorov & Erdős (1937-1942). If \( \phi(x) \) is non-decreasing

\[ \phi(x) \in \begin{cases} 
\text{upper class} & \Leftrightarrow \int_{t_0}^{t_0 + \phi(x)} e^{-\phi(x)} \, dt \text{ converges} \\
\text{lower class} & \Leftrightarrow \int_{t_0}^{t_0 + \phi(x)} e^{-\phi(x)} \, dt \text{ diverges}
\end{cases} \]

I choose a lower limit, so \( \phi(x) \) isn't too bad.
Brownian Motion: \[ E[F(\beta)] = \int_{\beta} e^{\frac{1}{2} \beta(s)^2} ds \]

This formalism we guessed as

\[ \lim_{\varepsilon \to 0} \log E\left( e^{\varepsilon F(\beta)} \right) = \sup_{\varepsilon \in \mathbb{R}} \left[ F(\varepsilon) - \frac{1}{2} \varepsilon F'(\varepsilon) \right] \]

Example: \( F(\beta) = \int_0^t \beta(s) ds \) by explicit calculation.

\[ E\left( e^{\frac{1}{2} F(\beta)} \right) = E\left( e^{\frac{1}{2} \int_0^t \beta(s) ds} \right) = e^{\frac{t^2}{8}} \]

\[ \lim_{\varepsilon \to 0} \log E\left( e^{\frac{\varepsilon}{2} \int_0^t \beta(s) ds} \right) = \frac{t^2}{8} \quad \text{The check is to calculate} \]

\[ \sup_{\varepsilon \in \mathbb{R}} \left[ \int_0^t \beta(s) ds - \frac{1}{2} \varepsilon \int_0^t \beta^2(s) ds \right] \quad \text{continuous with derivative in } L^2 \text{ and vanishing at the origin.} \]

Frechet differential: Suppose \( F: C_{0,\infty,\mathbb{R}} \to \mathbb{R} \). Let \( \psi \in C_{0,\infty,\mathbb{R}} \)

\[ SF = \frac{d}{dh} F(\beta + h\psi) \bigg|_{h=0} \]

Suppose \( F(\beta) = \int_0^t \beta(s)^2 ds \quad SF = 2 \int_0^t \beta(s) \psi(s) ds \quad \text{the old function} \]

\[ = \int_0^t \beta(s) \psi(s) ds \quad \text{of } \]
In the finite case

\[ SF = \sum \frac{dF}{dx} \cdot x. \]

So here \( \frac{dF}{dx} = 2 \). Let's apply such method.

So differentiating \( \frac{d}{dx} \int_0^t \psi(x) \, dx = 1 \) and

\[ G[w] = -\frac{1}{2} \int_0^t [w(x)]^2 \, dx, \quad G[w + h \psi] = -\frac{1}{2} \int_0^t [w'(x) + h \psi'(x)]^2 \, dx. \]

\[ \delta G = \frac{dG}{dh} \bigg|_{h=0} = -\int_0^t w'(x) \psi'(x) \, dx \]

Integrate by parts:

\[ = -\int_0^t \psi(x) w''(x) \, dx = \int_0^t \psi''(x) w'(x) \, dx \]

with the

natural b.c. \( w'(0) = 0 \) and \( w(0) = 0 \) \( \quad (\psi(0) = 0) \)

so our Euler-Equation is \( 1 + w''(x) = 0 \)

\[ w(0) = 0 \]

\[ w'(0) = 0 \]

\[ w'(x) = -x + t \Rightarrow w(x) = -\frac{x^2}{2} + t \]

but we want the sup.

We do indeed get \( \frac{t^2}{2} \) as was desired. Another example

consider the

\[ \text{Expected value for Brownian motion } E \left( e^{\sup_{0 \leq t \leq 3} \beta(t)} \right) = \int_0^3 e^{t} \, dP \{ \beta(t) \} \]

\[ = \int_0^\infty e^x \sqrt{\frac{2}{\pi t}} e^{-\frac{x^2}{2t}} \, dx = \sqrt{\frac{2}{\pi t}} \int_0^\infty e^{-\frac{u^2}{2t}} \, du \]

\[ \Rightarrow \lim_{t \to 0} E \left( e^{\sup_{0 \leq t \leq 3} \beta(t)} \right) = \frac{1}{2}. \]
Brownian Scaling: \( \beta(\tau) \) is \( N(0, \tau) \)
\( \beta(0) \) is \( N(0, c_0) \)
\( \sqrt{c} \beta(\tau) \) is \( N(0, c\tau) \)

\[ E\{\beta(\tau)\beta(c)\} = \min(\tau, c) \]
\[ E\{\beta(\tau)c\beta(c)\} = c\min(\tau, c) \quad \text{we claim that we can pull the} \]
\[ \text{out so that} \quad E\{\beta(\tau)c\beta(c)\} = E\{\beta(\tau)c\}E\{\beta(c)\} = c\min(\tau, c) \]

Now:
\[ E\{e^{t\sup_{0\leq\tau\leq1}\beta(\tau)}\} = E\{e^{t\sup_{0\leq\tau\leq1}\beta(\tau)}\} = E\{e^{t\sup_{0\leq\tau\leq1}\beta(\tau)}\} \]
\[ = E\{e^{\frac{1}{t}\sup_{0\leq\tau\leq1}\beta(\tau)}\} \quad \text{let} \quad t = \frac{1}{t} \]
\[ = E\{e^{\frac{1}{t}\sup_{0\leq\tau\leq1}\beta(\tau)}\} \quad \text{and so with our knowledge} \]

\[ \lim_{t \to \infty} \log E\{e^{\frac{1}{t}\sup_{0\leq\tau\leq1}\beta(\tau)}\} = \lim_{t \to \infty} \log E\{e^{t\sup_{0\leq\tau\leq1}\beta(\tau)}\} \]

\[ = \sup_{0 \leq \omega(\tau) \leq 1} \left[ \frac{(\omega(\tau) - \frac{1}{2})^2}{2} \right] \text{for} \quad \omega \in C(0, 1) \]
\[ = \max \left[ a - \frac{a^2}{2} \right] = \frac{1}{2} \quad \text{when} \quad a = 1 \quad \text{then} \]

We will eventually prove: Let \( \Omega \) be a separable metric space and let \( \mathcal{B} \) be the Borel sets in \( \Omega \). For any \( \varepsilon > 0 \), let \( P^\varepsilon \) be a probability measure on \( (\Omega, \mathcal{B}) \). Let \( A: \Omega \to [0, \infty] \) satisfying

\[ A(\varepsilon) = \inf_{\omega \in \mathcal{C}} A(\omega) \]

for every closed set \( \mathcal{C} \subset \Omega \) we want

\[ \lim_{\varepsilon \to 0} \varepsilon \log P^\varepsilon(\mathcal{C}) = -\inf_{\omega \in \mathcal{C}} A(\omega) \]
2) for every open set $G \in \mathcal{B}$ we also want

$$\lim_{\varepsilon \to 0} \varepsilon \log P^\varepsilon \{ G \} \geq -\inf_{w \in G} A[w].$$

Then for any $F : \Lambda \to \mathbb{R}$ which is bounded and continuous

$$\lim_{\varepsilon \to 0} \varepsilon \log E \left[ e^{\varepsilon F[\tau_{\varepsilon}]} \right] = \sup_{w \in \Lambda} \left[ F[w] - A[w] \right].$$

Brownian motion scaling gives $E \left[ e^{t F[\tau_{\varepsilon}]} \right].$
Let $Q = [0,1]$ with ±1 binary expansion and

$$S_n = X_1 + X_2 + \cdots + X_n$$

and a random walk

$$S_n$$ is the length of the random walk.

and $P \left[ \lim_{n \to \infty} \frac{S_n}{\sqrt{2n \log \log n}} = 1 \right] = 1$

$P \left[ \lim_{n \to \infty} \frac{S_n}{\sqrt{2n \log \log n}} = -1 \right] = 1$ for a.e. $w$ and $L_so$

$P \left\{ a < \frac{S_n}{n} < b \right\} = \int_a^b \frac{1}{x} \, e^{-u^2} \, du.$

Theorem (L. I. I. Khintchine). Let $X_1, X_2, \ldots$ be i.i.d. Bernoulli 
v.

$P \left[ X_1 = 1 \right] = p$

$P \left[ X_1 = 0 \right] = 1 - p$ and $S_n = \sum_{i=1}^{n} X_i$ then

(i) $P \left[ \lim_{n \to \infty} \frac{S_n - np}{\sqrt{np(1-p)}} = 1 \right] = 1$

(ii) $P \left[ \lim_{n \to \infty} \frac{S_n - np}{\sqrt{np(1-p)}} = -1 \right] = 1$

Recall the Laplace -De Moivre theorem: For fixed $x$

$$\lim_{n \to \infty} P \left\{ \frac{S_n - np}{\sqrt{np(1-p)}} > x \right\} = \frac{1}{\sqrt{2\pi}} \int_x^{\infty} e^{-u^2} \, du.$$

$$\lim_{n \to \infty} P \left\{ \frac{S_n - np}{\sqrt{np(1-p)}} > x \right\} = 1.$$ Does this hold if $x = x(n)$?

If $x_n \to 0$ as $n \to \infty$. This is a fact. Now the proof.

To prove (1) we must show

(3) If $\lambda > 1$ then for a.e. $w \in \mathbb{N}$ and $n > N$ implies
\( S_n \leq n p + 2 \sqrt{2n \log n}. \) And

(4) If \( \lambda < 1 \) then for almost all \( n \) there are infinitely many \( k \) for which \( S_n > n p + 2 \sqrt{2n \log n} \).

Remark: \( \exists a \) constant \( S > 0 \) depending on \( p \) but independent of \( n \).

\[
P\{ S_n > n p \} > \delta \quad \text{for} \quad n \to \infty.
\]

Proof: \[ P \{ S_n > n p \} = \sum_{n = k + 1}^{\infty} \binom{n}{k} p^k (1 - p)^{n-k} > 0 \quad \text{and} \]

\[
\lim_{n \to \infty} P\left( \frac{S_n - n p}{\sqrt{n p q}} > 0 \right) = \frac{1}{2}. \]

\text{Claim:} \quad P\left( \max_{1 \leq k \leq n} (S_k - k p) > x \right) \leq \frac{1}{2} P\{ S_n - n p > x \}

\text{Proof:} \quad P\left( \max_{1 \leq k \leq n} (S_k - k p) > x \right) = P\{ S_n - n p > x \} + \sum_{\gamma = 2}^{n} P\{ S_{\gamma} - \gamma p > x \}

\text{Consider} \quad \{ w : S_1 - p > 0, S_n - S > (n-1)p \}

\text{U} \{ w : S_\gamma - \gamma p > x \} \quad \text{for} \quad 1 \leq \gamma \leq n-1, \quad S_n - S > (n-1)p \}

\text{U} \{ w : S_n - n p > x \} \quad \text{for} \quad S_n - n p > x \}

\text{These are mutually disjoint on the left, because of the taggedged inequalities. Also we can factor the probabilities. So}

\[
P\{ S_1 - p > x \} \leq P\{ S_n - n p > x \} \leq \sum_{\gamma = 2}^{n} P\{ S_{\gamma} - \gamma p > x \} \quad \text{for} \quad 1 \leq \gamma \leq n-1
\]

\[
+ P\{ S_n - n p > x \} \leq P\{ S_n - n p > x \} \}

\text{So} \quad 2 + \max_{1 \leq k \leq n} (S_k - k p) > x \}

\text{O.E.D.
To prove (3), let \( \lambda > 1 \). Choose \( \delta \approx 2 > 8 > 1 \). And let
\[
\nu_r = \left\lceil \delta^{r} \right\rceil, \quad r = 1, 2, 3, \ldots \quad \text{(greatest integer)}.
\]
Let
\[
B_r = \left\{ \omega : \max_{n \leq r} (S_n - n \nu_r) > 2 \sqrt{2n \nu_r \log \log n} \right\}.
\]
We show that \( \sum_{r=1}^{\infty} P(B_r^c) < \infty \) and so by the first Borel-Cantelli lemma, for almost all \( \omega \) only finitely many of the events \( B_r \) can occur. Note that this will imply (3) because of the construction of \( B_r \). This means
\[
P \left( \limsup_{r \to \infty} B_r = 0 \right) \text{ or for almost all } \omega \text{ at most a finite number of the } B_r \text{ occur. So for almost all } \omega \in \mathbb{R} \exists R \in \mathbb{R} \text{ such that }
\]
\[
\max_{n \leq r} (S_n - n \nu_r) \leq 2 \sqrt{2n \nu_r \log \log n} < 2 \sqrt{2n \nu_r \log \log n}.
\]
To see that \( \sum_{r=1}^{\infty} P(B_r^c) < \infty \). From the lemma
\[
P(B_r) \leq \frac{1}{\delta} P \left( \max_{n \leq r} (S_n - n \nu_r) > 2 \sqrt{2n \nu_r \log \log n} \right),
\]
which is
\[
P(B_r^c) \leq \frac{1}{\delta} P \left( \frac{S_n - n \nu_r}{\sqrt{n \nu_r \log \log n}} > 2 \right).
\]
But this last probability is asymptotic to
\[
\frac{1}{\sqrt{\pi}} \int_{x_r}^{+\infty} e^{-u^2} \, du \quad \text{where } x_r = 2 \sqrt{2n \nu_r \log \log n} \\
\sim \frac{1}{\sqrt{x_r \sqrt{2n \nu_r \log \log n}}} e^{-x_r^2/2} \quad \text{as } r \to \infty.
\]
So what
\[
\frac{\nu_r}{n \nu_r} \sim 8 \quad \text{and} \quad \frac{1}{x_r} \sim \frac{1}{\sqrt{n \nu_r \log \log n}}.
\]
\[ P\{B_r\} \leq \frac{1}{6} e^{-\frac{1}{6} \log \log n} = \frac{1}{6} \frac{1}{(\log n)^2} \sim \frac{1}{6} (\log n)^2. \] So by comparison to the geometric series, since \( 1 > 1 \), then \( \sum_{r=1}^{\infty} P\{B_r\} < \infty \), and we have the case (3).

Now to prove (4) we will use the Second Borel-Cantelli Lemma.

Let \( j \leq 1 \), choose \( \lambda > 0 \) and so close to 1 that \( 1 - \lambda < (\frac{2}{3})^{\frac{1}{2}} \).

Then choose \( Y \) an integer \( \geq \frac{2}{\lambda} > \frac{2}{\lambda} \). Let \( n_r = \lambda^r, r = 1, 2, 3, \ldots \)
and let \( A_r = \{ w : (S_{n_r} - S_{n_{r-1}}) - (n_r - n_{r-1})p > \sqrt{2n_r \log \log n_r} \} \). Note that the \( A_r \) are independent. So if we can show that

\[ \sum_{r=1}^{\infty} P(A_r) = \infty \] we have by the second Borel-Cantelli lemma that \( P(\limsup A_r) = 1 \), i.e. for almost all \( w \) there are infinitely many \( r \) such that \( (S_{n_r} - S_{n_{r-1}}) - (n_r - n_{r-1})p > \sqrt{2n_r \log \log n_r} \).

From the first part of the proof with \( j = 2 \) we have that for almost all \( w \), \( \exists N : n > N \Rightarrow \)

\[ S_n > np - 2\sqrt{2np \log \log n} \quad \text{and, in particular there exists \( R \) so that for } r > R \Rightarrow \]

\[ S_{n_r} > n_r p - 2\sqrt{2n_r \log \log n_r} \]

Hence for almost \( w \) infinitely many \( r \) s \( \Rightarrow \) both inequalities hold. Adding them we have for almost all \( w \) infinitely many \( r > S_n - n_r p > \) (This in brackets is to be ignored).
Prob.

\[ i.e. \quad S_{n} - \eta_{n} \cdot p \geq - (q - \lambda) \sqrt{2 p q n_{0} \log \log n_{0}} \quad \text{because} \quad n_{0} > n_{0-1} \]

and

\[ \eta_{n_{0-1}} = \frac{\eta_{n}}{\eta_{n}} < \eta_{n} (1 - q) \quad \text{but} \quad 1 - q < (q - \lambda)^{2} n_{n_{0}} \]

so we may replace \( \eta_{n_{0-1}} \) and get this. Now add \( \times \) and \( \times^{T} \). For almost all \( \omega \) there are infinitely many \( \eta \).

\[ S_{n} - \eta_{n} \cdot p \geq \sqrt{2 p q n_{n} \log \log n_{n}} \quad \text{and so we have (4)} \]

and the theorem since this is a particular set of infinitely many \( \eta \)'s. But we must consider

\[ P(\{\text{Ar}\}) = P(\left\{ S_{n} - S_{n-1} - (n_{n} - n_{n-1}) \cdot p \geq \sqrt{\frac{2 n_{n} \log \log n_{n}}{(n_{n} - n_{n-1}) \cdot p}} \right\}) \]

but this is asymptotic

\[ \sim \frac{1}{\sqrt{2 \pi}} e^{-x_{n}^{2}/2} \quad \text{where} \quad x_{n} = \sqrt{\frac{2 n_{n}}{(n_{n} - n_{n-1}) \log \log n_{n}}} . \]

**Brownian Motion: Theorem:** Let \( \Omega \) be a separable metric space and let \( B \) be the Borel sets of \( \Omega \) for any \( \varepsilon > 0 \).

Let

\[ P^{\varepsilon} \] be a probability measure on \((\Omega, B)\). Let

\[ A : \Omega \rightarrow [0, \infty] \] be any functional satisfying

\[ \text{i. For any closed set } C \subset \Omega \]

\[ \lim_{\varepsilon \to 0} \varepsilon \log P^{\varepsilon}(C) \leq - \inf_{\omega \in C} A[\omega] \]

\[ \text{ii. For any open set } G \subset \Omega \]

\[ \lim_{\varepsilon \to 0} P^{\varepsilon}(C) = \inf_{\omega \in C} A[\omega] . \]
Suppose $V$ were not a given function and solve $u$ as a function of $V$. Then we would get an integral equation in function space.

**Theorem:**

If $V(y) \to \infty$ as $|y| \to \infty$ so that the eigenvalue problem

$$\frac{1}{2} \psi''(y) - V(y) \psi(y) = -\lambda^2 \psi(y) \quad \text{has a discrete spectrum}$$

Then the lowest eigenvalue is given by

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \log \int e^{-\frac{1}{2} \varepsilon^2 \int V(\beta(s)) ds} = -2.$$ 

**Proof:** Let $u(x,t) = E_x \{ e^{-\frac{1}{2} \varepsilon^2 \int V(\beta(s)) ds} \}$. Then by Feynman-Kac

$u(x,t)$ solves

$$u_t = \frac{1}{2} u_{xx} - V(x) u$$

$u(x,0) = 1$

In the other hand, by separating variables

$$u(x,t) = \sum_{j=1}^{\infty} e^{-\frac{1}{2} \lambda_j^2 t} \mathcal{F}_j(x) \mathcal{Y}_j(y) \quad \text{where}$$

$\lambda_1, \lambda_2, \ldots, \lambda_j, \ldots$ are the eigenvalues and normalized eigenfunctions of

$$\frac{1}{2} \psi'' - V \psi = -\lambda \psi.$$

So:

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \log u(x,t) = -2, \quad \text{child's play. But also by}$$

$$\lambda_1 = \inf \left\{ \int_{-\infty}^{\infty} V(y) \, \psi_j(y) dy \pm \frac{1}{2} \int_{-\infty}^{\infty} \left( \psi_j(y) \right)^2 dy \right\} \quad \text{Rayleigh-Ritz}$$
For \( \omega \in \mathbb{R} \), let \( F : \mathbb{R} \to \mathbb{R} \) be bounded and continuous such that

\[
\lim_{\epsilon \to 0} \epsilon \log E \left( e^{-\epsilon F(\omega)} \right) = \sup_{\omega \in \mathbb{R}} \left[ F(\omega) - \log A[\omega] \right]
\]

and of \( \omega \), because \( F \) is continuous and \( C_j^N \) is a closed subset of \( \mathbb{R} \) from \( i \) by

\[
\mathbb{P}^\epsilon \left( C_j^N \right) \leq \inf_{\omega \in C_j^N} A[\omega] \quad \text{Now}
\]

\[
\mathbb{P}^\epsilon \left( C_j^N \right) \leq \frac{N}{N-j} \mathbb{P}^\epsilon \left( C_j^N \right) \quad \text{therefore}
\]

\[
E \left( e^{-\epsilon F(\omega)} \right) \leq \max \left( \frac{N-j}{N} \inf_{\omega \in C_j^N} A[\omega] \right)
\]

\[
= \frac{M}{N} + \max \left( \frac{M}{N} - \inf_{\omega \in C_j^N} A[\omega] \right)
\]

\[
+ \max \left( \sup_{\omega \in C_j^N} A[\omega] \right)
\]

\[
e^{-\epsilon F(\omega)} \right) = \frac{M}{N} + \sup_{\omega \in C_j^N} \left( F(\omega) - A[\omega] \right)
\]

\[
E \left( e^{-\epsilon F(\omega)} \right) \leq \sup_{\omega \in \mathbb{R}} \left[ F(\omega) - A[\omega] \right]. \quad \text{So}
\]
And the A function will be $A(s) = \frac{1}{2} \int_{0}^{\infty} \beta'(s)^2 ds$ if $\beta \in \mathcal{C}_{0}^{k}(0,1)$, $k \geq 1$ are $\beta$'s which are absolutely continuous and have derivatives in $L^2$, i.e., $\int \beta'(s)^2 ds < \infty$ for these $\beta$'s. A subset of $\mathcal{C}_{0}^{k}(0,1)$.

Theorem: Let $\beta(s)$, $0 \leq s \leq \infty$ be B.M. on $(0, \infty)$ with $\beta(0)=0$.

Then $P \left( \lim_{t \to \infty} \frac{\beta(t)}{\sqrt{2t \log \log t}} = 1 \right) = 1$.

$P \left( \lim_{t \to \infty} \frac{\beta(t)}{\sqrt{2t \log \log t}} = -1 \right) = 1$ proved by Paul Levy. But

Then Chung (1948) proved $P \left( \lim_{t \to \infty} \frac{\max_{0 \leq s \leq t} \beta(s)}{\sqrt{2t \log \log t}} = 1 \right) = 1$. 


\[ A_r = \left\{ \frac{(S_{n_r} - S_{n_{r-1}}) - (n_r - n_{r-1}) \frac{1}{\sqrt{2 \log \log n_r}}}{\sqrt{(n_r - n_{r-1}) \log \log n_r}} > 2 \right\} \]

\[ n_r = y^r, \quad y < 1, \quad \frac{y}{\log y} > 2. \]

We need to show that \( \sum_{r=0}^{\infty} P(A_r) \) diverges,

\[ \frac{n_r}{n_r - n_{r-1}} = \frac{y^r}{y^r - 1} < 2 \]

\[ P(A_r) > P\left(\frac{1}{2} \log \log n_r \right) = \frac{1}{2} \log \log n_r + \frac{1}{2} \]

\[ = 2 \log \log n_r \left( \log (\log n_r) \right)^2 = 2 \log \left( (\log y)^2 \right) \left( \log (\log y) \right)^2 \]

which \( \frac{1}{r} \) diverges, and so by the second Borel-Cantelli lemma we prove it.

Back to Brownian motion:

\[ E\left\{ e^{tF(\beta(t))} \right\} \]

\[ \beta \text{ is B.M. on } 0 \leq t \]

\[ \beta(0) = 0 \]

Then \( \lim_{\epsilon \to 0} \epsilon \log E\left\{ e^{\frac{t}{\epsilon} F(\beta(t))} \right\} = \sup_{t \in C_0(0,1)} \int_{x \in C_0(0,1)} A_{[x]} = \frac{1}{2} \int_0^1 \left(x - \epsilon x^2\right)^2 ds \)
Consider \( \lim_{\varepsilon \to 0} \frac{E \left[ G \left[ t e^\varepsilon \beta c \right] \right]}{E \left[ e^{t \varepsilon \beta c} \right]} \)

\[
\int g(\varepsilon \beta \cdot e^{t \varepsilon \beta c}) - \frac{1}{2} \int \beta c^2 \ v \beta = \text{change of variable} \\
= \int e^{t \varepsilon \beta c} - \frac{1}{2} \int e^{t \varepsilon \beta c} \beta c^2 \ ds \\
= \frac{1}{t} \left[ F(x(t)) - \frac{1}{2} \int x(c) \beta c^2 \ ds \right] \\
\lim_{\varepsilon \to 0} \frac{1}{t} \int F(x(t)) - \frac{1}{2} \int x(c) \beta c^2 \ ds \\
\text{as } \varepsilon \to 0 \text{ this goes to the } S \text{ function kind of }
\]

\[ = G \left[ w^* \left( \cdot \right) \right] \text{ where } w^* \text{ maximizes } \sup_{w \in C^1 \left( \mathbb{R}^n \right)} \left[ F(w) - A[w, \varepsilon] \right] \] provided that things are unique. In many applied problems this situation actually occurs. If \( F \) depends on \( \alpha \) then sometimes the \( w^* \left( \cdot \right) \) will vary not nicely on \( x \). Sometimes this is because there is non-uniqueness as a function of \( x \).

Burger's Equation: \( u_t + uu_x = \frac{\varepsilon}{2} u_{xx} \quad -\infty < x < \infty \)

\( u(x,0) = u_0(x) \quad t > 0 \)

and \( \int u_0(x) dx = o(x^2) \) as \( |x| \to \infty \). This mimics fluid dynamics.

Hopf discovered \( u_t = \frac{\varepsilon}{2} V_{xx} \quad V(x,0) = e^{-\frac{1}{2} \int_0^x u_0(y)^2 dy} \) \( \text{or} \)

if \( \ u = -\frac{1}{2} V \) then we transform the equation. So
\[ V(x,t,e) = \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} \frac{y}{e^{y^2/2t}} dy \]

and hence
\[ u(x,t,e) = \int_{-\infty}^{\infty} \frac{y}{t} e^{-(y-x)^2/2t} dy \]

and so
\[ \lim_{|y| \to \infty} \frac{F_y}{y^2} = \frac{1}{2t} = F_y \]

Perhaps there exists more than one point \( y \) at which the minimum is achieved. Hopf showed that if at a point \((x,t)\) \( F \) has a single value \( y \) which minimizes \( F_y \), call this \( y, b \). Then

\[ \lim_{\varepsilon \to 0} u(x,t,e) = (\frac{x-b}{t}) = u_0(b). \]

Consider
\[ u_t + uu_x = \frac{\varepsilon}{2} u_{xx} + V(x), \quad u(x,0) = u_0(x), \quad -\infty < x < \infty, \quad t > 0 \]

and
\[ \int_{-\infty}^{\infty} u_x^2 dx = 0(x^2) \text{ as } x \to \infty. \] Try the Hopf–Cole transformation,

\[ v_{\varepsilon} = \frac{\varepsilon}{2} v_{xx} - \frac{v}{2} V(x) \]

\[ u(x,0) = e^{-\frac{1}{\varepsilon} \int_{-\infty}^{x} u_0(s) ds} \text{ which is lovely.} \]

So write down the Feynman–Kac formula:

\[ V(x,t,e) = E_x \left\{ e^{-\frac{1}{2} \int_{0}^{t} V(x(t)) dt} - \frac{1}{2} \int_{-\infty}^{\infty} u_0(x) dx \right\} \]
\[ \begin{align*}
\text{Prob.} & \quad e^\frac{t}{\beta(\varepsilon) + x} \\
& = E_0 \left\{ e^{-\int_0^t V(\beta s + x) \, ds + \int_0^t u_0(c) \, dz} \right\} \\
& \quad \text{so inverting to get } u(x, t; c) \\
& = E_0 \left\{ e^{-\int_0^t V(\beta s + x) \, ds + \int_0^t u_0(c) \, dz} \right\} \\
\frac{\partial}{\partial t} u(x, t; c) & = E_0 \left\{ e^{-\int_0^t V(\beta s + x) \, ds + \int_0^t u_0(c) \, dz} \right\} \\
F(\beta(t)) & = \int_0^t V(\beta s + x) \, ds + \int_0^t u_0(c) \, dz \\
G(\beta(t)) & = \int_0^t V(\beta s + x) \, ds + \int_0^t u_0(c) \, dz \\
& \quad \text{Then } \\
u(x, t; c) & = E \left\{ \frac{G(\sqrt{\beta(t)})}{E \left\{ e^{t F(\sqrt{\beta(t)})} \right\}} \right\} \\
\lim_{\varepsilon \to 0} u(x, t; c) & = G(w^*) \quad \text{provided } w^* \text{ is the unique function} \\
& \quad \text{which minimizes } \inf_{w^* \in C^0(0, T)} \left[ \int_0^T V(w^*(s) + x) \, ds + u_0(w^*(c) + x) \right] \\
& \quad i.e. \text{at a point } (x, t) \exists \text{ such a unique minimizing function } w^*, \text{ then the limit exists and is} \\
G(w^*(s)) & = \int_0^t V(w^*(s) + x) \, ds + u_0(w^*(c) + x). \\
\text{Now consider the variational problem:} \\
\inf_{w^* \in C^0(0, T)} \left[ \int_0^T V(w^*(s) + x) \, ds + \int_0^t u_0(c) \, dz + \frac{1}{2} \int_0^T \| w^*(s) \|^2 \, ds \right] \\
\right. \]
And consider the Freejet Derivative.

\[ \frac{\partial H}{\partial r} = \frac{dH}{dh} \bigg|_{h=0} = \int_0^t \sqrt{w''(s) + x} \, \psi''(s) \, ds \]

\[ + u_0(w(t) + x) \psi(t) + w(t) \psi(t) - \int_0^t w''(s) \psi''(s) \, ds \]

\[ = \frac{1}{2} \int_0^t [w'(s)]^2 \, ds \Rightarrow \frac{d}{dh} \int_0^t [w'(s) + h \psi(s)]^2 \, ds = \frac{1}{2} \int_0^t [w'(s)]^2 \, ds \text{ integration by parts} \]

\[ = w'(t) \psi(t) - \int_0^t w''(s) \psi''(s) \, ds \]

So we must have 
\[ V'(w(c)+x) = w''(c) \quad 0 \leq t \leq T \]
\[ w(0) = 0 \quad \text{since} \quad w \in C^1(0,T) \]
\[ w'(t) = -u_0(w(t)+x) \quad \text{to get rid of the other stuff.} \]

Now try this out on Hopf's result: Suppose \( V \equiv 0 \). Then

\[ w''(t) = 0 \quad w(0) = 0 \]
\[ w'(t) = -u_0(w(t)+x) \]

So 
\[ w(t) = ct \quad \text{for some constant} \quad c \]

\[ w'(t) = c = -u_0(ct+x) \quad \text{forall} \quad c = \frac{b-x}{t} \]

\[ \frac{b-x}{t} = -u_0(b) \quad \text{or} \quad u_0(b) = \frac{x-b}{t} \quad \text{So is there a unique} \quad b \quad \text{satisfies this and then we get a unique} \quad *c(s). \quad \text{So} \quad *c(s) = \left( \frac{b-x}{t} \right) \quad \text{when there is uniqueness.} \]
Therefore \( \lim_{t \to \infty} \frac{1}{t} \log E \left[ e^{\int_0^t V(y(s)) \, ds} \right] = \)

\[
- \inf_{\psi \in L^2_{\Omega}} \left[ \int_0^\infty V(y) \psi(y) \, dy + \frac{1}{2} \int_0^\infty \| \psi'(y) \|^2 \, dy \right].
\]
Consider \( u_t = \frac{1}{2} u_{xx} - V(x) u \quad u(x,0) = u_0(x), \quad t > 0 \)

\( V(x) \) measurable and bounded below.

The solution is given by \( E_x \left[ e^{-\int_0^t V(\beta(s)) ds} \right] \) where \( \beta(s) \) is \( B.M. \) with \( \beta(0) = x \). That is

\[
\phi_t(x,y) = \int_0^\infty u_0(x) \mathbb{E}_x \left[ e^{-\int_0^t V(\beta(s)) ds} \mathbb{I}(\beta(t)-y) \right] dy.
\]

\( \phi_t(x,y) \) satisfies I \( \phi_t = \frac{1}{2} \phi_{xx} - V(x) \phi \)

\[
\phi_t(x,y) = \mathbb{I}(x-y) \quad \text{and also}
\]

II \( \phi_t = \frac{1}{2} \phi_{yy} - V(y) \phi \)

\[
\phi_t(x,y) = \mathbb{I}(y-x).
\]

We prove II: We will say

\[
\phi_t(x,y) = E_x \left[ e^{-\int_0^t V(\beta(s)) ds} \mathbb{I}(\beta(t)-y) \right].
\]

\[
= \frac{1}{2\pi} \int_{\mathbb{R}} e^{-|y-x|^2} E_x \left[ e^{-\int_0^t V(\beta(s)) ds} \right] dx.
\]

Note: \( e^{-\int_0^t V(\beta(s)) ds} = 1 - \int_0^t V(\beta(s)) e^{-\int_0^s V(\beta(r)) dr} ds \).

So

\[
\phi_t(x,y) = E_x \left[ \mathbb{I}(\beta(t)-y) \right] - E_x \left[ V(\beta(t)) e^{-\int_0^t V(\beta(s)) ds} \right]
\]

\[
= \frac{1}{2\pi} e^{-\frac{|y-x|^2}{2t}} - \frac{1}{2\pi} \int_{\mathbb{R}} e^{-|y|^2/2t} E_x \left[ V(\beta(t)) e^{-\int_0^t V(\beta(s)) ds} \right] dy.
\]

Here with Markov process we use
\[ y(x, t, y) = \sqrt{2\pi} e^{-(y-x)^2/2t} \int_{-\infty}^{\infty} e^{-y^2} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left( \frac{t}{\sqrt{2\pi}} \right)^n \left( \frac{y-x}{\sqrt{2t}} \right)^n \right] e^{-u^2/2} du \]

\[
\int_{-\infty}^{\infty} e^{-y^2} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left( \frac{t}{\sqrt{2\pi}} \right)^n \left( \frac{y-x}{\sqrt{2t}} \right)^n \right] e^{-u^2/2} du = e^{-(y-x)^2/2t}
\]

This is an integral equation for \( y \) and is equivalent. Both sides:

\[
(\frac{3}{2t} - \frac{1}{2}) y(x, t, y) = 0 - V(y) y(x, t, y) \]

with careful analysis!

\( y(x, 0, y) = S(y-x) \). Now we have proved for \( II \) and we will show

\[ y(x, t, y) = y(y, t, x), \quad p(x, t, y) = \int_0^t V(\beta(t+s)) ds \]

\[
= E_y \left\{ e^{-\int_0^t V(\beta(t+s)) ds} S(\beta(t+s)) \right\}
\]

translating to zero

\[ E_y \left\{ e^{-\int_0^t V(\beta(t+s)) ds} S(\beta(t+s)) \right\}
\]

\[ = E_y \left\{ e^{-\int_0^t V(\beta(t+s)) ds} S(\beta(t+s)) \right\} \text{ because } \beta(t+s) \]

\[ = E_y \left\{ e^{-\int_0^t V(\beta(t+s)) ds} S(\beta(t+s)) \right\} \text{ for } \beta(t+s) \rightarrow \beta(t), \beta(t+s)
\]

\[ = E_y \left\{ e^{-\int_0^t V(\beta(t+s)) ds} S(\beta(t+s)) \right\} \text{ if } \beta(t+s)
\]

\[ = E_y \left\{ e^{-\int_0^t V(\beta(t+s)) ds} S(\beta(t+s)) \right\} \text{ this } \beta(t+s)
\]

\[ = E_y \left\{ e^{-\int_0^t V(\beta(t+s)) ds} S(\beta(t+s)) \right\} = y(x, t, x). \]
Le can also write: \( u(x,t) = E_x \left( e^{-\int_{0}^{t} \beta(s) + x - \beta(s) \, ds} \right) \) if \( V(x) = -x^4 \) then \( E_x \left( e^{\int_{0}^{t} \beta(s) \, ds} u_0(\beta(s)) \right) \) doesn't exist.

With Markov process we get: \( \beta = L_p - V(x) \) but need not be elliptic, local or differential operator. (review: \( u_t = \frac{1}{2} u_{xx} \) \( u(x,0,y) = f(x-y) \) \( \frac{1}{\sqrt{4\pi t}} e^{-\frac{(x-y)^2}{4t}} = \rho(\xi, x, y) \)

Suppose for another operator we get a fundamental solution \( \rho(\xi, x, y) \) and we build up the measure with this \( \rho(\xi, x, y) \). You then get a Feynman-Kac like formula and it goes on. \( L \) is the infinitesimal generator of a semi-group.

**Theorem. (LIL for Brownian Motion):** Let \( \beta(s), s \geq 0 \) be B.M., \( \beta_0 = 0 \)

1. \( P \left( \lim_{t \to \infty} \frac{\beta(t)}{\sqrt{2t \log \log t}} = 0 \right) = 1 \) symmetrically.
2. \( \frac{\beta(t)}{\sqrt{2t \log \log t}} = -1 \) \( \text{envelope} \)

A picture

We prove (1), we must show:

a) If \( c > 1 \) then for almost all paths there exists \( T \) \( \forall t > T \) \( \beta(t) \leq c \sqrt{2t \log \log t} \)

b) If \( c \leq 1 \) then for almost all paths there exists \( T \) \( \beta(t) > c \sqrt{2t \log \log t} \)

so ... \( \beta(t) > c \sqrt{2t \log \log t} \) arbitrarily large.
Proof: (a) Let \( c < 1 \), choose \( c' \geq \frac{1}{c} \), \( c'' \) and let \( \xi_n = \xi_n^{c''} \). Let \( \beta_n = \sup \beta_n \). We recall that we have proved
\[
P \left( \sup \beta_n \leq \xi \right) = \left( \int_0^\infty \frac{e^{-u^2}}{\sqrt{2\pi}} du \right)^{\xi^2}
\]
for \( \xi > 0 \).

Let \( \alpha_n = P \left( M_n \geq \sqrt{2\xi_n \log \log \xi_n} \right) \).

\[
P \left( M_n \geq \sqrt{2\xi_n \log \log \xi_n} \right) = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{e^{-u^2}}{\sqrt{u}} \frac{1}{x} e^{-\frac{x^2}{2u}} dx
\]

\[
= \frac{1}{\sqrt{2\pi}} \int_0^\infty \frac{1}{x} e^{-\frac{x^2}{2u}} \frac{1}{x} e^{-\frac{c^2}{2}}
\]

\[
= \frac{1}{\sqrt{2\pi}} \frac{1}{x} e^{-\frac{x^2}{2u}} \left[ (u-1) \log \xi \right]^{\frac{c^2}{2}}
\]

\[
\therefore \alpha_n < \left( \frac{1}{\sqrt{2\pi}} \right) \left( \frac{1}{x} e^{-\frac{x^2}{2u}} \right) \left[ (u-1) \log \xi \right]^{\frac{c^2}{2}} \quad \text{as } x_n \sim A \sqrt{\log n}
\]

\[
\therefore \alpha_n < \left( \frac{1}{\sqrt{2\pi}} \right) \left( \frac{1}{x} e^{-\frac{x^2}{2u}} \right) \left[ (u-1) \log \xi \right]^{\frac{c^2}{2}} \quad \text{but } \frac{c^2}{2} > 1
\]

\[
\therefore \sum_{n=1}^\infty \alpha_n < \infty. \quad \text{By the first Borel-Cantelli lemma, } \exists N \geq n\forall \Rightarrow M_n < c \sqrt{2\xi_n \log \log \xi_n}.
\]

Let \( T = T_n \) then for \( t > T \)

\[
t_n < t_n \text{ with } n > N \text{ so we have that for almost all paths } \exists T > t > T \text{ } P(\xi_n \leq M_n < c \sqrt{2\xi_n \log \log \xi_n})
\]

(b) Let \( c < 1 \) and choose \( c' \) so that \( c < c' \) and \( s \) large that
\[
\left( c' - \frac{c}{\sqrt{3}} \right) > 1 \quad \text{and fix } \xi \text{ and let } \xi_n = \xi_n^s.
\]

We must have \( \xi > 1 \) also. Let \( \beta_n = \beta(\xi_n) - \beta(\xi_{n-1}) \) and let
\[
\gamma_n = P \left( \beta_n \geq \xi_n \sqrt{\frac{c}{\sqrt{3}}} \xi_n \right)
\]
Let \( B_n \) is \( \mathcal{N}(0, t_n - t_{n-1}) = \mathcal{N}(0, e^{\kappa n(t - t)}} = \mathcal{N}(0, \left(\frac{e^{-1}}{\varphi} \right) t_n) \). Where
\[
\frac{c}{\sqrt{2 \log \log t_n}} = \frac{c}{\sqrt{2 \log n \log \varphi}} \sim \frac{c}{\sqrt{2 \log n}}
\]

So
\[
\gamma_n = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{u^2}{2}} du = \left[ x_n \sqrt{2\pi} e^{-\frac{x_n^2}{2}} \right]_{-\infty}^{\infty} \cdot \beta_n \sim \frac{1}{x_n \sqrt{2\pi} (n \log \varphi)} \times e^{-\frac{1}{2}} \cdot \beta_n \sim \frac{c}{\sqrt{\log n \log \varphi}},
\]

Therefore
\[
\gamma_n \sim \frac{1}{x_n \sqrt{2\pi} (n \log \varphi)} \cdot e^{-\frac{1}{2}} \cdot \beta_n \sim \frac{c}{\sqrt{\log n \log \varphi}},
\]

but \( c < 1 \) so the series \( \sum \gamma_n \) diverges by comparison. Since the \( \beta_n \)'s are independent, we have from the second Borel-Cantelli Lemma for almost all paths \( \beta_n = \beta(t_n) - \beta(t_{n-1}) > x_n \sqrt{2(\frac{t_n}{\varphi})} \). Therefore for infinitely many \( n \), From part (a) of this proof with \( c = 2 \) for almost all paths \( \exists N \geq n > N \Rightarrow 1/\beta(t_{n-1}) < 2 \sqrt{2 t_n \log \log t_{n-1}} \).

\[
< 2 \sqrt{\frac{2 t_n \log \log t_n}{\varphi}}
\]

Combining the two, for almost all paths infinitely often (there exist infinitely many \( n \)) such that
\[
\beta(t_n) > \beta(t_{n-1}) - 1/\beta(t_{n-1}) - \frac{c}{\sqrt{2 \log \log t_n}} \times 2 t_n \log \log t_n - 2 \sqrt{\frac{2 t_n \log \log t_n}{\varphi}}
\]

\[
= \left( \frac{c \sqrt{\frac{t_n}{\varphi}} - \frac{c}{\sqrt{\varphi}} \right) 2 t_n \log \log t_n > c \sqrt{2 t_n \log \log t_n}
\]

So we have proved b). What is coming:

1. Action asymptotics proof. (Few weeks)
2. Strassen's version of LIL.

**Review of Action Asymptotics:** \( \mathcal{C}_0, (0, 1) \), \( \beta(s), 0 \leq s < 1 \), \( \beta(0) = 0 \).

Let \( \mathbb{P} \) be B.M. measure on \( \mathcal{C}_0, (0, 1) \), then the generator is \( \frac{1}{2} \partial^2 \). Let \( \mathbb{P}^e \) be B.M. measure on \( \mathcal{C}_0, (0, 1) \) with the generator \( \frac{1}{2} \partial^2 \). This is the same except we multiply all the paths by \( \sqrt{t} \).

\[
\mathbb{P}^e \left( \mathcal{C}_0 \left( C^2 \left( 1 - \frac{3}{2} \right) \right) \right) = \mathbb{P} \left( \mathcal{C}_0 \left( C^2 \left( 1 - \frac{3}{2} \right) \right) \right).
\]
From formal manipulation of flat integral we were led to guess that

$$\lim_{\varepsilon \to 0} E \left[ e^{\frac{1}{\varepsilon} F[\beta]} \right] = \sup_{\nu \in C^1(0,1)} \left[ F[\nu] - \frac{1}{2} \int_0^1 \nu'(s)^2 \, ds \right]$$

where $C^1(0,1)$ is the set of absolutely continuous functions with derivatives in $L^2$.

We even tried out some examples and it worked. Then last semester towards the end we prove a theorem that said indeed:

If $F: C^1(0,1) \to \mathbb{R}$ is bounded and continuous, the

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \log E \left[ e^{\frac{1}{\varepsilon} F[\beta]} \right] = \sup_{\nu \in C^1(0,1)} \left[ F[\nu] - \frac{1}{2} \int_0^1 \nu'(s)^2 \, ds \right]$$

Providing:

1) If $S$ is a closed set in $C^1(0,1)$

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \log P^\varepsilon(\beta \in S) \leq - \inf_{\mu \in S} A[\mu]$$

$$P^\varepsilon(\beta \in \{ c \}) \approx e^{-\frac{1}{\varepsilon} A[\mu]}$$

2) If $G$ is an open set in $C^1(0,1)$

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \log P^\varepsilon(\beta \in G) \geq - \inf_{\mu \in G} A[\mu]$$

$$P^\varepsilon(\beta \in \{ c \}) \approx e^{-\frac{1}{\varepsilon} A[\mu]}$$
Theorem: If \( C \subseteq C_0(0,1) \) is closed then

\[
\lim_{n \to \infty} \log P^\varepsilon(C) = -\inf_{n \in C} A[\varepsilon]
\]

and if \( G \subseteq C_0(0,1) \) is open then

\[
\lim_{n \to \infty} \log P^\varepsilon(G) = -\inf_{n \in G} A[\varepsilon]
\]

where \( A: C_0(0,1) \to \mathbb{R}_+ \)

\[
A[\beta] = \left\{ \begin{array}{ll}
\frac{1}{2} \int \frac{(\beta'(s))^2}{s} \, ds & \beta \in C_0^*(0,1) \\
0 & \text{otherwise}
\end{array} \right.
\]

To prove the theorem we need some lemmas:

Let \( \beta \in C_0(0,1) \) and for \( s > 0 \) let \( S = S(\beta, S) \)

\[
= \{ t \in C_0(0,1) \mid \|t - \beta\| < S \}
\]

then 

\[
\lim_{n \to \infty} \log P^\varepsilon(S(\beta, S)) = -A[\beta].
\]

Proof: If \( \beta \in C_0^*(0,1) \) then \( A[\beta] = 0 \) and there is nothing to prove. Assume \( \beta \in C_0(0,1) \). Now

\[
P^\varepsilon(S(\beta, S)) = P(0, S(\beta, S))
\]

So

\[
P^\varepsilon \left( \frac{\beta}{\varepsilon}, \frac{S}{\varepsilon} \right) = \left\{ \begin{array}{ll}
P^\varepsilon \left( \frac{\beta}{\varepsilon} \right) & \beta \in C_0^*(0,1) \\
0 & \text{otherwise}
\end{array} \right.
\]

Cameron-Martin Translation Formula:

\[
E^\varepsilon \{ G[\beta + \psi] \} \quad \psi \in C_0(0,1)
\]

\[
= \int G[\beta + \psi] e^{-\frac{1}{2} \int(\psi'(s))^2 \, ds} \, d\beta
\]

\[
= \int G[\psi] e^{-\frac{1}{2} \int(\psi'(s))^2 \, ds} \, d\beta = e^{-\frac{1}{2} \int(\psi'(s))^2 \, ds} \int G(\psi) e^{-\frac{1}{2} \int(\psi'(s))^2 \, ds} \, d\beta
\]

So we guess

\[
E^\varepsilon \{ G[\beta + \psi] \} = e^{-\frac{1}{2} \int(\psi'(s))^2 \, ds} E^\varepsilon \{ G[\beta] e^{-\frac{1}{2} \int(\psi'(s))^2 \, ds} \} \]

So, also we get

\[
\int G(\psi) \, d\beta = \psi'(s) e B[V].
\]

We could also consider a stochastic integral to give meaning to it. We will not prove it however.
so \[ e^{-A[\beta]} E^P \left\{ X(\beta) e^{-\frac{1}{2} \int_0^s \rho_0(s') \, ds'} \right\} \] and another trick

Jensen's Inequality: \[ E \{ X^2 \} \geq E[E X^2] \]
\[ E g(X) \geq g(E X) \quad \text{if } g(x) = x^2 \]

\( g(x) \) is a convex function for any convex function.

From Jensen's Inequality:

\[ E \left\{ X(\beta) e^{-\frac{1}{2} \int_0^s \rho_0(s') \, ds'} \right\} \leq e^{-A[\beta]} P \left\{ S(0, \frac{s}{4}) \right\} \]

we have both Bond \( B \) in our expectation is zero.

so \[ e^{-A[\beta]} E \left\{ X(\beta) e^{-\frac{1}{2} \int_0^s \rho_0(s') \, ds'} \right\} \]

\[ = e^{-A[\beta]} P \left\{ S(0, \frac{s}{4}) \right\} \]

\[ e \log P \left\{ S(\beta, s) \right\} \geq -A[\beta] + e \log P \left\{ S(0, \frac{s}{4}) \right\} \]

therefore

\[ \lim_{\epsilon \to 0} e \log P \left\{ S(\beta, s) \right\} \geq -A[\beta] \]

1) to prove 2) let \( G \) be an open set in \( C_0(0,1) \) and \( \beta_0 \in G \) then
\[ \epsilon \log P^G (\epsilon) \geq e \log P^G (S(\beta_0, s)) \]

from lemma just proved

\[ \lim_{\epsilon \to 0} \epsilon \log P^G (\epsilon) = -A[\beta_0] \]

since \( \beta_0 \) is any element of \( G \), we see.
Lemma 2

Let \( \lim_{\varepsilon \to 0} P^\varepsilon (C) \geq -\inf_{\beta \in \mathcal{C}} A[\beta]. \) Now we prove 1). Let \( C(0,0,1) \) be closed and \( S \geq 0. \) Let \( \mathcal{C} = \bigvee_{\beta \in \mathcal{C}} S(\beta, S) \) the \( S \)-savage of \( \mathcal{C} \) then

4) \( \lim_{\varepsilon \to 0} \log P^\varepsilon (C) \leq -\inf_{\beta \in \mathcal{C}} A[\beta]. \) This is interesting.

Let \( m \) be a positive integer. For each \( \beta \in (0,1) \) let \( \beta_n \) be the polygonal function connecting \( (0,0), (\frac{1}{n}, \beta(\frac{1}{n})), (1, \beta(1)) \) clearly \( 3 \beta_n \in C(0,0,1). \) For every \( n \) and any \( S \geq 0 \)

5) \( P^\varepsilon (C) = P^\varepsilon (\bigvee_{\beta \in \mathcal{C}} : ||\beta - \beta_n|| \geq S \beta_n) + P^\varepsilon (\bigvee_{\beta \in \mathcal{C}} : ||\beta - \beta_n|| < S \beta_n) \)

We now estimate each of the probabilities on the right-hand side of 5). Let \( l \varepsilon = \inf_{\beta \in \mathcal{C}} A[\beta] \) then

\[
P^\varepsilon \left( \bigvee_{\beta \in \mathcal{C}} : \beta_n \in C \right) \leq P^\varepsilon \left( \bigvee_{\beta \in \mathcal{C}} : A[\beta_n] \geq l \varepsilon \right)
\]

But \( A[\beta_n] = \frac{1}{2} \int_{\mathcal{C}} (\beta_n - \beta) ds = \frac{1}{2} \sum_{n} \left( \beta(\frac{1}{n}) - \beta \left( \frac{1}{n} \right) \right)^2. \) Therefore

\[
P^\varepsilon \left( \bigvee_{\beta \in \mathcal{C}} : A[\beta_n] \geq l \varepsilon \right) = P^\varepsilon \left( \bigvee_{\beta \in \mathcal{C}} : \frac{1}{2} \sum_{n} \left( \beta(\frac{1}{n}) - \beta \left( \frac{1}{n} \right) \right)^2 \geq \frac{l \varepsilon}{2} \right)
\]

But \( (\beta(\frac{1}{n}) - \beta(\frac{1}{n})) \) is a stochastic \( (Y_1, Y_2, ..., Y_n) \) are independent \( N(0,1). \)
So, \( Y_1^2 + Y_2^2 + \cdots + Y_n^2 \) is \( \chi^2(n) \). So
\[
P\{p \in (0,1) : A[p] > t_d\} = \frac{1}{\Gamma\left(\frac{n}{2}\right)} \int_0^{t_d} x^{\frac{n}{2} - 1} e^{-\frac{x}{2}} \, dx.
\]

Therefore, \( \frac{1}{\Gamma\left(\frac{n}{2}\right)} \int_0^{t_d} e^{-\frac{x}{2}} \, dx \leq \frac{1}{\Gamma\left(\frac{n}{2}\right)} \int_0^{t_d} x^{\frac{n}{2} - 1} e^{-\frac{x}{2}} \, dx \). \( \frac{1}{\Gamma\left(\frac{n}{2}\right)} \int_0^{t_d} e^{-\frac{x}{2}} \, dx \) we use the following:
\[
\frac{1}{\Gamma\left(\frac{n}{2}\right)} \int_0^{t_d} e^{-\frac{x}{2}} \, dx \to 1 \quad \text{as} \quad t_d \to \infty.
\]

Therefore, \( \frac{1}{\Gamma\left(\frac{n}{2}\right)} \int_0^{t_d} e^{-\frac{x}{2}} \, dx \leq \frac{1}{\Gamma\left(\frac{n}{2}\right)} \int_0^{t_d} x^{\frac{n}{2} - 1} e^{-\frac{x}{2}} \, dx \) \( \frac{1}{\Gamma\left(\frac{n}{2}\right)} \int_0^{t_d} e^{-\frac{x}{2}} \, dx \) we use the following:
\[
\frac{1}{\Gamma\left(\frac{n}{2}\right)} \int_0^{t_d} e^{-\frac{x}{2}} \, dx \to 1 \quad \text{as} \quad t_d \to \infty.
\]

\[
\log P\{c\} \leq \max (-t_d, -n \frac{\delta^2}{2}) \quad \text{for any} \quad n \quad \text{this holds,}\]

\[
\text{lemma:} \quad A \quad \text{is lower semi continuous on} \quad (0,1).
\]
Proof: Let \( \beta_n \) be a sequence in \( \mathcal{C}(0,1) \) which converges to \( \beta \in \mathcal{C}(0,1) \). Suppose \( A[\beta_n] \leq \epsilon \rightarrow \infty \) as \( n \rightarrow \infty \), we must show \( A[\beta] \leq \epsilon \). This implies lower semi-continuity. It must be that \( \beta \in \mathcal{C}^1(0,1) \) since \( A[\beta_n] \leq \epsilon \) for each \( n \), we show now that \( \beta \), the uniform limit of \( \beta_n \) is absolutely continuous.

Let \( [x_i', x_i''] \), be disjoint intervals on \([0,1]\). Then

\[
\sum_{i=1}^{m} |\beta_n(x_i'') - \beta_n(x_i')| = \sum_{i=1}^{m} \int_{x_i'}^{x_i''} \beta_n'(s) \, ds
\]

Schwarz inequality \( \leq \sum_{i=1}^{m} \sqrt{\int_{x_i'}^{x_i''} [\beta_n'(s)]^2 \, ds} \sqrt{\int_{x_i'}^{x_i''} (x_i'' - x_i')}
\]

the whole integral \( \leq \sqrt{\sum_{i=1}^{m} \int_{x_i'}^{x_i''} [\beta_n'(s)]^2 \, ds} \sqrt{\sum_{i=1}^{m} (x_i'' - x_i')}
\]

\( \leq \sqrt{2 \epsilon} \sqrt{\sum_{i=1}^{m} (x_i'' - x_i')} \). Let \( n \rightarrow \infty \) we see

\[
\sum_{i=1}^{m} |\beta(x_i'') - \beta(x_i')| \leq \sqrt{2 \epsilon} \sqrt{\sum_{i=1}^{m} (x_i'' - x_i')}
\]

Therefore \( \beta \) is absolutely continuous and \( \beta' \) exists a.e. on \([0,1]\). Let \( h > 0 \) and for each \( s \in [0,1-h] \) we have

\[
|\frac{\beta(s+h) - \beta(s)}{h} - \beta'(s)| = \frac{1}{h} \int_{s}^{s+h} \beta_n'(x) \, dx \leq \frac{1}{h} \int_{s}^{s+h} [\beta_n'(x)]^2 \, dx
\]

\( \leq \frac{1}{h} \int_{s}^{s+h} \sum_{i=1}^{m} [\beta_n'(x)]^2 \, dx \leq Z \epsilon \) true for each \( h > 0 \), we take \( n \rightarrow \infty \) and uniform convergence implies for each \( h > 0 \),
\[ \int f = \lim_{E \to \infty} \int f \]

\[ \int \left( \frac{1}{|s-t|} \right)^2 ds = 2\pi \text{ now we use Fatou's lemma} \]

\[ \int |\beta '(s)| ds = 2\pi \text{ taking } \lim_{n \to \infty} \text{ inside} \]

Using the lower semi-continuity of \( A \), the Arzela-Ascoli theorem and the fact that \( C \) is closed one can show

\[ \lim_{s \to 0} \inf A \omega = \inf \left( \lim_{s \to 0} \omega \right) \text{ so we can prove the part 1) and we have the action asymptotics.} \]

Recall some classical analysis:

\[ \Delta v = \lambda v \quad \text{in } \mathbb{R}^2 \quad \lambda \Delta u + \lambda u = 0 \]

\[ u = 0 \text{ on } \partial \Omega \]

We can prove that there is a discrete set of eigenvalues \( \lambda_1, \lambda_2, \lambda_3, \ldots \)

(normalized) complete set of eigenvectors \( u_1, u_2, u_3, \ldots \)

\[ \sum_1 = \# \text{ eigenvalues } \leq \lambda \text{ When } \lambda \to \infty \text{ Weyl proved } \]

\[ \frac{d}{d\xi} \sum_1 \sim \frac{1}{2\pi} \lambda \]

\[ \lambda \sim \frac{1}{2\pi} \lambda \]

\[ \lambda \sim \frac{1}{2\pi} \lambda \]

\[ \lambda \sim \frac{1}{2\pi} \lambda \]

Another famous result is

\[ \sum_{j=1}^\infty \hat{u}_j (x,y) \sim \frac{1}{2\pi} \lambda \text{ as } \lambda \to \infty \text{ for } (x,y) \in \Omega \]

Sobolev's Theorem: \[ \int_\Omega e^{-it}\Delta u \left( \frac{x}{\lambda} \right) = f(t) \text{ where } \]

\[ f \text{ is a } C^0 \text{ function on } [0, \infty) \text{ and the integral exists for } t > 0. \]

If we know how \( f(t) \) behaves for \( t \to 0 \), we know how \( u(x, \lambda) \) behaves as \( \lambda \to \infty \).
Theorem: (Karamata) : If for some non-negative $y$, suppose

$$ y f(x) \sim \frac{e^x}{x} \quad \text{as} \quad t \to 0^+, \text{ or as } t \to \infty. $$

Then $\alpha(x) \sim \xi^y (y+1)$ as $x \to 0^+$ or as $x \to \infty$.

Try it out on a trivial case: Let $n$ be a positive integer,

$$ \alpha(x) = \int x^n \, dx = \frac{x^{n+1}}{n+1}. $$

Then

$$ f(x) = \int_0^x e^{-\lambda t} t^n \, dt \quad \text{at} \quad x = u. $$

$$ = \frac{1}{n+1} \int_0^u e^{-u} u^n \, du = \frac{\Gamma(n+1)}{x^{n+1}}. \text{ Clearly } f(x) = \frac{\Gamma(n+1)}{x^{n+1}}, \text{ as } x \to 0. $$

Then $\gamma = n+1 \quad \xi = \Gamma(n+1)$ so $\frac{\Gamma(n)}{\Gamma(n+2)} = \frac{\lambda^{n+1}}{n+1}. \text{ This exact. Whooper dooper we have it.}$

(AC) Can you hear the shape of a drum?
Recall the Tauberian theorem (Karamata): Widder Laplace transform

\[ F(s) = \int_0^\infty e^{-st} f(t) \, dt \quad \text{exists for } t > 0 \]

if \( f(0+) \geq \frac{A}{t^p} \quad \text{for } t > 0^+ \) and \( f(t) \) is non-decreasing for \( t \to \infty \)

Then \( f(t) \sim \frac{A t^{p-1}}{p(1+p)} \quad \text{as } t \to \infty \) (in contrast with Abelian theorem treats the inverse)

In \( \mathbb{R}^2 \) we have \( \nabla \cdot \nabla u + \lambda u = 0 \)

\[ u = 0 \text{ on } \Gamma \]

With discrete eigenvalues \( \lambda_1, \lambda_2, \ldots \)

and normalized eigenfunctions \( u_1(x, y), u_2(x, y), \ldots \)

as \( \lambda \to \infty \):

\[ \sum \frac{\lambda}{2\pi} \frac{1}{\lambda^{3/2}} \quad \text{Weyl} \]

and Carleman proved:

\[ \sum \frac{\lambda u^2(x, y)}{2\pi} \sim \frac{1}{\lambda} \quad \text{as } \lambda \to \infty. \]

Where does this come from: Vibrating membrane

\[ F(x, y, t) \text{ satisfies } \frac{\partial^2 F}{\partial x^2} = \frac{1}{\lambda} \Delta F \quad \text{in appropriate domain we seek } F(x, y, t) = u(x, y) e^{-i\omega t}, \text{ then } u \text{ satisfies} \]

\[ \frac{1}{\lambda} \Delta u + u = 0 \quad \text{in } \Omega \quad \text{with } u = 0 \quad \text{on } \partial \Omega. \]

Let \( C^{(1,1)} \) be smooth interior and set \( \Gamma(x, y) \) and going hopefully to \( \infty \) and set
\[ p^*(x_0, y_0, x, y, t) = \text{the probability that a B.M. path in } \mathbb{R}^2 \text{ starting from } (x_0, y_0) \text{ to } (x, y) \text{ in time } t \text{ without touching } \mathbb{R}^2 \]

\[ p(x_0, y_0, x, y, t) = \quad \text{(Free Brownian motion)} \]

\[ p(x_0, y_0, x, y, t) = \frac{1}{2\pi t} e^{-\frac{(x-x_0)^2 + (y-y_0)^2}{2t}} \]

But our intuition says that \( p^* \) satisfies \( \frac{\partial p^*}{\partial t} = \frac{1}{2} \Delta p^* \), \( p^*(x_0, y_0, x, y, t) \) must be singular as \( t \to 0 \). Or

\[ \lim_{t \to 0} \int \int f(x, y) p^*(x_0, y_0, x, y, t) \, dx \, dy = f(x_0, y_0) \]

and \( p^*(x_0, y_0, x, y, t) = 0 \) if \( (x, y) \in \mathbb{R}^2 \). We can assume a separation argument

\[ p^*(x_0, y_0, x, y, t) = \sum_{\alpha \in \mathbb{R}} e^{-\lambda_t} u_\alpha(x, y) u_\alpha(x_0, y_0). \]

This obviously satisfies the above. Now consider

\[ p(x_0, y_0, x_0, y, t) = \frac{1}{2\pi t} \text{ returning in time } t. \]

\[ p^*(x_0, y_0, x_0, y, t) = \sum_{\alpha = 1}^{\infty} e^{-\lambda_t} u_\alpha^2(x_0, y_0) \]

Obviously for short time

\[ \sum_{\alpha = 1}^{\infty} e^{-\lambda_t} u_\alpha^2(x_0, y_0) \sim \frac{1}{t} \text{ as } t \to 0. \]

Let \( \alpha(x) = \sum_{\alpha} u_\alpha^2(x, y_0) \), then

\[ \int e^{-\frac{\lambda t}{2}} \, d\alpha(x) \sim \frac{1}{t} \quad \text{as } t \to 0. \]

So by Kawasaki's theorem, \( \delta = 1 \), \( A = \frac{1}{4\pi} \), we get
\( \alpha(t) \sim \frac{1}{2\pi} t \) as \( t \to \infty \). Now integrate \( \otimes \) over \( \Omega \)

we get

\[
\sum_{j=1}^{\infty} e^{-\frac{2}{j} t} \sim \frac{1}{2\pi t} e^{-\frac{1}{t}} \quad \text{as} \quad t \to 0^+ \\
\]

\[
\alpha(t) = \sum_{\lambda \leq \lambda} \frac{1}{\lambda} \\
\]

So again by the Tauberian theorem \( \alpha(t) \sim \frac{1}{2\pi} t \) as \( t \to \infty \).

We can get new results if we go on. In short time you don't see the boundary, but if you do you see only a piece of straight line.

Using the next order of approximation in this way you get

\[
\sum_{j=1}^{\infty} e^{-\frac{2}{j} t} \sim \frac{1}{2\pi t} - \frac{1}{4 \sqrt{\pi t}} \\
\]

Also, I. Singer \& H. McKean later. On a manifold we have a Laplace-Beltrami operator and with it a diffusion. One can explore the diffusion etc. and get geometrical data.

We proved \( \Pr\{ \lim_{t \to \infty} \frac{\beta(s)}{\sqrt{2t \log t}} = 1 \} = 1 \) by Khintchine and we can formulate: For each Brownian motion path \( \beta(0) \in \mathbb{R}^n \)

\( \beta(0) = \alpha \), define \( \beta_t(s) = \beta_{t-} \) for \( s \in [0,1] \), so

\[
\beta_t(s) = \frac{\beta(s) - \beta(s-)}{\sqrt{t \log t}} \\
\]

Consider the family \( \{ \beta_t(s) \mid 0 \leq t \leq 1 \} \). Theorem (Stress): For almost all \( \beta(s) \), the family \( \beta_t \) \( \mathbb{R}^3 \) is compact and

\[
1 = \mathbb{E}\{ \chi_0(s) \cdot e^{\xi_0(0,1)} > \frac{1}{2} \sqrt{K(s)} \} ds \\
\]

If this were to be proved we have a corollary,
For each \( \beta(x) \in (0,1) \), consider the family \( \{ B_n(x) \}_{n \geq 3} \) on \((0,1)\). For almost all \( \beta(x) \), the family \( \{ B_n(x) \}_{n \geq 3} \) is compact and the set \( K \) of limit points is \( K = \{ x \in (0,1) : A(x) \neq \emptyset \} \).
Consider the sequence \( \{ \beta_{n_j}(\cdot) \} \), for each \( d > 0 \) let \( K_d \) be the sausage of \( K \), i.e., \( K_d = \bigcup_{x \in K} S(x, d) \). Consider
\[
P\left\{ \beta_{n_j}(\cdot) \in \tilde{K}_d \right\} = P\left\{ \frac{\beta_{n_j}(\cdot)}{\sqrt{\log \log n_j}} \in \tilde{K}_d \right\}
\]
\[
= P\left\{ \frac{\beta_{n_j}(\cdot)}{\sqrt{\log \log n_j}} \in \tilde{K}_d \right\} = P\left\{ \\sqrt{\log \log n_j} = \frac{1}{c} \right\}
\]
\[
P\left\{ \exists \epsilon \beta(\cdot) \in \tilde{K}_d \right\}, \epsilon = \frac{1}{c} \log 2 = P\left\{ \epsilon \tilde{K}_d \right\}, \tilde{K}_d \text{ is closed}
\]
and by action asymptotics \( \lim_{c \to 0} \log P\left\{ \epsilon \tilde{K}_d \right\} \leq -\inf_{w \in \tilde{K}_d} A[w] \)

Therefore for an \( r > 0 \) \( J > 0 \) small enough, i.e., \( \epsilon \geq 1 \) large enough,
\[
P\left\{ \beta_{n_j}(\cdot) \in \tilde{K}_d \right\} = P\left\{ \tilde{K}_d \right\} \leq e^{-\epsilon \left( \inf_{w \in \tilde{K}_d} A[w] - r \right)}
\]

Let \( Y = \inf_{w \in \tilde{K}_d} A[w] \). By def. of \( \tilde{K}_d \) we see \( Y > 1 \). Now

choose \( r > 0 \) so small that \( \epsilon = Y - r > 1 \). Thus for sufficiently large \( j \)
\[
P\left\{ \beta_{n_j}(\cdot) \in \tilde{K}_d \right\} \leq e^{-Y \log \log n_j}
\]

\[
1 - e^{-Y \log \log n_j} \approx \frac{1}{n_j^{\frac{1}{Y}}} - \frac{1}{n_j^{Y}}
\]
since \( Y > 1 \). This is the general term of a convergent series, therefore
\[
\sum_{j=1}^{\infty} P\left\{ \beta_{n_j}(\cdot) \in \tilde{K}_d \right\} < \infty, \ \text{since} \ \epsilon > 1.
\]
Therefore by the first Donsker-Contelli lemma, for almost all \( \beta \in \mathcal{F} \) \( \forall j > J \Rightarrow \beta_{n_j}(\cdot) \in \tilde{K}_d \). This conclusion holds for any sequence \( n_j = [q_j] \) where \( q > 1 \).
For any integer \( n \) let \( N(n) \) be the first \( n \) \( \tau \)-free. Now \( \beta(n) \)

\[
\frac{\beta(n)}{\sqrt{n \log \log n}} \quad \beta_N(n) = \frac{\beta(n)}{\sqrt{N \log \log n}}
\]

so we conclude

\[
\beta_N(n) = \beta(n) \left( \frac{n \log \log n}{N \log \log N} \right)^{1/2}
\]

Now choose \( g \) so close to one that \( \frac{n}{N} \) is very close to one. Therefore for almost all \( \beta \) for some \( n \) large enough \( \beta(n) \in K \), therefore the limit points of \( \{ \beta(n) \} \) are at most the set \( K \).
Probability

Consider \( \beta \in (0,1) \) and \( \beta_c(3) = \frac{\beta(3)}{\sqrt{\log \log e}} \leq 3.3 \) 0.5 \leq 1

For each \( \beta \in (0,1) \) we have \( E \beta c(3) \leq 3.3 \) (we proved that every sequence \( \beta \in (1) \) has a convergent subsequence and the set of limit points is at most \( K = \{ x : \mathcal{E} (\gamma x) : A(x) = \frac{1}{x} \mathcal{E} (\gamma x) \}_{x \in (1)} \)

Consider the classical L.I.L. : \( P \lim_{x \to \infty} \frac{\beta_c(x)}{\sqrt{x \log x}} = 1^2 = 1 \). There are other iterated logarithm results. What about large deviation in the small direction? We get an action-esque quantity. What is it?

Recall: \( E \left[ e^{-\frac{1}{2} \beta_c(3)^2} \right] = \sqrt{\pi e} \), so therefore

\[ \lim_{x \to \infty} \frac{1}{\log E \left[ e^{-\frac{1}{2} \beta_c(3)^2} \right]} = -\frac{1}{2} \] which was explained by (1944)

Locally if \( V(0) = 0 \) and \( V(x) \to \infty \) as \( x \to \infty \) then

\[ \frac{1}{2} \log (x) - V(x) \to -\frac{1}{2} \] has a discrete spectrum

eigenvalues \( \lambda_1, \lambda_2, \ldots \)

Thm: \( \lim_{x \to \infty} \frac{1}{\log E \left[ e^{-\frac{1}{2} \beta_c(3)^2} \right]} = -1 \)

(1949)

Proof: By Feynman-Kac

\[ u(x, t) = E_x \left[ e^{-\frac{1}{2} \beta_c(t)^2} \right] \] satisifies

\[ u_t = \frac{1}{2} u_{xx} - V(x), \quad u(x, 0) = 1. \]

On the other hand by separation of variables \( u(x, t) = \sum_{j=1}^{\infty} e^{-\lambda_j^2 t} y_j(x) \)

But they must be equal \( E_x \left[ e^{-\frac{1}{2} \beta_c(t)^2} \right] = \sum_{j=1}^{\infty} e^{-\lambda_j^2 t} y_j(x) \) so

\[ \lim_{t \to \infty} \frac{1}{\log E_x \left[ e^{-\frac{1}{2} \beta_c(t)^2} \right]} = -1 \] since it controls it.
Prob.

$$E \left\{ e^{-\int_{0}^{t} V(\beta(s)) ds} \right\} \quad \text{as } t \to \infty,$$

the paths that stay around zero contribute the most when \( V(x) > 0 \) to the asymptotic behavior of this integral as \( t \to \infty \). This is because \( V(x) \to \infty \) as \( |x| \to \infty \). Other problems in mathematical physics were

$$E \left\{ e^{-\int_{0}^{t} \frac{1}{2} \dot{\psi}^2(\psi \psi_{xx}, \psi_{xx}) ds} \right\}, \quad \text{positive definite in } \int_{1}^{2} \quad \text{as } t \to \infty.$$

What else do we know about \( \lambda_1 \)?

$$\lambda_1 = \inf_{\psi \neq 0} \left\{ \int_{-\infty}^{\infty} V(\psi) \psi^2(y) dy + \frac{1}{2} \int_{-\infty}^{\infty} [\dot{\psi}^2(\psi \psi_{xx}, \psi_{xx})] dy \right\}. \quad \text{We use the}

\text{Euler equation,}

which is

$$\frac{1}{2} \ddot{\psi}(x) - V(x) \psi(x) = -2 \psi(x).$$

Since we know this stuff we kind of know:

$$\lim_{t \to \infty} \frac{1}{t} \log E \left\{ e^{-\int_{0}^{t} V(\beta(s)) ds} \right\} = -\inf_{\psi \neq 0} \left\{ \int_{-\infty}^{\infty} V(\psi) \psi^2(y) dy + \frac{1}{2} \int_{-\infty}^{\infty} [\dot{\psi}^2(\psi \psi_{xx}, \psi_{xx})] dy \right\}, \quad \text{for } \psi \in L^2, \int_{-\infty}^{\infty} \psi(0) = 1,$$

and we want to prove it without differential equations. From flat integrals

$$E \left\{ e^{-\int_{0}^{t} V(\beta(s)) ds} \right\} = \int_{-\infty}^{\infty} e^{-\frac{1}{2} \int_{0}^{t} V(\beta(s)) ds} \cdot e^{-\frac{1}{2} \int_{0}^{t} [\dot{\beta}^2(s)] ds} = \int_{-\infty}^{\infty} e^{-\frac{1}{2} \int_{0}^{t} \left\{ V(\beta(s)) + [\dot{\beta}^2(s)] \right\} ds} = \int_{-\infty}^{\infty} e^{-\int_{0}^{t} \frac{1}{2} \left\{ \beta^2(s) + \dot{\beta}^2(s) \right\} ds} \left( \int_{0}^{t} e^{-\frac{1}{2} \int_{0}^{s} \dot{\beta}^2(s) ds} \right) \dot{\beta}.$$

This is not the way. We must be deeper.

**Local Time of Brownian Motion:** \( \beta(0) = x \), B.M. For \( t > 0 \) consider

proportion of time = \( \frac{1}{t} \int_{0}^{t} \chi_A(\beta(s)) ds \)

\( A \subset \mathbb{R} \), \( \chi_A(x) \) is

a 1-to-1 function of \( A \).

$$L_t(\beta(s), A) = \frac{1}{t} \int_{0}^{t} \chi_A(\beta(s)) ds, \quad \text{paths starting at } x$$
Notice that for fixed $t > 0$, fixed $x \in \mathbb{R}$ and particular path $\beta(\cdot)$,

\[ L_t(\beta(\cdot), \cdot) \] is a countably additive, non-negative \textit{g} function and \[ L_t(\beta(\cdot), \mathbb{R}) = 1. \] So $L_t(\beta(\cdot), \cdot)$ for fixed $x \in \mathbb{R}$, $t > 0$ maps $C_{x+0,00}$ into $M$ the space of probability measures on the real line. Now as a set function, i.e., for fixed $x \in \mathbb{R}$ and $t > 0$ or almost all $\beta(\cdot)$, $L_t(\beta(\cdot), \cdot)$ has a density function, call it $L_t(\cdot)$. We can represent

\[ L_t(\beta(\cdot), y) = \int_{-\infty}^{\infty} \delta(\beta(s)-y) \, ds \quad \text{notice} \]

\[ L_t(\beta(\cdot), A) = \int_{-\infty}^{\infty} \mathbb{1}_A(y) \, L_t(\beta(\cdot), y) \, dy \]

$L_t(\beta(\cdot), y)$ is the normalized local time. Look at a picture

Consider

\[ \mathbb{E}_x \left[ \exp \left( - \int_0^t V(\beta(s)) \, ds \right) \right] = \mathbb{E}_x \left[ \exp \left( - \int_0^\infty V(y) \, L_t(\beta(\cdot), y) \, dy \right) \right] \]

\[ \int_0^t V(\beta(s)) \, ds = \mathbb{E}_x \left[ \int_0^\infty V(y) \, L_t(\beta(\cdot), y) \, dy \right]. \]

For fixed $x \in \mathbb{R}$ and $t > 0$

$L_t(\beta(\cdot), \cdot)$ maps $C_{x+0,00} \rightarrow M$ the space of probability measures on $\mathbb{R}$. For fixed $x \in \mathbb{R}$ and $t > 0$ we now define a probability measure on $M$ call it $Q_{x,t}$.

\[ Q_{x,t} = PL_t^{-1} \quad \text{i.e. if } C \subset M \]
\[ Q_{x,t}(C) = P(\beta(\cdot) \in (x, c_0, \infty) : L_t(\beta(\cdot), \cdot) \in C) \]

\[ L_t(\beta(\cdot), \cdot) \] is the occupation measure. So we can write

\[ E_x \left( e^{-t \int V(\beta(s)) ds} \right) = E_x \left( e^{-t \int_{V} L_t(\beta(s), y) dy} \right) = E_x \left( e^{-t \int_{V} \int_{\mathbb{R}} \mu(dy) \mu(dy)} \right) = E_x \left( e^{-t \int_{V} \Phi(y) dy} \right) \]

\[ \Phi(y) = \int_{V(x)} e^{-t \int_{V} \Phi(y) dy} \]

Here, \( V \) is the space of probability densities on \( \mathbb{R} \). We integrate on \( V \) here.

We can now show by looking at this as \( t \to \infty \). We want to see what \( Q_{x,t} \) becomes as \( t \to \infty \).

To see how \( E_x \left( e^{-t \int_{V} \Phi(y) dy} \right) \) behaves as \( t \to \infty \) we must see how \( Q_{x,t} \) behaves as \( t \to \infty \). First of all, how does \( L_t(\beta(\cdot), \cdot) \) as \( t \to \infty \)? Answer:

\[ L_t(\beta(\cdot), \cdot) \to 0 \text{ for } A \subset \mathbb{R} \text{ compact for a.a. } \beta(\cdot) \]

so for almost all \( \beta(\cdot) \), \( L_t(\beta(\cdot), \cdot) \to 0 \text{ as } t \to \infty \). This is a special case of the ergodic theorem for one-dimensional Brownian motion.

If this were not B.M. with an invariant measure this for a.a. \( \beta \) converges to the invariant measure.

So \( Q_{x,t} \) measure decays as \( t \to \infty \) if \( C \neq \emptyset \).

How fast? Theorem: (on \( \mathbb{F} \), we need to put Levy topology)

If \( C \in \mathbb{F} \) is closed

\[ \lim_{t \to \infty} t \log Q_{x,t}(C) = \inf_{C \in \mathbb{F}} I(C) \]

If \( C \in \mathbb{F} \) is open

\[ \lim_{t \to \infty} t \log Q_{x,t}(C) = \inf_{C \in \mathbb{F}} I(C) \]

where,
Prob. \( Q_{x,t}(c) \sim e^{-c t} \)

\( \text{const} = \inf I(f) \)

\[
I(f) = \frac{1}{\lambda} \int_{\mathcal{F}} \frac{(f(y))^2}{f(y)} dy. \text{ Suppose } f(y) = \frac{1}{\sigma} \exp \left( -\frac{y^2}{2\sigma^2} \right) \]

then \( I(f) = \frac{1}{2\sigma^2} \).

As \( \sigma \to 0 \), \( I(f) \to \infty \)

As \( \sigma \to \infty \), \( I(f) \to 0 \). So I tell you how much information is of it, it is called the entropy.

\[
I : \mathcal{F} \to [0, \infty] \bigg| \text{ as a functional.} 
\]

Roughly speaking, as \( t \to \infty \), \( Q_{x,t} (A) \sim e^{-t I(f)} \) for nice \( A \).

With a particular density function \( f \in \mathcal{F} \) and let \( A \) shrink on \( A \), so at the point \( f \):

\[
"Q_{x,t} (f) \sim e^{-t I(f)}" \bigg| \text{ roughly again.}
\]

\[
\lim_{t \to \infty} \log \mathbb{E} \left( e^{-\int_{\mathcal{F}} V(y) dy} \right) = -\inf I(f)
\]

Choose asymptotics:

\[
\lim_{t \to \infty} \log \mathbb{E} \left( e^{-\int_{\mathcal{F}} V(y) dy} \right) = -\inf I(f)
\]
Using the general structure theorem: Theorem.

Let $\Phi : \mathbb{F} \to \mathbb{R}$ which is bounded and continuous. Then by theorem 0 we have

$$\lim_{t \to 0^+} \frac{1}{t} \log E \left\{ e^{-t \Phi(f)} \right\} = -\inf_{f \in \mathbb{Z}} \left[ \Phi(f) + I(f) \right]$$

equivalently

$$\lim_{t \to 0^+} \frac{1}{t} \log E_n \left\{ e^{-t \Phi(A_0(\beta, \mathcal{S}))} \right\} = g(0).$$

But in theorem 0 we can only prove that with $C$ is compact and $f$ must have special hypotheses. This is more subtle than action asymptotics.

$$\lim_{t \to 0^+} \frac{1}{t} \log E \left\{ e^{+t \Phi(f)} \right\} = \sup_{f \in \mathbb{Z}} \left[ \Phi(f) - I(f) \right]$$

In statistical mechanics this is $-\Phi(0)$. There may be an $\alpha$ where there is a phase transition.

This is due to a non-uniqueness to the $f$ that maximizes this. How do you check uniqueness?
let \( \beta(t) \) be B.M. on \([a, \infty)\), \( \beta(0) = 0 \). For \( t \geq 3 \) we defined

\[
\beta_c(s) = \frac{\beta_c(t)}{\sqrt{1 + \gamma^2 s^4}} \quad 0 \leq s \leq 1
\]

For each \( \beta \in (a, \infty) \), we have a family \( \{\beta(s)\}_{s \geq 3} \). We showed that for almost all \( \beta \in (a, \infty) \) every sequence in the family has a convergent subsequence and the set of limit points is at most

\[
K = \{x \in (\alpha, \infty) : M_x = \frac{1}{2} \int \sqrt{x + s^4} \, ds \leq 1\}
\]

We now prove that every \( x \in K \) is a limit point of some sequence in \( \{\beta(s)\}_{s \geq 3} \) for almost all \( \beta \in (a, \infty) \).

From the first part, we that the set of limit points is at most \( K \), we infer that if \( \beta : (0, \alpha) \to \mathbb{R} \) which is continuous, then

\[
P \left( \lim_{n \to \infty} \Phi(\beta(n)) \leq \sup_{x \in K} \Phi(x) \right) = 1.
\]

For purposes latter consider \( \sup \sup |x(s)| \leq e^{\sqrt{T}} \) \( \forall T \leq 1 \)

Let \( x \in K \) be any element of \( K \) we must exhibit a subsequence of \( \{\beta(s)\}_{s \geq 3} \) that converges to \( x \). Let \( \delta > 0 \) be given. Choose \( k \in \mathbb{Z}^+ \) so large and \( x \in (0, \alpha) \) which is \( \delta \) for \( 0 \leq s \leq k \) and

\[
\frac{1}{2} \int x(s)^2 \, ds \leq 1 \Rightarrow x \in S(x_0, \frac{1}{k}). \text{ In fact choose } \frac{1}{k} \text{ is } \delta^2 \text{ where } c \text{ is the constant of the lemma.} \text{ Now choose } \delta \text{ so small so that } \frac{1}{k} + \delta < \frac{1}{2}. \text{ We show now that for almost all } \beta(s) \in (a, \infty), \exists \text{ infinitely many } n \Rightarrow \beta_n(s) \in S(x_0, \frac{1}{k} + \delta). \text{ This will mean that for almost}
\]
$\beta_n(s) \in \mathcal{S}(\alpha, \mathcal{S})$. For each $\beta_n(s) \in (0, \infty)$ let

$$f_n(s) = \frac{\beta_n(s) - \beta_n^{-1}(s)}{\sqrt{\log \log \mathcal{S} + k}}$$

Now

$$\mathbb{P}\left\{ f_n(s) \in \mathcal{S}(\alpha_0, \mathcal{S}) \right\} = \mathbb{P}\left\{ \sup_{\frac{1}{k} \leq s \leq 1} \left| \frac{\beta(s) - \beta(\frac{1}{k})}{\sqrt{\log \log \mathcal{S} + k}} - \xi_0(s) \right| < \mathcal{S} \right\}$$

$$= \mathbb{P}\left\{ \sup_{\frac{1}{k} \leq s \leq 1} \left| \frac{\beta(s) - \beta(\frac{1}{k})}{\sqrt{\log \log \mathcal{S} + k}} - \xi_0(s) \right| < \mathcal{S} \right\} \text{ by Brownian Scaling}$$

$$= \mathbb{P}\left\{ \sup_{\frac{1}{k} \leq s \leq 1} \left| \frac{\beta(s) - \beta(\frac{1}{k})}{\sqrt{\log \log \mathcal{S} + k}} - \xi_0(s) \right| < \mathcal{S} \right\} \text{ as we need.}$$

From action asymptotics

$$\lim_{\epsilon \to \infty} \epsilon \log \mathbb{P}\left\{ \sup_{\frac{1}{k} \leq s \leq 1} \left| \frac{\beta(s) - \beta(\frac{1}{k})}{\sqrt{\log \log \mathcal{S} + k}} - \xi_0(s) \right| < \mathcal{S} \right\}$$

$$\geq -A[\alpha_0] \text{ (actually } > -\inf_{x \in \mathbb{R}} A(x))$$

Choose $\rho > 0$ so small that $\lambda = A[\alpha_0] + \rho < 1$. Therefore

$$\mathbb{P}\left\{ f_n(s) \in \mathcal{S}(\alpha_0, \mathcal{S}) \right\} \geq e^{-\frac{\rho}{2k}} \text{ for } n \text{ large enough}.$$
\( f_n(\cdot) \in \mathcal{S}(\Delta_n, \delta) \). Now consider the uniform norm of 
\[
\| \beta \|_{\Delta_n} - X(\cdot) \| \leq \| \beta \|_{\Delta_n} - f_n(\cdot) \| + \| f_n(\cdot) - X(\cdot) \|. 
\]
What is 
\[
\beta_n(s) = \left( \frac{\beta(s)}{\sqrt{n \log \log n}} \right)_{0 \leq s \leq 1} (f_{n}(\cdot) = \left\{ \begin{array}{ll}
\frac{\beta(s)}{\sqrt{n \log \log n}} & 0 \leq s \leq \frac{1}{n} \\
0 & \frac{1}{n} \leq s \leq 1
\end{array} \right.) 
\]
\[
\leq \sup_{0 \leq s \leq 1} \left| \frac{\beta(s)}{\sqrt{n \log \log n}} \right| + \| f_n(\cdot) - X(\cdot) \|. 
\]
From part one of the theorem for \( 0 \leq t \leq 1 \)
\[
\left( \overline{\Phi_{\Delta_n}(\cdot)} = \sup_{0 \leq s \leq 1} \| X(s) \| \right) 
\]
\[
\lim_{n \to \infty} \sup_{0 \leq s \leq t} \| \beta_n(s) \| = \sup_{\epsilon \in K} \| \epsilon(\cdot) \| 
\]
\[
\left\{ \begin{array}{l}
\epsilon(\cdot) \in C_c(0, \infty) \text{ infinitely many } n \Rightarrow \| \epsilon(\cdot) \| \leq \frac{\epsilon}{2} + \delta < \frac{\epsilon}{2} \text{ recall.}
\end{array} \right.
\]
So we have it. Q.E.D. \( \frac{\epsilon}{2} \in C_c(0, \infty)! \)

**Entropy Asymptotics:** (A calculation)

Recall \( P \{ \sup_{0 \leq s \leq t} \beta(s) \leq \epsilon \} = \sqrt{\frac{2}{\pi t}} \int_0^\epsilon e^{-u^2} \, du \). \( \beta(s) \in C_c(0, \infty) \) for almost all \( \beta(\cdot) \in C_c(0, \infty) \) \( \epsilon \) infinitely many \( n \Rightarrow \| \beta_n(\cdot) - X(\cdot) \| \leq \frac{\epsilon}{2} + \delta < \frac{\epsilon}{2} \). \( \beta(\cdot) \in C_c(0, \infty) \)

Therefore
\[
E \left( e^{-\sup_{0 \leq s \leq t} \beta(s)} \right) = h(\epsilon) = \int_0^\infty e^{-\epsilon^2} \, dP \{ \sup_{0 \leq s \leq t} \beta(s) \leq \epsilon \}
\]
\[
= \int_0^\infty e^{-\sqrt{\frac{2}{\pi t}} \epsilon} \, d\epsilon = \sqrt{\frac{2}{\pi t}} \int e^{-\frac{1}{4t} \epsilon^2} \, d\epsilon - \frac{1}{4t} \epsilon^2 d\epsilon
\]
Let \( \alpha = -\frac{1}{4t} \epsilon^2 \). So we conclude
\[ \lim_{t \to \infty} t \log h(t) = \frac{1}{2}. \] Now we will derive this from action-entropy asymptotics. The theorems are:

\[ \lim_{\epsilon \to 0} \epsilon \log E \left\{ e^{\epsilon \frac{1}{2} F[\beta(t,\cdot)]} \right\} = \sup \left\{ F[w] - A[w] \right\} \]

\[ \lim_{\epsilon \to 0} \epsilon \log E \left\{ e^{\epsilon \frac{1}{2} F[\beta(t,\cdot)]} \right\} = \sup \left\{ \epsilon \log \left( \sup_{\omega \in \Omega} \frac{e^{\epsilon \frac{1}{2} F[\beta(t,\cdot)]}}{e^{\epsilon \frac{1}{2} F[\beta(t,\cdot)]}} \right) \right\} \]

Brownian Scaling

\[ \epsilon t = E \left\{ e^{\epsilon \sup_{x \in \Omega} \beta(x)} \right\} = E \left\{ e^{\epsilon \sup_{x \in \Omega} \beta(x)} \right\} = E \left\{ e^{\epsilon \sup_{x \in \Omega} \beta(x)} \right\} = E \left\{ e^{\epsilon \sup_{x \in \Omega} \beta(x)} \right\} \]

\[ = E \left\{ e^{\epsilon \sup_{x \in \Omega} \beta(x)} \right\} \]

let \( \epsilon = \frac{1}{t} \) then by AA

\[ \lim_{t \to \infty} \frac{1}{t} \log h(t) = \lim_{\epsilon \to 0} \epsilon \log E \left\{ e^{\epsilon \frac{1}{2} F[\beta(t,\cdot)]} \right\} = \sup \left\{ \sup_{a \in \mathbb{R}} \left( a - \frac{1}{2} a^2 \right) \right\} \]

We see on the right there is a fight between these two.

We reduce to this picture:

\[ \begin{array}{c}
\text{Entropy: Let } \mathcal{H} \text{ be the space of probability density functions on } \\
\mathbb{R} \text{ in } f_{\text{to}} \text{ and } \int_{-\infty}^{\infty} f(x) dx = 1 \text{ and } \mathcal{F}: \mathcal{H} \to \mathbb{R} \text{ with hypotheses on } \mathcal{F}. \text{ Let } \]

\[ \mathcal{L}_t \left( \beta(s), y \right) = \frac{1}{t} \int_{-\infty}^{\infty} (\beta(s) - y)^2 ds \text{ is } \mathcal{F} \text{ measurable.}\]

\[ \lim_{t \to \infty} \frac{1}{t} \log E \left\{ e^{\mathcal{L}_t \left( \beta(s), y \right)} \right\} = \sup \left\{ F[f] - I[f] \right\} \text{ when} \]

\[ I[f] = \frac{1}{2} \int_{-\infty}^{\infty} \left[ f(x) \right]^2 dx. \]
For example, if \( V(y) \to 00 \) as \( y \to 00 \) and \( \Phi(t) = \frac{1}{2} \int V(y) \phi(y) \, dy \), then \( \Phi \) satisfies the unspoken hypothesis. Then

\[
\lim_{t \to 00} \frac{1}{t} \log E \left[ e^{-\int \Phi(y) \, dy} \right] = \sup_{f \in \mathcal{F}} \left[ -\int V(y) \phi(y) \, dy \right]
\]

Also,

\[
\lim_{t \to 00} \frac{1}{t} \log E \left[ e^{-\int \Phi(y) \, dy} \right] = -\inf_{f \in \mathcal{F}} \left[ \int V(y) \phi(y) \, dy \right]
\]

or

\[
\lim_{\psi \to 00} -\inf_{\psi \in \mathcal{L}} \frac{1}{\psi} \log E \left[ e^{-\int \Phi(y) \, dy} \right] = -\inf_{\psi \in \mathcal{L}} \left[ \int V(y) \psi(y) \, dy + \frac{1}{\psi} \int \psi(y) \, dy \right] \text{.}
\]

But by Rayleigh-Ritz \( \lambda = \lambda \), where \( \lambda \) is the lowest e.v. of \( \frac{1}{2} \psi \cdots \psi = \lambda \psi \).

Now back to the sup: It was the discovery of Paul Levy that the probability distribution of \( \sup \beta(x) \) is exactly the same the p.d.f. of \( t \in \mathcal{F}(\mathcal{L}, \varnothing, R) \).

\[ P \left( \sup \beta(x) = \infty \right) = P \left( t \in \mathcal{F}(\mathcal{L}, \varnothing, R) \right) \]

We check the

First moments at the end of the lecture. Therefore:

\[ h(x) = E \left[ e^{-\int \phi(y) \, dy} \right] = E \left[ e^{-\int \phi(y) \, dy} \right] = E \left[ e^{-\int \phi(y) \, dy} \right] \]

where \( \phi \{ \phi \} = \varnothing \). Therefore from entropy asymptotic

\[
\lim_{t \to 00} \frac{1}{t} \log h(x) = \lim_{t \to 00} \frac{1}{t} \log E \left[ e^{-\int \phi(y) \, dy} \right] = \sup_{f \in \mathcal{F}} \left[ \int \phi(y) \, dy \right] - \frac{1}{2} \int \phi(y) \, dy \]

The maximizing family is \( \phi(y) = a e^{-\frac{1}{2} y^2} \), \( a > 0 \), \( \left( \sup \left[ a - \frac{1}{2} y^2 \right] = \frac{1}{2} \right) \).

How do you get the maximizing family? Go to \( L^2(\varnothing, \mathcal{L}) \)

\[ \sup_{f \in \mathcal{F}} \left[ \int \phi(y) \, dy \right] = \sup_{f \in \mathcal{F}} \left[ \int \phi(y) \, dy \right] = \sup_{f \in \mathcal{F}} \left[ \int \phi(y) \, dy \right] \]
So $\psi''(y) = 2L\psi$ from the constrained Euler-Lagrange problem. So the density that maximizes it is $e^{-2y}$.

Many equations of mathematical physics can be pinched:

$$\int_0^\infty \log E \left[ \int e^{t \sqrt{2L} \psi(y)} \, dy \right] = \sup_{f \in F} \left[ \frac{1}{2} \int_0^\infty \frac{[f(y)]^2}{\sqrt{2L} \psi(y)} \, dy \right].$$
Let \( (X, (0, \infty)) \) be continuous \( \beta \)-12 with \( \beta \in (0, \infty) \). Let \( P_x \) and \( E_x \) denote B.M. measure and expectation on \( (X, (0, \infty)) \). Let \( t \geq 0 \) and \( x \in \mathbb{R} \), and define
\[
E_t(\beta \cdot, y) = \int 1_t(\beta(s)) dP_x(s)
\]
which is fixed \( t \) and fixed \( y \) is a point in \( y \) with \( \beta \). Therefore, for fixed \( x \) and \( t \) be maps \( (X, (0, \infty)) \rightarrow \mathcal{D} \), the space of prob. dist. functions on \( \mathbb{R} \).

Use this mapping to define a probability measure on \( \mathcal{D} \).

Let \( A \) be \( \mathcal{D} \), we define the probability measure \( Q_{x,t} \)
\[
Q_{x,t}(A) = P_x(\{ \beta \cdot \in (X, (0, \infty)) : E_t(\beta \cdot, 1) \in A \}).
\]

This is obviously countably additive in \( A \) and \( Q_{x,t}(\mathcal{D}) = 1 \).

Recall the topology on \( \mathcal{D} \), the Lévy topology, the topology of weak convergence.

**Definition:** Let \( U \) be the space of functions \( u(x) \) on \( \mathbb{R} \) which are twice continuously differentiable and for each of which \( E \) constants \( \alpha \) and \( \beta \Rightarrow \alpha \leq u''(x) \leq \beta < \infty \) for all \( x \in \mathbb{R} \).

For all \( F \in \mathcal{D} \) define
\[
I(F) = -\inf_{u \in U} -\int_{-\infty}^{\infty} \frac{1}{2} (u(x)^2) dF(x)
\]
so \( I : \mathcal{D} \rightarrow [0, \infty] \) also. \( I \) is lower semi-continuous on \( \mathcal{D} \) because of the Lévy topology of weak convergence.

\[
\int \delta_{gF_n} \rightarrow \int \delta_{gF} \quad \forall g \text{ bounded continuous}
\]
We will prove: Let \( f \) be a p.d.f. which is continuously differentiable on \( \mathbb{R} \) and on the interval \(-\infty < a \leq b < \infty\) \( f \) is strictly positive and \( f = 0 \) outside \([a, b]\). For such a density, \( I(F) \) where \( F \) is the corresponding p.d.f. to \( F \) = \( \frac{1}{b-a} \int_a^b f(t) dt \) in the sense that if one side is finite then so is the other and they are equal.

**Theorem:** Let \( \Phi : \mathcal{D} \to \mathbb{R} \) satisfying hypotheses I to V, then

\[
\lim_{t \to 0} \frac{1}{t} \log E_x \left[ e^{-t \Phi(L_t(x, \cdot))} \right] = \sup_{F \in \mathcal{D}} [\Phi(F) - I(F)]
\]

**Lemma:** Let \( C \subset \mathcal{D} \) be compact

\[
\lim_{t \to 0} \frac{1}{t} \log Q_{x,t}(C) \leq -\inf_{F \in \mathcal{C}} I(F)
\]

So \( Q_{x,t}(C) \sim e^{-\inf_{F \in \mathcal{C}} I(F)} \), as a remark.

**Proof:** Let \( \Phi(x, \cdot) = E_x \left[ e^{-\frac{1}{2} \int_0^t \frac{u''(s)}{u(s)} \, ds} \right] \), we see Feynman-Kac coming, so \( \Phi(x, t) \) satisfies

\[
\psi_t = \frac{1}{2} \psi_{xx} - \frac{1}{2} \frac{u''(x)}{u(x)} \psi_x \quad \psi(x, 0) = u(x)
\]

However the solution is trivially \( \psi(x, t) \equiv u(x) \), therefore

\[
E_x \left[ e^{-\frac{1}{2} \int_0^t \frac{u''(s)}{u(s)} \, ds} \right] = u(x). \quad \text{Since } u \in \mathcal{U}, \text{ we have}
\]

\[
E_x \left[ e^{-\frac{1}{2} \int_0^t \frac{u''(s)}{u(s)} \, ds} \right] \leq \frac{u(x)}{x}
\]

i.e.

\[
\frac{u(x)}{x} \leq \frac{u(x)}{x}
\]

because
Prob.

\[ L_t(\beta c, y) = \frac{t}{\alpha} \int_0^t \mathcal{L}(\beta c s, y) ds \quad \text{with density} \]

\[ L_t(\beta c, y) = \frac{t}{\alpha} \int_0^t \mathcal{L}(\beta c s, y) ds. \quad \text{By the defn. of Q, we measure} \]

\[ E \int_{\mathcal{F}_t} E e^{-\int_0^t \frac{1}{2} \frac{1}{u(\psi y)} dFy} \leq \frac{ue^{\frac{\alpha^2}{2}}}{\alpha^2} \leq \frac{\beta}{\alpha^2}. \]

This gives us an upper bound on the measure of any Borel set \( C \subset \mathcal{F} \), by a Chebychev-like inequality

\[ Q_{\alpha, \xi}(C) \leq \frac{\beta}{\alpha^2} \int_{\mathcal{F}_t} \mathbf{1}_C \mathbf{1}_{\mathcal{F}_t} dFy, \]

Therefore

\[ \lim_{t \to \infty} \frac{1}{t} \log Q_{\alpha, \xi}(C) \leq \sup_{x \in \mathcal{F}_t} \int_{\mathcal{F}_t} \frac{1}{2} \frac{1}{u(\psi y)} dFy, \]

Then, since this holds for all \( x \in U \), we can conclude that

\[ \lim_{t \to \infty} \frac{1}{t} \log Q_{\alpha, \xi}(C) \leq \inf_{x \in U} \sup_{F \in \mathcal{F}_t} \int_{\mathcal{F}_t} \frac{1}{2} \frac{1}{u(\psi y)} dFy. \]

Suppose \( C \subset \bigcup_{j=1}^k C_j \) where each \( C_j \) is a Borel set. Then trivially

\[ Q_{\alpha, \xi}(C) \leq \sum_{j=1}^k Q_{\alpha, \xi}(C_j) \]

and so

\[ \lim_{t \to \infty} \frac{1}{t} \log Q_{\alpha, \xi}(C) \leq \max_{1 \leq j \leq k} \inf_{x \in U} \sup_{F \in \mathcal{F}_t} \int_{\mathcal{F}_t} \frac{1}{2} \frac{1}{u(\psi y)} dFy \]

But this is true for all such coverings by Borel sets of \( C \) so

\[ (1) \lim_{t \to \infty} \frac{1}{t} \log Q_{\alpha, \xi}(C) \leq \inf_{x \in U} \max_{1 \leq j \leq k} \inf_{F \in \mathcal{F}_t} \sup_{y \in \mathcal{F}_t} \int_{\mathcal{F}_t} \frac{1}{2} \frac{1}{u(\psi y)} dFy. \]
\[ V_\varepsilon \text{ of course have (1) for this } C. \text{ Let } \lambda = \inf \{ F(C) \} \text{ by definition.} \]

\[ \lambda = \sup \inf \int_0^\infty \frac{1}{\sqrt{2\pi} \sigma(y)} dF(y) \quad \text{for every } F \in C \]

\[ \inf \int_0^\infty \frac{1}{\sqrt{2\pi} \sigma(y)} dF(y) \leq \lambda \quad \text{for any } \varepsilon > 0 \text{ and } \forall F \in C \]

\[ \forall \varepsilon \in \mathbb{R}^+ \quad \exists \sup_{C \in C} \int_0^\infty \frac{1}{\sqrt{2\pi} \sigma(y)} dF(y) \leq \lambda + \varepsilon \quad \text{since } \frac{1}{\sqrt{2\pi} \sigma(y)} \text{ is bounded and continuous.} \]

\[ \forall \varepsilon \in \mathbb{R}^+ \quad \exists \sup_{C \in C} \int_0^\infty \frac{1}{\sqrt{2\pi} \sigma(y)} dF(y) \leq \lambda + \varepsilon \quad \text{by optimality.} \]

\[ \int_0^\infty \frac{1}{\sqrt{2\pi} \sigma(y)} dF(y) \text{ is a continuous functional on } C \text{ thus for very } F \in C \exists \text{ a neighborhood } N_F = \{ G \in N_f \}

\[ \int_0^\infty \frac{1}{\sqrt{2\pi} \sigma(y)} dG(y) \leq \lambda + \varepsilon \quad \text{now the collection } \]

\[ N_F \text{ for } F \in C \text{ is an open covering of } C, \text{ but by compactness:} \]

\[ \exists F_1, F_2, \ldots, F_k \in C \cap \bigcup_{j=1}^k N_j \quad \text{Let } C_j = N_{F_j} \quad C \subseteq \bigcup_{j=1}^k C_j \]

and we have

\[ \sup \int_0^\infty \frac{1}{\sqrt{2\pi} \sigma(y)} dF_j(y) \leq \lambda + \varepsilon \quad \text{and we have:} \]

\[ \inf \sup \int_0^\infty \frac{1}{\sqrt{2\pi} \sigma(y)} dF_j(y) \leq \lambda + \varepsilon \quad \text{and so also} \quad \text{since it is true for all } C_j \]

\[ \max \inf \sup \int_0^\infty \frac{1}{\sqrt{2\pi} \sigma(y)} dG_j(y) \leq \lambda + \varepsilon \quad \text{and so this is true for all such covers}\]

\[ \inf \max \inf \sup \int_0^\infty \frac{1}{\sqrt{2\pi} \sigma(y)} dG_j(y) \leq \lambda + \varepsilon \quad \text{and by } C_j \text{ and } G_j \text{ we have it.} \]
Pseudo: Let \( f_{xy} \) be a probability density function on \( \mathbb{R} \) which is strictly positive on \( \mathbb{R}^2 \) where \(-\infty < a < b < \infty\) and \( f = 0 \) outside of \([a, b]^2\), and \( fx \) is continuously differentiable. Then for \( F \) the corresponding distribution function:

\[
I(F) = \frac{1}{2} \int_a^b \left( \frac{f_{xy}}{f_{xy}} \right)^2 \, dy
\]

in the sense that if the right-hand side is finite so is the left-hand side.

Proof: Assume that \( \int_{-\infty}^{\infty} f_{xy} \, dy < \infty \). We show now that for \( U \)

\[
\int_{-\infty}^{\infty} \frac{1}{2} \left( \frac{f_{xy}}{f_{xy}} \right)^2 \, dy = \frac{1}{2} \int_a^b \left( \frac{f_{xy}}{f_{xy}} \right)^2 \, dy
\]

which will imply

\[ I(F) = \frac{1}{2} \int_a^b \frac{f_{xy}}{f_{xy}} \, dy. \]

For any \( u \) let \( h = \log u \)

and

\[ \int_{-\infty}^{\infty} \frac{1}{2} \left( \frac{f_{xy}}{f_{xy}} \right)^2 \, dy = \frac{1}{2} \int_a^b \left( \frac{h_{xy}}{h_{xy}} + \left( \frac{h_{xy}}{h_{xy}} \right)^2 \right) f_{xy} \, dy. \]

Now consider

\[ \int_a^b \left( \frac{h_{xy}}{h_{xy}} \right) f_{xy} \, dy = \int_a^b \frac{h_{xy}}{f_{xy}} \sqrt{f_{xy}} \, dy \]

by Schwartz

\[ \leq \int_a^b h_{xy} |f_{xy}|^{1/2} \, f_{xy}^{1/2} \, dy \]

since \( 2ab = a^2 + b^2 \):

\[ \leq \int_a^b |h_{xy}| |f_{xy}| + \frac{1}{4} \int_a^b \frac{f_{xy}}{f_{xy}} \, dy. \]

Rearranging,

\[ \frac{1}{2} \int_a^b \left( h_{xy} \right)^2 f_{xy} \, dy - \frac{1}{2} \int_a^b h_{xy} f_{xy} \, dy = \frac{1}{8} \int_a^b \frac{f_{xy}}{f_{xy}} \, dy \]

integration by parts

\[ \frac{1}{2} \int_a^b \left( \frac{h_{xy}}{f_{xy}} + \frac{f_{xy}}{f_{xy}} \right) f_{xy} \, dy = -\frac{1}{8} \int_a^b \frac{f_{xy}}{f_{xy}} \, dy \]

so we have it by \( \circ \).
Now we prove the other way, to show: If \( I(F) \) is finite then
\[
\frac{1}{4} \int_{a}^{b} \left( \frac{h' (y)}{f (y)} \right)^{2} dy \leq I(F) \text{ which will complete the theorem.}
\]

From the definition of \( I(F) \) that \( h \in U \) and therefore any

\[
eq \log u \int_{a}^{b} \left( \frac{u^{1/n}}{u^{1/n} + F (y)} \right) dy = \frac{1}{2} \int_{a}^{b} \left[ \frac{h (y)}{f (y)} \right]^{2} \left( \frac{h (y) f (y)}{h' (y) f (y)} \right) dy \geq -I(F).
\]

In particular this last inequality holds for those \( h \)'s which are constant outside of a closed interval contained in \((a, b)\). If we let \( H \) denote this latter class of functions \( h \)'s, we have that for any \( h \in H \)

\[
\frac{1}{2} \int_{a}^{b} \left\{ \left[ h (y) \right]^{2} f (y) - h' (y) f (y) \right\} dy \geq -I(F)
\]

If \( h \in H \), then \( 2h \in H \) for any real \( 2 \) and therefore we have

\[
\frac{1}{2} \int_{a}^{b} \left[ h (y) \right]^{2} f (y) dy - \frac{1}{2} \int_{a}^{b} h' (y) f (y) dy + I(F) \geq 0.
\]

This cannot have distinct real roots \( 0 \), so the discriminant of this quadratic is \( \leq 0 \). This is the \( P \)

\[
\int_{a}^{b} \left| h' (y) f (y) \right| dy \leq \sqrt{8 I(F)} \left[ \int_{a}^{b} \left[ h (y) \right]^{2} f (y) dy \right]^{1/2}.
\]

Let \( g (y) = h (y) \)

then we have

\[
\int_{a}^{b} \left| g (y) f (y) \right| dy \leq \sqrt{8 I(F)} \left[ \int_{a}^{b} \left[ g (y) \right]^{2} f (y) dy \right]^{1/2}
\]

this is true for all \( g \) continuously differentiable and with
compact support contained in $(a,b)$. But this is also true for any $g$ which is continuous with compact support in $(a,b)$ by Wierstrass approximation.

Let $\psi \in C^\infty$ with compact support in $(a,b)$ and $0 \leq \psi(x) \leq 1$ for all $x \in \mathbb{R}$. We now apply the preceding inequality to

$$f(x) \psi(x)$$

which is continuous and has compact support in $(a,b)$.

So,

$$\int_a^b \frac{1}{2} \frac{x^2}{\psi(x)} \psi(x) dy \leq \sqrt{8I(F)} \left[ \int_a^b \frac{x^2}{\psi(x)} \psi(x) dy \right]^{1/2} \quad \text{get rid of one}$$

We get

$$\frac{1}{8} \int_a^b \frac{x^2}{\psi(x)} \psi(x) dy \leq I(F)$$

and by approximation of the function by the $\psi_i$ we get by bounded convergence

$$\frac{1}{8} \int_a^b \frac{x^2}{\psi(x)} dy \leq I(F).$$

If instead of B.M. if we have $u_c = L u$, then

$$I = -\inf_{u \in D^L} \int \frac{u(x)^2}{\psi(x)} dy$$

if $L$ is a self-adjoint operator there is an analogous argument that gives you some theorem similar to the lemma.
where $T:F \to R$ is defined by

$$T(F) = \frac{1}{2} \int_{\mathbb{R}^d} F(x) \, dx$$

and we have

$$I(F) = \int_{\mathbb{R}^d} F(x) \, dx$$

We also proved that

$$\lim_{I(F) \to \infty} C = \frac{1}{2}$$

and $C$ is bounded.

Also, let $R$ be a fixed element in $\mathbb{R}^d$. Then

$$\mathbb{P}(T(F) \leq R) = \frac{1}{2}$$

If $C$ is compact, then

$$\mathbb{P}(T(F) \leq R) \to \frac{1}{2}$$

as $F \to \infty$.

In particular, if $F$ is a bounded distribution, then

$$\mathbb{P}(T(F) \leq R) = \frac{1}{2}$$

for all $R$. This means that $T(F)$ is uniformly distributed on $[0,1]$. Theorems about the occupation distribution of $R$ with $C$.

For fixed $R$ in $\mathbb{R}$,

$$E = \cdots$$

where

$$E = \cdots$$

and $E$ is compact.

Thus, we have

$$E = \cdots$$

for all $R$. The proof is complete.
Prob. 6

I. \( \alpha \leq \Phi(F) \leq \infty \)

II. \( \Phi \) is lower semi-continuous on \( \mathcal{D} \)

III. For any \( K \in F \in \mathcal{D} ; \Phi(F) \leq K \) is compact in \( \mathcal{D} \)

Example: If \( \Phi(F) = \sup \{ V_N(V) \mid F \geq V_N \} \), then III forces \( V_N \to \infty \) or \( y_N \to \infty \).

Then \( \lim_{\epsilon \to 0} \frac{1}{\epsilon} \log E_x \left[ e^{-\epsilon \Phi(F) \left( \beta(x, \cdot) \right)} \right] \leq -\inf_{F \in \mathcal{D}} \left[ \Phi(F) + I(F) \right] \).

Proof: For any \( K \) let \( A_K = \{ F \in \mathcal{D} : \Phi(F) \leq K \} \) so by III \( A_K \) is compact.

Consider

\[ E_{A_K} \left\{ e^{-\epsilon \Phi(F)} \right\} = E_{A_K} \left\{ e^{-\epsilon \Phi(F)} \right\} + E_{A_K} \left\{ e^{-\epsilon \Phi(F)} \right\} \]

\[ \leq E_{A_K} \left\{ e^{-\epsilon \Phi(F)} \right\} + e^{-2K} \]

\[ \Rightarrow \lim_{\epsilon \to 0} \frac{1}{\epsilon} \log E_x \left[ e^{-\epsilon \Phi(F) \left( \beta(x, \cdot) \right)} \right] \leq \max \left( \lim_{\epsilon \to 0} \frac{1}{\epsilon} \log E_{A_K} \left\{ e^{-\epsilon \Phi(F)} \right\} ; -K \right) \]

We note that by the definition of \( I(F) \) it is lower semi-continuous on \( \mathcal{D} \) as is \( \Phi(F) \) by II. Let \( F^* \in A_K \). Assume first \( I(F^*) < 0 \), since \( I \) is l.s.c. \( F^* \) is a neighborhood of \( N_{F^*} = \{ F \in A_K \mid I(F) < 0 \} \).

\[ I(F^*) \leq I(F) + \epsilon \]

\[ \Phi(F^*) \leq \Phi(F) + \epsilon \]

Let \( G_{F^*} \) be another neighborhood of \( F^* = G_{F^*} \in N_{F^*} \). Consider
\[ E_{Q_{n,t}} \left\{ e^{-t \Phi(F)} \right\} \leq e^{-t (\Phi(F^*) - \varepsilon)} Q_{n,t} (G_{F^*} \cap \Lambda_K) \]

\[ \lim_{t \to \infty} \frac{1}{t} \log E_{Q_{n,t}} \left\{ e^{-t \Phi(F)} \right\} \]

\[ \leq -\Phi(F^*) + \varepsilon - \inf_{F \in G_{F^*} \cap \Lambda_K} I(F) \leq -\Phi(F^*) + \varepsilon - \inf_{F \in F^* \cap \Phi} \left[ I(F) + \frac{\varepsilon}{2} \right] \]

In the other hand, if \( I(F^*) = \infty \), then by l.s.c. of \( I \) in \( b \) of \( F^* \)

\[ \text{Let } G_{F^*} \text{ be another nbhd of } F^* \text{ and we have} \]

\[ \leq -\Phi(F^*) + \varepsilon - \inf_{F \in F^* \cap \Phi} I(F) \leq -\Phi(F^*) + \varepsilon - \inf_{F \in F^* \cap \Phi} \left[ I(F) + \frac{\varepsilon}{2} \right] \]

\[ \lim_{t \to \infty} \frac{1}{t} \log E_{Q_{n,t}} \left\{ e^{-t \Phi(F)} \right\} \leq -\inf_{F \in \Phi} I(F) \leq -K \]

\[ \text{Since each point } F^* \text{ of } \Lambda_K \text{ is covered by a nbhd } G_{F^*} \text{ (whether } I(F^*) \text{ is infinite or finite) and since } \Lambda_K \text{ is compact, } I \text{ is finite sub-collection} \]

\[ F^*_1, F^*_2, \ldots, F^*_n \supseteq \Lambda_K = \bigcup_{j=1}^n G_{F^*_j} \quad \text{so we have that} \]

\[ \lim_{t \to \infty} \frac{1}{t} \log E_{Q_{n,t}} \left\{ e^{-t \Phi(F)} \right\} \leq \max_{F \in \Phi} \left( \lim_{t \to \infty} \frac{1}{t} \log E_{Q_{n,t}} \left\{ e^{-t \Phi(F)} \right\} \right) \]

\[ \leq \max_{F \in \Phi} ( -K, -\inf_{F \in \Phi} \left[ \Phi(F) + I(F) + \frac{\varepsilon}{2} \right] ) \]

\[ \text{by} \quad \lim_{t \to \infty} \frac{1}{t} \log E_{Q_{n,t}} \left\{ e^{-t \Phi(F^*)} \right\} \leq \max_{F \in \Phi} \left( \frac{1}{t} \log E_{Q_{n,t}} \left\{ e^{-t \Phi(F)} \right\} \right) \]

\[ \text{by} \quad \lim_{t \to \infty} \frac{1}{t} \log E_{Q_{n,t}} \left\{ e^{-t \Phi(F^*)} \right\} \leq \max_{F \in \Phi} \left( \frac{1}{t} \log E_{Q_{n,t}} \left\{ e^{-t \Phi(F)} \right\} \right) \]
So \[ = -\min \left( K, \inf_{\mathcal{F}} \left[ \Phi(F) + I(F) \right] \right) - 2\varepsilon \]

If \( K \) and \( \varepsilon \) are arbitrary, let \( K \to \infty \) as \( \varepsilon \) is fixed. Now let \( \varepsilon \to 0 \) and we have

\[
\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \log E_x \left[ e^{-\varepsilon \Phi'(\xi)} \right] \leq -\inf_{\mathcal{F}} \left[ \Phi(F) + I(F) \right].
\]

You cannot drop any of the hypotheses in the theorem and have

weak convergence.

IV: \( F_n \Rightarrow F \) and the support of \( F_n \) is contained in the finite closed interval \([a, b] \) in \( \mathbb{R} \), then \( \Phi(F_n) \to \Phi(F) \).

Let \( \mathcal{D}_x \) be the set of all density functions with two continuous derivatives and for which \( F \) is closed in \([a, b] \) and \( F' \) is in \( \mathcal{L}^2 \) outside \([a, b] \), for which \( f=0 \) and \( f \geq 0 \) on \([a, b]\) and \( \int_{-\infty}^{\infty} \frac{y^2}{f(y)} \, dy < \infty \).

Assume: Let \( f \) satisfy IV. Then

\[
\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \log E_x \left[ e^{-\varepsilon \Phi'(\xi)} \right] \geq -\inf_{\mathcal{D}_x} \left\{ \Phi(f) + \frac{1}{2} \int_{a}^{b} f(y)^2 \, dy \right\}
\]

And so \( I \) (came) this equals \( \Phi \). (Ventsel + Freidlin).

Proof: Let \( f \in \mathcal{D}_x \) and let \( F \) be the corresponding distribution function on each \( \beta \in \mathcal{P} \). \( \alpha(\beta) = a \leq \beta \leq b \). 

To see, its occupation distribution function whose support is contained in \([a, b]\).

\( L_t(\beta_{t_{\cdot}}(\cdot)) \).
We will show that
\[
\lim_{t \to \infty} \frac{1}{t} \log P_x e^{-t \mathbb{E} \left( L_c (\beta_c) \right)} \geq -\mathbb{E} (F) - \frac{1}{2} \int_{D_1} \left[ \Phi(f) + \frac{1}{2} \int_{D_1} \left[ \Phi(f) + \frac{\mathcal{F}[\phi]}{\mathcal{F}[\phi]} dy \right] \right]
\]

We will show this by proceeding as young man.

Now the sticky proof: Recall that \( u_t = \frac{1}{2} \partial^2 u / \partial x^2 \) has a solution

\[
p(x, y) = \frac{1}{(2\pi t)^{1/2}} e^{-\frac{(x-y)^2}{2t}}.
\]
\[ E_x \mathbb{E}[\mathbb{E}[\beta]] = E_x \{ \mathbb{E}[\beta] e^{-\int_0^t b(\beta(s)) ds + \frac{1}{2} \int_0^t b^2(\beta(s)) ds} \} \]

Not classically defined.

We are trying to prove: Let \( f \in \mathcal{D}_x \) and let \( F \) be the c.d.f. and let \( N_F \) be any Levy par of \( F \). Let \( \psi(y) = \frac{1}{2} f(y^2) \) and consider \( 3 \).  

\[ (t) \]

\[ \text{It can't get out.} \]

\[ \text{Look at} \ F, \text{McKean under Feller's test.} \]

The deepest result of Markov chains is ergodic theory. The invariant measure above is f(y). That is

\[ \psi(y) \xrightarrow{\text{weak}} \text{Feyn} \]  

for a.e \( \beta \) in drift measure.

\[ \text{From Cameron-Martin formula} \]

\[ P_x \mathbb{E}_\beta (x, \phi e): L_t (\beta(c), \cdot) \in N_F, \beta(c) \in C_{\alpha, \beta} \]

\[ \cdots \]

we can drop this because

\[ \cdots \]

\[ = E_x \left\{ e^{-\int_0^t b(\beta(s)) ds + \frac{1}{2} \int_0^t b^2(\beta(s)) ds} \cdot L_t (\beta(c), \cdot) \in N_F, \beta(c) \in C_{\alpha, \beta} \right\} \]

\[ = E_x \left\{ e^{-\frac{1}{2} \int_0^t \left( \frac{d}{ds} (\beta(s)) \right)^2 + \frac{1}{2} \int_0^t \left( \frac{d}{ds} (\beta(s)) \right)^2 ds} \cdot L_t (\beta(c), \cdot) \in N_F \right\} \]
To's Formula: Let \( g(y) \) be twice continuously differentiable.

\[
dg(\beta(s)) = g'(\beta(s)) \frac{d\beta(s)}{ds} + \frac{1}{2} g''(\beta(s)) ds.
\]
Problem 8. Let $Z_o$ and consider

$$
E \left[ e^{\int_0^t \beta_s \, ds} - \frac{1}{2} \int_0^t \beta_s^2 \, ds \right]
$$

Find $\lim_{t \to \infty} \frac{1}{t} \log E \left[ e^{\int_0^t \beta_s \, ds} - \frac{1}{2} \int_0^t \beta_s^2 \, ds \right]$ using entropy asymptotics.

9. Check your result in #8 by actually calculating the function space $N\varepsilon$ lies left of the $D_X$ integral.

**Lemma.** $P \left\{ \beta (c_s) \in \left( x (c_0), \ldots, L (c (c_s)), 1 \right) \in \mathbb{N}^e, \beta (c) \in \mathbb{C} \right\}$

$$
\lim_{t \to \infty} \frac{1}{t} \log \left( 1 - \frac{1}{t} \int_0^t \frac{\langle \beta (x) \rangle^2}{f (x)} \, dx \right)
$$

Recall: $\frac{1}{2} \frac{dy^2 + b y^2}{dx} \frac{1}{b} \frac{dy^2}{dy} = \frac{1}{2} \frac{d y^2}{dy}$, w.r.t. the drift property the measure is $\pi_x \in E$ and the invariant measure is $\nu$ itself.

Using the Cameron-Martin formula we got,

$$
P \left\{ \beta (c_s) \in \left( x (c_0), \ldots, L (c (c_s)), 1 \right) \in \mathbb{N}^e, \beta (c) \in \mathbb{C} \right\}
$$

$$
= \mathbb{E}_x \left[ e^{\frac{1}{2} \int_0^t \left( f' (\beta (c_s)) \right)^2 \, dc_s - \frac{1}{2} \int_0^t \beta (c_s)^2 \, dc_s \right]
$$

**Ito's Formula.** Given any $g$ with two continuous derivatives,

$$
dg (\beta (c_s)) = g' (\beta (c_s)) d \beta (c_s) + \frac{1}{2} g'' (\beta (c_s)) d \sigma
$$

Consider $\int_0^t g' (\beta (c_s)) \, dc_s$, in particular let $g (y) = \log y$.
\[ d \log(\beta(s)) = (\log f_x)^{(\beta(s))} \, d\beta(s) + \frac{1}{2} \left( (\log f_x)^{(\beta(s))} \right)^{''} \, ds \]

Integrate from 0 to \( t \) with \( \beta(0) = x \):

\[ \log(\beta(t)) - \log(\beta(0)) = \int_0^t (\frac{1}{2} \left( (\log f_x)^{(\beta(s))} \right)^{''} \, ds - \int_0^t (\frac{1}{2} \left( (\log f_x)^{(\beta(s))} \right)^{'} \, ds \]

Now we use this to replace the nasty term & \( e^{\sigma(x, N_x, t, a, b)} \)

Let \( E(\beta) : L_x(\beta(0)) \in N_x, \beta(0) \in C_{a, b} \)

\[ = \mathbb{E}_x \left\{ \sqrt{f_x(\beta(0))} e^{-\int_0^t (\frac{1}{2} \left( (\log f_x)^{(\beta(s))} \right)^{''} \, ds + \int_0^t (\frac{1}{2} \left( (\log f_x)^{(\beta(s))} \right)^{'} \, ds \right\} \}

Let \( \bar{J}(x) = \frac{1}{2} (\frac{1}{2} \left( (\log f_x)^{(\beta(s))} \right)^{'} \, ds \)

and \( x \epsilon \mathbb{E} \) be given.

Let \( S(e, x) : \mathbb{E} \beta(0) \in C_{a, b} : \frac{1}{2} \int \bar{J}(x) \, d\mu(\beta(0), x) = \bar{J}(x) \}

P(S(e, x)) \to 1 \text{ as } t \to \infty \because \text{invariant measure}.

Let \( S'(e, x) = S(e, x) \cap \mathbb{E} \beta(0) \in C_{a, b} : L_x(\beta(0), \cdot) \in N_x \}

\[ \text{Note: } \int \bar{J}(x) \, d\mu(\beta(0), x) = \int_{\mathbb{R}} \left( \frac{1}{2} \int \bar{J}(x) \, d\mu(\beta(0), y) \right) \, dy \]

\[ = \frac{1}{2} \int_{\mathbb{R}} \bar{J}(y) \, dy \]

\[ e = \mathbb{E}_x \left\{ \sqrt{f_x(\beta(0))} e^{-\int_0^t \bar{J}(y) \, dy} \right\} \}

\[ = \mathbb{E}_x \left\{ \sqrt{f_x(\beta(0))} e^{-\int_0^t \bar{J}(x) \, d\mu(\beta(s), y)} \right\} \}

\[ \geq \mathbb{E}_x \left\{ \sqrt{f_x(\beta(0))} e^{-\int_0^t \bar{J}(x) \, d\mu(\beta(s), y)} \right\} \]

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Proba.  

\[ \lim_{t \to 0} \left( \int \text{exp} \left( -t \int \text{exp} \left( \frac{b}{\text{exp}} \right) \right) \text{d}y \right) = e^{-t} \int \text{exp} \left( \frac{b}{\text{exp}} \right) \text{d}y \]

Now \( \int \text{exp} \left( \frac{b}{\text{exp}} \right) \text{d}y \geq \text{exp} \left( \frac{b}{\text{exp}} \right) \text{d}y \)

Lower bound by definition

For almost all \( \beta (c) \) (\( \beta \)-measure) \( \int \text{exp} \left( \frac{b}{\text{exp}} \right) \text{d}y \geq \epsilon \).

So

\[ \sigma \geq e^{-t} \int \text{exp} \left( \frac{b}{\text{exp}} \right) \text{d}y \]

Now let \( S'(c, \epsilon) \) be the complement of \( S(c, \epsilon) \). Then

\[ \lim_{t \to 0} \text{P}_x \left( S'(c, \epsilon) \right) = \lim_{t \to 0} \text{P}_x \left( \beta ; \epsilon \right) \in \left( \epsilon, \epsilon \right) ; \]

Because the ergodic theorem which says that \( L^2 \) weak conv.

Moreover \( \lim_{t \to 0} \text{P}_x \left( \beta ; \epsilon \right) \in \left( \epsilon, \epsilon \right) ; \)

by the ergodic theorem since this is a Leb neighborhood.

But \( S'(c, \epsilon) = S(c, \epsilon) \cup \left\{ \beta ; \epsilon \right\} \in \left( \epsilon, \epsilon \right) ; \)

\[ \lim_{t \to 0} \text{P}_x \left( S(c, \epsilon) \right) = 0 \]

\[ \lim_{t \to 0} \frac{\log \sigma}{\epsilon} \geq -\epsilon - \frac{\frac{1}{2} (\text{exp}^{-1})^2}{\int \text{exp} \left( \frac{b}{\text{exp}} \right) \text{d}y} \]

Let \( \epsilon \to 0 \) and we have the desired result. With a mild extra hypothesis one can get the upper and the lower estimates are the same. So what we have proved is
\[
\lim_{t \to \infty} \frac{1}{t} \log E^t e^{+t \Phi(L_0(\beta_t, \omega))} = \sup_{F \in \mathcal{G}^+} \left[ I(F) - \mathcal{I}(F) \right]
\]

\[
I(F) = \mathcal{I}(F) = \frac{1}{2} \int_{-\infty}^{\infty} \frac{d^2}{dx^2} dF
\]

Malliavin Calculus: Frechet, Volterra, Gelfand. The derivative in function space.

\[ G : (0, 1) \to \mathbb{R} \quad \text{and from the Calculus of Variations} \]
\[
\delta G(x) = \frac{d}{d\beta} G[\beta + h\mathcal{V}] \bigg|_{\beta=0}. \quad \text{This is a linear functional of } \delta \]
\[
\delta \mathcal{V} = \int \left( \frac{d\delta}{d\beta} \right) Y(s) ds
\]

Levy called this "concrete functional analysis," also Daniell worked.

Is there a nice relationship between the derivative and the integral?

\[ G : (0, 1) \to \mathbb{R} \]
\[
E \left( \frac{dG}{ds} \right) \quad \text{on scratch paper}
\]
\[
E \left( \frac{dG}{ds} \right) = \int \frac{dG}{ds} e^{-\frac{1}{2} \left( \beta - c \right)^2} ds \quad \text{integrate by parts}
\]
\[
= - \left[ G[\beta] \beta''(s) e^{-\frac{1}{2} \left( \beta - c \right)^2} ds \right]_{-\infty}^{\infty}
\]
\[
= - \frac{d^2}{ds^2} \left[ G[\beta] \beta''(s) e^{-\frac{1}{2} \left( \beta - c \right)^2} ds \right] = \frac{d^2}{ds^2} E \left( \beta'''(s) G[\beta] \right) \quad \text{so}
\]
\[
E \left( \frac{dG}{ds} \right) = - \frac{d^2}{ds^2} E \left( \beta'''(s) G[\beta] \right) \quad \text{we also see by using inverse}
\]
\[
\int_{-\infty}^{\infty} \min (\beta, s) E \left( \frac{dG}{ds} \right) ds = E \left( \beta''(s) G[\beta] \right)
\]
Examples

\[ G[\beta] = \beta c_0, \quad \sigma \in [0,1] \]

\[
E \left( \beta c \sigma \right) G[\beta] = E \left( \beta c \sigma \beta(\sigma) \right) = \min(c_0, \sigma)
\]

\[
\frac{S[G[\beta]]}{S[\beta(\sigma) + \delta(\sigma)]} = \frac{S[\delta(\sigma)]}{S[\beta(\sigma)]} \quad \text{then} \quad \int \min(c_0, \sigma) E[\delta(\sigma)] d\sigma = \min(c_0, \sigma).
\]

The proof is four lines long using the Cameron translation formula.

The n-dimensional Laplacian is

\[
\sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} \quad \text{"going all the way"} \quad \int \frac{\partial^2}{\partial \beta(\sigma)^2} d\sigma \quad \text{but this is tiring.}
\]

Now we work a little

\[
E \left( \beta c_0 \sigma \right) G[\beta] = \int \min(c_0, \sigma) E \left( \beta c_0 \sigma \right) \frac{S[G[\beta]]}{S[\beta(\sigma)]} d\sigma
\]

\[
+ \int \min(c_0, \sigma) E \left[ G[\beta] \delta(\sigma) \right] d\sigma \quad \text{repeating}
\]

\[
= \int \min(c_0, \sigma) \int \min(c_0, \sigma) E \left[ \frac{S[G[\beta]]}{S[\beta(\sigma)]} \right] d\sigma d\sigma_2 + \int E \left[ G[\beta] \right] d\sigma
\]

Consider \( E \left[ h(\beta c_0 \sigma) G[\beta] \right] = E \left[ \frac{1}{2 \pi} \right] \delta y \left[ \frac{\min(c_0, \sigma)}{\frac{\partial^2}{\partial \beta(\sigma)^2}} \right] \]

\[ G[\beta] \]

If we put in an exponential we get a nice formula. Let

\[ h(y) = e^{my} \]

moment generating function

\[ E \left[ e^{\eta \left( \beta c_0 \sigma \right)} G[\beta] \right] = E \left[ e^{\eta \min(c_0, \sigma) \frac{S[G[\beta]]}{S[\beta(\sigma)]}} G[\beta] \right] \cdot e^{\frac{\eta^2 \sigma}{2}}.
\]

Why do we want such a thing? Let \( \sigma \in (0, \sigma_0) \) and...
\[ u(q) = E \left( e^{\frac{i}{\hbar^2} q \cdot \phi(x) \cdot \rho(x) - \frac{i}{\hbar} q \cdot \dot{\rho}(x)} \right) \]

Let \( c \in [0,1] \)
\[
\frac{\delta u}{\delta q} = i E \left[ \rho(x) e^{\frac{i}{\hbar^2} q \cdot \phi(x) \cdot \rho(x) - \frac{i}{\hbar} q \cdot \dot{\rho}(x)} \right]
\]

\[
\begin{align*}
&= i \int \min(t, \sigma) E \left[ \phi(x) \cdot \rho(x) - \rho(x) \cdot \dot{\rho}(x) \right] e^{\frac{i}{\hbar^2} q \cdot \phi(x) \cdot \rho(x) - \frac{i}{\hbar} q \cdot \dot{\rho}(x)} \, dx \\
&= -i \left( \int \min(t, \sigma) \phi(x) \cdot \rho(x) \, dx \right) \frac{\delta u}{\delta q} - \int \min(t, \sigma) \frac{\delta \rho}{\delta q} \, dx
\end{align*}
\]

with \( \rho(x) = \sqrt{\cosh(t)} \). How do we solve these? This is a Fourier transform in function space. This is a tough thing.
Let $\beta(s) ; 0 \leq s \leq \infty$, $\beta(0) = 0$ and $\alpha \in (0, \infty)$ and define $\beta$.

I: $\varphi(x, t) = E \left\{ \exp \left( i \int_0^t g(s) \cos \beta(s) ds - \frac{1}{2} \int_0^t \beta^2(s) ds \right) \delta(\beta(t) - x) \right\}$

we can explicitly get, by random Fourier Series

$I$: also $u(x, t)$ solves

$$\frac{du}{dt} = \frac{1}{2} \frac{d^2 u}{dx^2} - \frac{x^2}{2} u + x \varphi(x, t) u$$

$$u(x, t) \to 0 \text{ as } t \to \infty$$

$$u(x, t) \to \delta(x) \text{ as } t \to 0$$

Theorem: $u(x, t)$ is the unique solution of

$$\frac{du}{dt} = \int_0^\infty g(s) ds \delta(s) \frac{du}{dx}$$

$$\lim_{t \to \infty} \frac{du}{d\beta(s)} = x u$$

$$\frac{d}{dt} u(x, t) = -\frac{1}{2} \frac{d^2 u}{dx^2} + \frac{x^2}{2} u + x \varphi(x, t)$$

$$u(0, x, t) \to \delta(x) \text{ as } t \to 0.$$
\[ u(t, x, t) = \mathbb{E} \left[ e^{-\int_0^t \beta(s) \, ds} \int_0^t e^{\int_0^s \beta(r) \, dr} \, ds - \frac{1}{2} \int_0^t \beta^2(r) \, dr \right] \]

\[ = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\mu x} \mathbb{E} \left[ e^{\int_0^t \beta(s) \, ds} e^{-\int_0^t \beta(s) \, ds} \right] \, d\mu \]

\[ \text{at } t \in (0, T) \text{ } \text{differentiate w.r.t. } \mu: \]

\[ \frac{\partial u}{\partial \mu} = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\mu x} \mathbb{E} \left[ \beta(t) e^{\int_0^t \beta(s) \, ds} - \frac{1}{2} \int_0^t \beta^2(s) \, ds + i \mu \beta(t) \right] \, d\mu \]

Recall:

\[ \int_0^t \min(c, s) \mathbb{E} \left[ \frac{\mathbb{E} \left[ \mathcal{G} \mathcal{F} \right]}{\mathbb{E} \left[ \mathcal{G} \mathcal{F} \right]} \right] \, ds = \mathbb{E} \left[ \beta(t) \mathcal{G} \mathcal{F} \right] \]

\[ = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\mu x} \int_0^t \min(c, s) \mathbb{E} \left[ \frac{\mathbb{E} \left[ \mathcal{G} \mathcal{F} \right]}{\mathbb{E} \left[ \mathcal{G} \mathcal{F} \right]} \right] \, ds \]

\[ = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\mu x} \int_0^t \min(c, s) \mathbb{E} \left[ \mathcal{G} \mathcal{F} \right] \, ds \, d\mu \]

\[ \frac{\partial u}{\partial \mu} = \left( -\int_0^t \min(c, s) \, ds \right) u \]

\[ \left( -\int_0^t \min(c, s) \, ds \right) \frac{\partial u}{\partial \mu} - \frac{1}{2} \int_0^t \beta^2(s) \, ds - i \mu \frac{\partial u}{\partial x} \]

\[ \text{computing } \]

\[ \frac{\partial u}{\partial t} = \mathbb{E} \left[ \beta(t) e^{\int_0^t \beta(s) \, ds} - \frac{1}{2} \int_0^t \beta^2(s) \, ds \right] \]

\[ = i \mu u \text{ The rest follows from the Feynman–Kac Formula.} \]

\[ \text{that if instead of } \int_0^t \beta^2(s) \, ds \text{ was replaced by } \int_0^t V(\beta(s)) \, ds \]
uniquely determine $K(z) = \sqrt{2wz} \sinh t$ and so if we iterate this argument we get $c_1, c_2, c_3, \ldots$ etc and we get the whole expression for $w(y, t)$.

Markov Chains: The Gambler’s ruin problem:

\[ \text{prob. of winning} = p \]
\[ \text{prob. of losing} = q \]
\[ p + q = 1 \]

Starts with 2 dollars and plays until he has 0 or 1 dollars

\[ g(z) = \text{probability of his being ruined} \]  
\[ h(z) = \text{probability of his succeeding} \]

\[ g(z) = pqg(z+1) + qg(z-1) \quad \text{where} \quad g(0) = 0, \quad g(1) = 1 \]

Case 1: \( p = g = \frac{1}{2} \)
\[ g(z) = \frac{1}{2} (g(z+1) + g(z-1)) \quad \text{for} \quad z = 0, 1, 2 \]

Particular solutions: 1, 2, 2, 2, so general solution
\[ g(z) = A + Bz \]

\[ g(z) = 1 - \frac{z}{2} \]

which is what it should be intuitively.

Case 2: \( p < g \)
\[ g(z) = pqg(z+1) + (1-p)qg(z-1) \quad g(0) = 0, \quad g(1) = 1 \]

Particular solutions 1, and \( \left( \frac{q}{p} \right)^z = g(z) \)

Does \( \left( \frac{p}{q} \right)^z = p \left( \frac{q}{p} \right)^{z+1} + q \left( \frac{q}{p} \right)^{z-1} \) so the general solution is:
Recall: Let \( x_1, x_2, \ldots \) be i.i.d. with means 0 and variances \( \sigma^2 \).
\[ x_i = \varepsilon_1 + \cdots + \varepsilon_i, \quad \sigma^2 = \varepsilon_2^2 + \cdots + \varepsilon_i^2, \]
then
\[ P \left( \max_{1 \leq i \leq n} \left| x_i \right| \geq 2 \sigma \sqrt{n} \right) \leq \left( 2 - \sqrt{2} \right) \sigma \sqrt{n} \]

Note: \( \sum_{i=1}^{\infty} P(\varepsilon_i^2 > 3) = P(\max_{1 \leq i \leq n} 3 \geq 2 \sigma \sqrt{n}, i \text{ follows}. \)

Cor: Let \( \varepsilon_1, \ldots \) be i.i.d. r.v.'s with mean 0 and variances \( \sigma^2 \), for \( \sigma^2 \) sufficiently large,
\[ \lim_{n \to \infty} P \left( \max_{1 \leq i \leq n} \left| x_i \right| \geq 2 \sigma \sqrt{n} \right) \leq \frac{3}{2}. \]

Proof: If \( \lambda > 2\sqrt{2} \), then \( \lambda - \sqrt{2} \geq \frac{3}{2} \) and from the lemma with \( \lambda > 2\sqrt{2} \),
\[ P \left( \max_{1 \leq i \leq n} \left| x_i \right| \geq 2 \sigma \sqrt{n} \right) \leq 2 P \left( \left| x_i \right| \geq \frac{3}{2} \sigma \sqrt{n} \right) \text{ from C.L.T.} \]

\[ P \left( \varepsilon_i^2 \geq \frac{3}{2} \right) \to P \left( \varepsilon_i^2 \geq \frac{3}{2} \right) \quad N \sim N(0,1) \]
\[ \leq 8 \left( \frac{1}{2} \right) E[\varepsilon_i^3]^2 \]

Now choose \( \lambda \) so large that \( \varepsilon > \frac{3}{2} \), \( E[\varepsilon_i^3]^2 \) and we have it.

To establish why this should be true: \( F: [0,1] \to \mathbb{R} \) then
\[ P \left( \varepsilon_i \leq \frac{3}{2} \right) = \lim_{n \to \infty} P \left( \varepsilon_i \leq \frac{3}{2} \right) \quad \varepsilon_i = \left\{ \begin{array}{ll} \frac{3}{2} & \text{if } c = 0 \\ \frac{3}{2} & \text{if } c = 0 \end{array} \right. \]

It suffices to show weak convergence:
\[ E[\varepsilon_i^2] = \lim_{n \to \infty} E[\varepsilon_i^2] \quad \text{for bdd. cont. } F. \]

Let \( G \) be continuous and \( \varepsilon \) is bdd and continuous, but continuity of characteristic functions then does it.
A. T. P.

\[ E \{ F[\beta] \} = \lim_{k \to \infty} E \{ f_k(\beta(1), \ldots, \beta(1)) \} \text{ for polynomialization.} \]

Suppose \( F[\beta] = \int f(x) \text{ d}x \) then \( E \{ F[\beta] \} = \lim_{k \to \infty} E \{ f_k(\beta(1), \ldots, \beta(1)) \} \), where \( \beta(1) = \left[ \frac{c_k}{k} \right] \).

(Conjecture \( k = n \)) = \lim_{n \to \infty} E \{ f_k(\frac{c_k}{k}, \frac{c_k}{k}, \ldots, \frac{c_k}{k}) \} \) which is our theorem, so you gave the result.

1-dimensional C.L.T.: Let \( X_1, X_2 \) be i.i.d. r.v.'s mean 0 and variance 1 and \( \mu \) be their common d.f. and \( [a, b] \) be a half open interval then

\[
\lim_{n \to \infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(c_1, \ldots, c_n) \text{ d}x_1 \cdots \text{ d}x_n = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{e^{-x^2/2}} \text{ d}x.
\]

k-dimensional C.L.T.: Let \( X_1, X_2, \ldots, X_n \) be i.i.d. r.v.'s mean 0 variance 1 and \( S_n, \mu^{(n)} = X_1 + X_2 + \cdots + X_n \).

the \( \mu^{(n)} \) depend on \( n \)

Let \( N = u_1 + \cdots + u_n \) let \( P_N = P \{ \mu < S_n \} = P_{\beta_1 < \beta_2 < \cdots < \beta_k} \).

\[
\lim_{n \to \infty} P_N = \frac{1}{(2\pi)^{k/2}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \frac{1}{(2\pi)^{k/2}} \text{ d}u_1 \cdots \text{ d}u_k.
\]

This leads by routine approximation arguments to

\[
\lim_{n \to \infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f \left( \frac{S_1, \mu^{(1)}}{\sqrt{n}}, \frac{S_2, \mu^{(2)}}{\sqrt{n}}, \ldots, \frac{S_k, \mu^{(k)}}{\sqrt{n}} \right) \text{ d}y(X_1) \cdots \text{ d}y(X_k) = \frac{1}{(2\pi)^{k/2}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(u_1, \ldots, u_k) e^{-\frac{1}{2} u_1^2} \text{ d}u_1 \cdots \text{ d}u_k.
\]
Let $u_j = \sqrt{\nu} v_j$, and $v_j = \sqrt{\nu} \left( V_j - V_{j-1} \right)$, $j = 2, 3, \ldots, k$.

Then,

$$u_j = \sqrt{\nu} \left( V_j - V_{j-1} \right) = \sqrt{\nu} \int_0^\infty f \left( \sqrt{\nu} v, \sqrt{\nu} (v_2 - v_1), \ldots, \sqrt{\nu} (v_k - v_{k-1}) \right) \left( \frac{v - v_j}{2 \sqrt{\nu}} \right)^{1/2} e^{-\left( \frac{v-v_j}{2 \sqrt{\nu}} \right)^2} dv, \ldots, dv_k$$

$$= E \left\{ g \left( \beta \left( \frac{1}{2} \right), \beta \left( \frac{3}{2} \right), \ldots, \beta (1) \right) \right\}$$

and the left hand side is

$$= \lim_{n \to \infty} \int_{\mathbb{R}^n} g \left( \frac{s_1}{\nu_1}, \frac{s_2}{\nu_2}, \ldots, \frac{s_n}{\nu_n} \right) d\mu(x_1) \ldots d\mu(x_n)$$

**Special Case:** Let $n_j = \left[ \frac{\nu}{\nu_j} \right]$ and $n_{j+1} = n_j - n_j - 1$, then we get the result. If $g(t_1, \ldots, t_n)$ is a bdd. Borel measurable function and continuous a.e. on every finite $k$-dimensional interval then

$$\lim_{n \to \infty} \int_{\mathbb{R}^n} g \left( \frac{s_1}{\nu_1}, \frac{s_2}{\nu_2}, \ldots, \frac{s_n}{\nu_n} \right) d\mu(x_1) \ldots d\mu(x_n)$$

$$= E \left\{ g \left( \beta \left( \frac{1}{2} \right), \beta \left( \frac{3}{2} \right), \ldots, \beta (1) \right) \right\}.$$

**Kac's Drum:** In $\mathbb{R}^2 \cap \mathbb{N}$ with $\partial N$ consider $\frac{1}{2} \Delta u + \lambda u = 0$ with $u = 0$ on $\partial N = 0$. $\exists$ a discrete spectrum $\lambda_1, \ldots$ and $u(x, y)$, the normalized eigenfunctions. Consider

$$\sum_{\lambda \leq \lambda} \# \text{ of eigenvalues } \leq \lambda$$

This is an increasing function of $\lambda$. 

Hermann Weyl: \( \sum_{\lambda \leq \lambda} \frac{1}{\lambda^{1/2}} \sim \frac{2\lambda^{1/2}}{2\pi^2} \) as $\lambda \to \infty$.

Consider the numbers \( \sum_{x+y \leq \lambda} |u(x, y)| = \frac{1}{2\pi^2} \) as $\lambda \to \infty$. 

...
Start a B.M. at \((x_0, y_0)\) and \(p(x_0, y_0; x, y, t)\) be the p.d.f. of a 2-d B.M. starting at \((x_0, y_0)\) reaching \((x, y)\) in time \(t\) without crossing \(\partial \Omega\).

Einstein-Smoluchowski: \(p(x_0, y_0; x, y, t)\) solves
\[
\begin{cases}
\frac{\partial p}{\partial t} = \frac{1}{2} \Delta p & \text{in } \Omega \\
p = 0 & \text{on } \partial \Omega \\
\int_{\partial \Omega} p(x_0, y_0; x, y, t) = g(x_0, y_0, x, y, t)
\end{cases}
\]

Assume \(p = T(t) U(x, y)\) gives
\[
T(t) U = \frac{-1}{2} \Delta U \quad U = 0 \text{ on } \partial \Omega \quad U(0) = \delta(0)
\]

So \(T(t) = e^{-\frac{1}{2}t}\) and \(U\) is an eigenfunction of the Laplacian, i.e.,
\(U(\alpha_j, \beta_j) = u_j(\alpha_j, \beta_j)\) \(j = 1, 2, \ldots\)

So \(p(x_0, y_0; x, y, t) = \sum_{j=1}^{\infty} e^{-\frac{1}{2}j} u_j(x_0, y_0) u_j(x, y)\), and we know
\(p(x_0, y_0, x, y, t) = \sum_{j=1}^{\infty} e^{-\frac{1}{2}j} u_j^2(x_0, y_0)\).

Let \(p^*(x_0, y_0; x, y, t)\) be the p.d.f. for unrestricted B.M. starting at \((x_0, y_0)\) getting to \((x, y)\) at \(t\)
\[
p^*(x_0, y_0; x, y, t) = \frac{1}{2\pi t} e^{-\frac{1}{2t} - \frac{(x-x_0)^2}{2t} - \frac{(y-y_0)^2}{2t}}
\]
\[
\sum_{j=1}^{\infty} \sum_{\beta_j} e^{-\frac{1}{2}j} u_j^2(x_0, y_0) \sim \frac{1}{2\pi t} \text{ as } t \to 0.
\]

Karamata Tauberian Theorem: \(\int e^{-\lambda t} \, d\alpha(t)\) and let
\(\alpha(t)\) be non-decreasing on \(0, \infty\) and assume this Laplace-Stieltjes transform exists, then if \(f(t) = \int_{\infty}^{\infty} e^{-\lambda t} \, d\alpha(t)\) is asymptotic to \(A t^0\) as \(t \to 0\) where \(Y\) and \(A\) are constants, then
\[
\sum_{j=1}^{\infty} \sum_{\beta_j} e^{-\frac{1}{2}j} u_j^2(x_0, y_0) \sim \frac{1}{2\pi t} \text{ as } t \to 0.
\]
\( \chi(\lambda) \sim \frac{A \lambda^d}{\Gamma(\gamma + 1)} \) as \( \lambda \to \infty \) \((\lambda \to 0)\).

\[
\int_0^\infty e^{-t} \, d\chi(t) = \sum_{j=1}^\infty e^{-\frac{\lambda^2}{4j}} \nu_j(x_0, y_0) \quad \text{where} \quad \alpha(\gamma) = \sum_{j=1}^\infty \nu_j(x_0, y_0)
\]

by K-T theorem \( \chi(\lambda) \sim \frac{\lambda^2}{2\pi} \) as \( \lambda \to \infty \). Integrating over \( \Omega \) we get Weyl's theorem.
Karamata Tauberian Theorem: Let \( \langle \alpha \rangle \) be nondecreasing such that \( f(s) = \int_0^\infty e^{-st} \alpha(t) \, dt \) converges for \( s > 0 \). If for some \( \gamma > 0 \)
\[ f(s) \sim A s^{-\gamma} \quad s \to +\infty, \quad (s \to \infty) \text{ the-} \]
\[ g(t) \sim A t^{\gamma-1} \quad t \to \infty, \quad (t \to +\infty). \]

Note: 1. Choose \( \alpha(t) = \frac{t^\gamma}{\Gamma(\gamma+1)} \), then \( \alpha(t) = \frac{t^{\gamma-1}}{\Gamma(\gamma)} \) then
\[ f(s) = \int_0^\infty e^{-st} t^{\gamma-1} \, dt = \frac{\Gamma'(\gamma)}{s^\gamma} = \frac{\Gamma(\gamma+1)}{s^\gamma}. \]

2. If \( \alpha(t) = P \{ X \leq t \} \) and \( f(s) = E \{ e^{-sX} \} \)
\[ = \int_0^\infty e^{-st} dP \{ X \leq t \}. \]

Proof: (Lemma) Let \( g \) be Riemann integrable on \([0, 1]\) and let \( \gamma > 0 \).
For any \( \varepsilon > 0 \) polynomials \( p(x) \) and \( P(x) \rightarrow p(x) < g(x) < P(x) \)
\( x \in [0, 1] \) and
\[ \int_0^1 e^{-t} t^{\gamma-1} [P(e^{-t}) - P(e^{-t})] \, dt < \varepsilon. \]
(Proof: Consider \( g \) the indicator function of \( (\alpha, \beta) \) when \( (\alpha, \beta) \in [0, 1] \) Let \( \gamma > 0 \) be given. \( \exists \) a continuous function \( h(x) \Rightarrow \)
\( h(x) = g(x) \) \( x \in [0, 1] \) and \( \int_0^1 e^{-t} t^{\gamma-1} [h(e^{-t}) - g(e^{-t})] \, dt < \varepsilon \)
By the W-approximation theorem \( \exists \) a polynomial \( Q \Rightarrow \)
\[ |Q(x) - h(x)| \leq \varepsilon \quad \forall x \in [0, 1]. \]
Let \( P(x) = Q(x) + g \) \( x \in [0, 1] \)
Now \( \int_0^\infty e^{-t} t^{\gamma-1} [P(e^{-t}) - g(e^{-t})] \, dt \leq \int_0^\infty e^{-t} t^{\gamma-1} [Q(e^{-t}) - h(e^{-t})] \, dt \]
\[ + \int_0^\infty e^{-t} t^{\gamma-1} [h(e^{-t}) - g(e^{-t})] \, dt \]
\[ + \gamma \int_0^\infty e^{-t} t^{\gamma-1} \, dt. \]
we can also get $p(x) \leq g(x)$ in an analogous manner $\Rightarrow$

\[
\int_0^\infty e^{-t} e^{-t'} \left[ g(e^{-t}) - p(e^{-t}) \right] dt \leq 2 \int_0^\infty e^{-t} e^{-t'} dt.
\]

So, the lemma is true for indicator functions of intervals in $[0,1]$. Therefore, the lemma is also true for step functions on $[0,1]$ having a finite number of jumps.

Now let $g$ be R.I. on $[0,1]$. Let $M = \sup g(x)$. Choose $\delta > 0$ with $R > 2M \int_0^\delta e^{-t} e^{-t'} dt + \int_\delta^\infty e^{-t} e^{-t'} dt < \frac{\epsilon}{6}$. Since $g$ is R.I. on $[0,1]$ 3 step functions $g_1$ and $g_2$ on $[0,\delta]$ each with a finite number of jumps $\Rightarrow g_1(e^{-t}) \leq g(e^{-t}) \leq g_2(e^{-t})$ $t \in [0,\delta]$ and

\[
\int_\delta^\infty [g_2(e^{-t}) - g_1(e^{-t})] e^{-t} e^{-t'} dt < \frac{\epsilon}{6}.
\]

Now let $g_2(e^{-t}) = M$ $0 \leq t \leq \delta$ and $R < t < \infty$ $g_1(e^{-t}) = M$ $\Rightarrow$ then $\Omega$ holds everywhere. By the first part of the proof 3 polynomials $p(x)$ and $P(x) \Rightarrow$

\[
p(x) + g_1(x) \leq g(x) \leq g_2(x) < P(x) \quad x \in [0,1]
\]

$\Rightarrow \int_0^\infty e^{-t} e^{-t'} \left[ p(e^{-t}) - g_1(e^{-t}) \right] dt \leq \frac{\epsilon}{3}$

$\int_0^\infty e^{-t} e^{-t'} \left[ g_2(e^{-t}) - p(e^{-t}) \right] dt \leq \frac{\epsilon}{3}$
\[ \int_0^\infty \int_{\frac{\epsilon}{6} + \frac{\epsilon}{6}}^R g_3(e^{-t}) - g_1(e^{-t}) \, dt = \int_0^\infty \int_{\frac{\epsilon}{6} + \frac{\epsilon}{6}}^R g_3(e^{-t}) - g_1(e^{-t}) \, dt + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \frac{\epsilon}{3} \]

and finally
\[ \int_0^\infty \int_{\frac{\epsilon}{6} + \frac{\epsilon}{6}}^R [P(e^{-t}) - p(e^{-t})] \, dt \]

\[ = \int_0^\infty \int_{\frac{\epsilon}{6} + \frac{\epsilon}{6}}^R [P(e^{-t}) - g_2(e^{-t}) + g_2(e^{-t}) - g_1(e^{-t}) + g_1(e^{-t}) - p(e^{-t})] \, dt \]

\[ \leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon \]

(Theorem) Let \( \langle x \rangle \) be non-decreasing and \( \epsilon > 0 \) \( \int_0^\infty e^{-st} \langle x \rangle \, dx = \int_0^\infty f(s) \, ds \) converges for all \( s > 0 \). Assume for some \( Y > 0 \) \( \int_0^Y f(s) \, ds \rightarrow 0 \) \( s \rightarrow 0^+ \) \( (s \rightarrow 0^+ ; \, s \rightarrow 0^+ \), \( \langle x \rangle \) be of bdd variation on \( [0,1] \), then:

1. \( \int_0^\infty e^{-st} \langle x \rangle \, dx = \int_0^Y \frac{1}{p(y)} \int_0^\infty e^{-su} \langle x \rangle \, du \, ds \) as \( s \rightarrow 0^+ \) \( (s \rightarrow 0^+ \).

(Proof) Let \( \langle x \rangle \) be continuous except perhaps at \( x_1, x_2, \ldots \), and let \( g_3 (e^{-t}) \) be continuous except perhaps at \( y_0, y_1, \ldots \) and let \( E \) be the set of points \( x_i / x_j \), \( i,j = 0,1,2, \ldots \). If \( S \notin E \) the integral on the l.h.s. of (0) exists. Since \( E \) is countable, \( E^c \) is dense in \( (0,\infty) \) and so we have no problem in letting \( s \rightarrow 0^+ \) or \( s \rightarrow \infty \) through points in \( E^c \).

Let \( x_0 \) be given. From the lemma \( \exists \text{per} \) and \( \text{per}x \) \( \exists \)

\[ p(x) < g(x) < \text{per}x \] \( \quad x \in E, x \in \exists \)

\[ (2) \quad \frac{1}{\text{per}x} \int_0^\infty \int_{\frac{\epsilon}{6} + \frac{\epsilon}{6}}^R [P(e^{-t}) - p(e^{-t})] \, dt \leq \epsilon. \]

Now
\[ (3) \quad \frac{1}{\text{per}x} \int_0^\infty \int_{\frac{\epsilon}{6} + \frac{\epsilon}{6}}^R \langle x \rangle \, dx \leq \frac{1}{\text{per}x} \int_0^\infty \int_{\frac{\epsilon}{6} + \frac{\epsilon}{6}}^R \langle x \rangle \, dx \]

and \( \langle x \rangle \) is non-decreasing so

\[ (4) \quad \int_0^\infty \int_{\frac{\epsilon}{6} + \frac{\epsilon}{6}}^R \langle x \rangle \, dx \leq \int_0^\infty \int_{\frac{\epsilon}{6} + \frac{\epsilon}{6}}^R \langle x \rangle \, dx \leq \int_0^\infty \int_{\frac{\epsilon}{6} + \frac{\epsilon}{6}}^R \langle x \rangle \, dx \]
In the hypothesis we had $f(s) \sim \frac{1}{s^2}$ as $s \to 0^+$ ($s \to \infty$). Replace $s$ by $(nt)^+$, where $n \in \mathbb{Z}^+$ we get

$$
\int_0^\infty e^{-st} s^r \, dt = \frac{1}{\Gamma(r)} \int_0^\infty e^{-t} t^{r-1} \, dt
$$

$s \to 0^+$ ($s \to \infty$)

It follows then that

$$
\int_0^\infty e^{-st} Q(e^{-st}) \, dt = \frac{1}{\Gamma(r)} \int_0^\infty e^{-t} t^{r-1} Q(e^{-t}) \, dt
$$

Multiply through (4) by $s^r$ and let $s \to 0^+$ ($s \to \infty$) through values of $E$. So we get

$$
\frac{1}{\Gamma(r)} \int_0^\infty e^{-t} t^{r-1} p(e^{-t}) \, dt \leq \lim_{s \to 0^+} s^r \int_0^\infty e^{-st} g(e^{-st}) \, dt
$$

$$
\leq \lim_{s \to 0^+} s^r e^{E^2} \int_0^\infty e^{-t} t^{r-1} p(e^{-t}) \, dt
$$

the limit exists from (3) and (2), and the fact that $e^{E}$ is arbitrary

$$
\lim_{s \to 0^+} s^r \int_0^\infty e^{-st} g(e^{-st}) \, dt = \frac{1}{\Gamma(r)} \int_0^\infty e^{-t} g(e^{-t}) \, dt.
$$

Cauchy–Schwarz Theorem is a corollary.

Proof: Let $g(x) = \begin{cases} 0 & 0 \leq x \leq \frac{1}{e} \\ \frac{1}{x} & \frac{1}{e} < x < 1 \\ 0 & 1 < x < \infty \end{cases}$, i.e $g(e^{-t}) = \begin{cases} e^t & 0 < t < \frac{1}{s} \\ 0 & -t > 0 \end{cases}$

or $g(e^{-st}) = \begin{cases} e^{st} & 0 < t < \frac{1}{s} \\ 0 & -t > 0 \end{cases}$. So (a) becomes
Potential Theory: In $\mathbb{R}^3$, let $\Omega$ be a bounded closed region, and $C$ is the space of all continuous functions $\tilde{v}(\cdot)$ starting from the origin. Let $X(\tilde{v})$ be the indicator function of $\Omega$ and consider the functional on $C$ given by

$$T_\omega (\tilde{v}, \tilde{v}(\cdot)) = \int_0^\infty X(\tilde{v} + \tilde{v}(\cdot)) \, d\tau \quad \tilde{v} \in \mathbb{R}^3$$

the total occupation time of $\tilde{v}(\cdot)$ in $\Omega$, translated by $\tilde{v}$. (3d B.M. starting at $0$)

Impose on $C$ the Wiener measure. Consider

$$E \{ T_\omega (\tilde{v}, \tilde{v}(\cdot)) \} = \int_0^\infty P \{ \tilde{v} + \tilde{v}(\cdot) \in \Omega \} \, d\tau$$

by Fubini's theorem we get

but $P \{ \tilde{v} + \tilde{v}(\cdot) \in \Omega \} = \frac{1}{(2\pi \tau)^3} \int e^{-\frac{|\tilde{v} - \tilde{v}|^2}{2\tau}} \, d\tilde{v}$ so that

$$E \{ T_\omega (\tilde{v}, \tilde{v}(\cdot)) \} = \int_0^\infty \int e^{-\frac{|\tilde{v} - \tilde{v}|^2}{2\tau}} \, d\tilde{v} \, d\tau$$

here we see why we are in $\mathbb{R}^3$. This is convergent

$$= \frac{1}{2\pi} \int \frac{d\tilde{v}}{|\tilde{v} - \tilde{v}|} < \infty$$

We see that in $\mathbb{R}^3$ a.e. path starting from $\tilde{v}$ spends a finite time in $\Omega$. 

We see now the the $k$th moment of the occupation time is
\[
E\left\{ T_n^k (\hat{\gamma}, \hat{\nu} (\cdot)) \right\} = \frac{k!}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{d\hat{\gamma}}{1 - \hat{\gamma}} \frac{d\hat{\nu}}{1 - \hat{\nu}} \cdots \frac{d\hat{\nu}}{1 - \hat{\nu}}
\]

We examine $k = 2$:
\[
E\left\{ T_n^2 (\hat{\gamma}, \hat{\nu} (\cdot)) \right\} = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} P_e \left\{ \hat{\gamma} + \hat{\nu} (\cdot) \right\} \left\{ \hat{\nu} (\cdot) \right\} \in \mathbb{R}^3 \, d\hat{\gamma}, d\hat{\nu}
\]
\[
= 2 \int_0^{\infty} d\gamma \int_0^{\infty} d\nu \int_0^{\infty} \frac{1}{(2\pi)^n} e^{-\frac{\gamma^2 + \nu^2}{2\gamma}} \left[ \frac{1}{2\pi (\gamma + \nu)} \right]^{\frac{3}{2}} e^{-\frac{\gamma^2 + \nu^2}{2\pi (\gamma + \nu)}} d\gamma, d\nu
\]
\[
= \frac{2}{(2\pi)^n} \int_0^{\infty} \frac{1}{\sqrt{\gamma} \sqrt{\nu} \sqrt{\gamma + \nu}} d\gamma, d\nu, \text{ etc.}
\]
The formula for the $k$th moment suggests we consider the eigenvalue problem:
\[
\frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{\varphi (\rho)}{i \rho - \hat{\rho}} \, d\rho = \lambda \varphi (\rho), \quad \rho \in \mathbb{R}.
\]
This kernel is Hilbert-Schmidt as
\[
\int_{\mathbb{R}^2} \frac{1}{i \rho - \hat{\rho}} \, d\rho, d\hat{\rho} < \infty
\]
and will now show it is positive definite:
\[
\begin{align*}
\therefore \frac{1}{2\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{\varphi (\rho) \varphi (\hat{\rho})}{i \rho - \hat{\rho}} \, d\rho, d\hat{\rho} &> 0, \quad \varphi (\rho) \neq 0 \text{ in } L^2 (\mathbb{R}) \\
\text{note } \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{1}{i \rho - \hat{\rho}} \, d\rho, d\hat{\rho} &= \int_0^{\infty} \frac{1}{(2\pi)^n} e^{-\frac{\gamma^2 + \nu^2}{2\gamma}} \, d\gamma \int_0^{\infty} \frac{1}{(2\pi)^n} \left( \frac{\gamma^2}{2\gamma} \right)^{\frac{3}{2}} \int_0^{\infty} \frac{e^{-\frac{\gamma^2 + \nu^2}{2\gamma}} \, d\gamma}{i \gamma - \hat{\rho}} \, d\nu
\end{align*}
\]
Therefore we have $\lambda_1, \lambda_2, \ldots$ discrete positive spectrum and a complete orthonormal basis for $L^2(\mathbb{R})$ in the eigenfunctions.

**Lemma.** \( \frac{1}{k!} E \{ T_n^k (\bar{y}, \bar{r}(\cdot)) \} = \sum_{j=1}^{\infty} \lambda_j^{k-1} \int_\mathbb{R} \psi_j(\bar{r}) d\bar{r} \int_\mathbb{R} \frac{1}{1 + \frac{\bar{y}^2}{\bar{r}^2}} \) \( \psi_j(\bar{r}) d\bar{r} \) \( \bar{y} \in \mathbb{R}^3 \) and if $\bar{y} \in \mathbb{R}$ \( \lambda_j \psi_j(\bar{r}) \).

**Proof:** \( \frac{1}{k!} E \{ T_n^k (\bar{y}, \bar{r}(\cdot)) \} = \frac{1}{2\pi} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \sum_{j=1}^{\infty} \lambda_j^{k-1} \int_\mathbb{R} \bar{r}^{j-1} \psi_j(\bar{r}) \bar{r}^{j-1} \phi_j(\bar{r}) d\bar{r} \) \( \bar{r} \) \( \int_\mathbb{R} \psi_j(\bar{r}) d\bar{r} \).

Consider the moment generating function, with $z \in \mathbb{C}$

\[
E \left\{ e^{z T_n (\bar{y}, \bar{r}(\cdot))} \right\} = \sum_{k=0}^{\infty} \frac{z^k}{k!} E \{ T_n^k (\bar{y}, \bar{r}(\cdot)) \} = 1 + \frac{z}{2\pi} \sum_{j=1}^{\infty} \left( \frac{1}{\lambda_j^2} \right) \int_\mathbb{R} \psi_j(\bar{r}) d\bar{r} \int_\mathbb{R} \frac{\psi_j(\bar{r})}{1 + \frac{\bar{y}^2}{\bar{r}^2}} d\bar{r}.
\]

This series converges if $|z|$ is small enough ($< \frac{1}{\lambda_{\max}}$). The left hand side is analytic if $\text{Re} \, z > 0$ since $T_n \geq 0$. The right hand side is analytic for $\text{Re} \, z > 0$, so by analytic continuation this identity holds with $\text{Re} \, z > 0$. 
Review: A closed bdd. set in $\mathbb{R}^3$ and we have

$$T_n = \int_0^\infty \mathcal{X}_n(\vec{y} + \vec{r}(\omega))\,d\omega,$$

with Wiener measure on $\omega \in \mathcal{C}$ with $\vec{r}(0) = \vec{0}$. (Total occupation time in $\Omega$ starting at $\vec{y}$).

$$E\left\{ T_n (\vec{y}, \vec{r}(\omega)) \right\}^3 = \frac{1}{2\pi} \int_{|\vec{y}|}^\infty \frac{d\rho}{|\vec{y} - \vec{\rho}|} < \infty$$

and therefore all paths starting at $\vec{y}$ spend at most a finite time in $\Omega$. Also we have

$$E\left\{ T_n^k (\vec{y}, \vec{r}(\omega)) \right\} = \frac{k!}{(2\pi)^k} \int_0^\infty \int_{|\vec{y}|}^\infty \cdots \int_{|\vec{y}|}^\infty \frac{d\vec{\rho}}{|\vec{y} - \vec{\rho}|} \frac{d\vec{\rho}}{|\vec{y} - \vec{\rho}|} \cdots \frac{d\vec{\rho}}{|\vec{y} - \vec{\rho}|},$$

so let $\lambda_1, \lambda_2, \ldots$ and $\varphi_1, \varphi_2, \ldots$ be the eigenvalues and normalized eigenfunctions of

$$\frac{1}{2\pi} \int \frac{\varphi(\vec{\rho})\,d\vec{\rho}}{|\vec{y} - \vec{\rho}|} = \lambda \varphi(\vec{\rho}), \quad \vec{\rho} \in \mathcal{C},$$

and also

$$\frac{1}{k!} E\left\{ T_n^k (\vec{y}, \vec{r}(\omega)) \right\} = \sum_{j=0}^\infty \lambda_j^{k+1} \int \varphi_j(\vec{\rho})\,d\vec{\rho} \frac{1}{2\pi} \int \frac{\varphi_j(\vec{\rho})\,d\vec{\rho}}{|\vec{y} - \vec{\rho}|} \varphi_j(\vec{y}) \,d\vec{y}, \quad \vec{y} \in \mathcal{C}.$$

Set $u_{\lambda}$ and define

$$h(\vec{y}, \vec{u}) = E\left\{ e^{-u T_n (\vec{y}, \vec{r}(\omega))} \right\} =$$

$$1 - \frac{u}{2\pi} \sum_{j=1}^\infty \left( \frac{1}{1 + \lambda_j u} \right) \int \varphi_j(\vec{\rho})\,d\vec{\rho} \int \frac{\varphi_j(\vec{\rho})\,d\vec{\rho}}{|\vec{y} - \vec{\rho}|}$$

(in $\mathcal{C}$).

Note: The series converges uniformly on compact sets because

$$\frac{1}{1 + \lambda_j u} < 1 \quad \text{and} \quad \left( \sum_{j=1}^\infty \int \varphi_j(\vec{\rho})\,d\vec{\rho} \int \frac{\varphi_j(\vec{\rho})\,d\vec{\rho}}{|\vec{y} - \vec{\rho}|} \right)^2 \leq$$
A.T.P. by Parceval

\[
\leq \sum_{j=1}^{\infty} \left( \int_{\mathcal{V}} \varphi_j(\textbf{v}) \, d\textbf{v} \right)^2 \sum_{j=1}^{\infty} \left( \int_{\mathcal{V}} \frac{\varphi_j(\textbf{\rho})}{|\textbf{\rho} - \textbf{y}|} \, d\textbf{\rho} \right)^2 = 1 \mathcal{L} \left( \int_{\mathcal{V}} \frac{d\textbf{\rho}}{|\textbf{\rho} - \textbf{y}|} \right) < \infty,
\]

so we have uniform convergence by Weierstrass M-test and hence analyticity.

If \( \textbf{y} \in \mathbb{R}^2 \) this last becomes \( h(\textbf{y}, \textbf{u}) = 1 - \sum_{j=1}^{\infty} \frac{2j}{1+\lambda_j} \int_{\mathcal{V}} \varphi_j(\textbf{v}) \, d\textbf{v} \varphi_j(\textbf{y}) \), so

\[
\frac{1}{2\pi} \int_{\mathcal{V}} \frac{h(\textbf{y}, \textbf{u})}{|\textbf{y} - \textbf{\rho}|} \, d\textbf{y} = \frac{1}{2\pi} \int_{\mathcal{V}} dy \varphi_{j}(\textbf{\rho}) \frac{1}{1+\lambda_j} \int_{\mathcal{V}} \frac{\varphi_j(\textbf{v})}{|\textbf{\rho} - \textbf{y}|} \, d\textbf{\rho} 2\pi \int_{\mathcal{V}} \frac{\varphi_j(\textbf{\rho})}{|\textbf{\rho} - \textbf{y}|} \, d\textbf{\rho}.
\]

But

\[
\frac{1}{2\pi} \int_{\mathcal{V}} \frac{dy \varphi_j(\textbf{\rho})}{|\textbf{y} - \textbf{\rho}|} = \sum_{j=1}^{\infty} \int_{\mathcal{V}} \frac{\varphi_j(\textbf{\rho})}{|\textbf{\rho} - \textbf{y}|} \, d\textbf{\rho} \int_{\mathcal{V}} \frac{\varphi_j(\textbf{\rho})}{|\textbf{\rho} - \textbf{y}|} \, d\textbf{\rho}.
\]

Thus we have

\[
\frac{1}{2\pi} \int_{\mathcal{V}} \frac{h(\textbf{y}, \textbf{u})}{|\textbf{y} - \textbf{\rho}|} \, d\textbf{y} = \sum_{j=1}^{\infty} \frac{1}{1+\lambda_j} \int_{\mathcal{V}} \varphi_j(\textbf{\rho}) \, d\textbf{\rho} \int_{\mathcal{V}} \frac{\varphi_j(\textbf{\rho})}{|\textbf{\rho} - \textbf{y}|} \, d\textbf{\rho}.
\]

i.e.

\[
\frac{1}{2\pi} \int_{\mathcal{V}} \frac{h(\textbf{y}, \textbf{u})}{|\textbf{y} - \textbf{\rho}|} \, d\textbf{\rho} = \frac{1}{u} (1 - h(\textbf{y}, \textbf{u})) \quad \forall \textbf{y} \in \mathbb{R}^3,
\]

or renaming the variables and any \( \textbf{y} \in \mathbb{R}^3 \)

\[
\frac{1}{2\pi} \int_{\mathcal{V}} \frac{h(\textbf{y}, \textbf{u})}{|\textbf{y} - \textbf{\rho}|} \, d\textbf{\rho} = \frac{1}{u} (1 - h(\textbf{y}, \textbf{u})).
\]  (2)

Observations:

If \( \textbf{y} \neq \textbf{0} \), \( h(\textbf{y}, \textbf{u}) \) is harmonic since (1) shows each term in the series is harmonic in \( \textbf{y} \), and the series converges uniformly on compact \( \mathcal{V} \) 's. Moreover, from (1):

\[
h(\textbf{y}, \textbf{u}) > 1 - \frac{1}{2\pi} \left( \sum_{j=1}^{\infty} \left( \int_{\mathcal{V}} \varphi_j(\textbf{\rho}) \, d\textbf{\rho} \right)^2 \right)^{\frac{1}{2}}
\]

\[
> 1 - \frac{1}{2\pi} \lim_{R \to \infty} \left( \int_{|\textbf{\rho}| > R} \frac{d\textbf{\rho}}{|\textbf{\rho} - \textbf{y}|} \right)^{\frac{1}{2}}
\]

and as

\[
0 \leq h(\textbf{y}, \textbf{u}) \leq 1 \quad \text{and so} \quad \lim_{|\textbf{y}| \to \infty} h(\textbf{y}, \textbf{u}) = 1.
\]  (3)
And for $\vec{y} \in \Omega \quad \Delta \left( \frac{1}{2} \int_{\partial \Omega} \frac{h(\vec{y}, u)}{|\vec{y} - \vec{\gamma}|^2} d\vec{\gamma} \right) = -4\pi h(\vec{y}, u)$.  
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So applying the Laplacian to both sides of (2) we get:

$$-2 \ h(\vec{y}, u) = -\frac{1}{u} \ \Delta \ h(\vec{y}, u) \quad \text{or we have}$$

$$\frac{1}{2} \ \Delta \ h(\vec{y}, u) - u h = 0 \quad \text{for} \quad \vec{y} \in \Omega.$$  

Consider $V(\vec{y}) = \lim_{u \to \infty} (1 - h(\vec{y}, u)) = P \mathbb{E} T_2 (\vec{y}, \vec{r}(c)) > 0$, this is the potential.

We can readily from the definition of the moment generating function, by example: Let $\Omega$ be a sphere of radius centered at $0$. It is clear that in this special case $h(\vec{y}, u)$ is spherically symmetric, one sees this from the moments of $T_2 (\vec{y}, \vec{r}(c))$. Since, as we have seen above, $h(\vec{y}, u)$ is harmonic outside $\Omega$, we have that

$$h(\vec{y}, u) = \frac{\alpha(u)}{|\vec{y}|} + \beta(u) \quad \vec{y} \notin \Omega.$$  

But since, from (3') we see $\beta(u) = 1$, we have $h(\vec{y}, u) = \frac{\alpha(u)}{|\vec{y}|} + 1 \quad \vec{y} \in \Omega.$

But from (3) for $\vec{y} \in \Omega \quad h(\vec{y}, u) = V(u) \frac{\sinh \left( \frac{|\vec{y}|}{2u} \right)}{|\vec{y}|}$, substituting this into the equation

$$\frac{1}{2\pi} \int_{\Omega} \frac{h(\vec{y}, u)}{|\vec{y} - \vec{\gamma}|^2} d\vec{\gamma} = \frac{1}{u} \left( 1 - h(\vec{y}, u) \right) \quad \text{we get}$$

$$V(u) = \sqrt{2u} \cosh(\sqrt{2u}) \quad \text{Now } h(\vec{y}, u) \text{ is continuous at } \vec{y},$$

from the uniform convergence of that series, therefore
\[ \frac{1}{\sqrt{2u}} \frac{\sinh(\sqrt{2u} \, a)}{\cosh(\sqrt{2u} \, a)} \cdot \frac{1}{a} = \frac{x\cos y}{a} + 1. \text{ Thus we finally get} \]

\[ h(\tilde{y}, u) = \begin{cases} 1 - \frac{a}{1 + \frac{\tanh(\sqrt{2u} \, a)}{a}} & \tilde{y} \notin S(0, a) \\ \frac{a}{1 + \frac{\tanh(\sqrt{2u} \, a)}{a}} & \tilde{y} \in S(0, a) \end{cases} \]

\[ \text{Thus } U(\tilde{y}) = \lim_{u \to 0} (1 - h(\tilde{y}, u)) = PE T_{S(0, a)} (\tilde{y}, \tilde{w} c) > 0 \]

Thus this is the capacity potential at $S(0, a)$, a very well known fact.

Back to the general case, we have $\tilde{y} \in \mathbb{R}^3$

\[ 1 - E \mathbb{E} e^{-u T_2 (\tilde{y}, \tilde{w} \alpha)} = \sum_{j=1}^{\infty} \frac{1}{2^{j+1}} \int y(x) \, dx \, \frac{1}{2\pi} \int_{\mathbb{R}^3} \frac{s(x, y)}{x \cdot \tilde{y} - \tilde{y} \cdot y} \, dy \]

As noted before, $0 \leq 1 - h(\tilde{y}, w) \leq 1$, and moreover,

\[ 1 - h(\tilde{y}, w) \leq 1 - h(\tilde{y}, w) \quad \text{if } w_1, w_2. \]

Since

\[ 0 \leq e^{-u T_2 (\tilde{y}, \tilde{w} \alpha)} \leq 1 \quad \text{and since } \lim_{u \to 0} e^{-u T_2 (\tilde{y}, \tilde{w} \alpha)} = \frac{\tilde{y} \cdot \tilde{w}}{\tilde{w} \cdot \tilde{w}}, \]

we have from the bounded convergence theorem.

\[ U(\tilde{y}) = \lim_{u \to 0} (1 - h(\tilde{y}, u)) = PE T_2 (\tilde{y}, \tilde{w} \alpha) > 0, \text{ hence also} \]

\[ U(\tilde{y}) = \lim_{u \to 0} \sum_{j=1}^{\infty} \frac{1}{2^{j+1}} \int y(x) \, dx \, \frac{1}{2\pi} \int_{\mathbb{R}^3} \frac{s(x, y)}{x \cdot \tilde{y} - \tilde{y} \cdot y} \, dy \]

and this holds $\forall \tilde{y} \in \mathbb{R}^3$.

Case 1: Consider $1^{st}$ $\tilde{y} \in S$. 


Clearly, the continuity of the paths $\mathfrak{p}$ immediately implies
\[ U(\mathfrak{p}) = PET_{\mathfrak{p}}(\mathfrak{p}, \mathfrak{p}(\cdot)) > 0 \] 3 = 1.

Remark on the side: Let $\mathfrak{p} \in \Sigma$ then we have $U(\mathfrak{p}) = 1$, so
\[ 1 = \lim_{n \to \infty} \sum_{j=1}^{\infty} \frac{3}{3 + \frac{a}{j}} \int_{2}^{\infty} \mathfrak{p}(\mathfrak{p}) \mathfrak{p}(\mathfrak{p}) d\mathfrak{p}, \quad \text{this is a summability result.} \]

Case 2: $\mathfrak{p} \not\in \Delta$, we have already seen that the function $1 - h(\mathfrak{p}, u)$ are harmonic in $\mathfrak{p}$ and we noted they are monotone decreasing in $u$ and moreover
\[ \lim_{u \to \infty} (1 - h(\mathfrak{p}, u)) = U(\mathfrak{p}) = PET_{\mathfrak{p}}(\mathfrak{p}, \mathfrak{p}(\cdot)) > 0 \] 3 exists, therefore by Harnack's theorem $U(\mathfrak{p})$ is harmonic for $\mathfrak{p} \not\in \Delta$.

Let $S_{0,0}$ be a sphere of radius $a$ centered at $0$ which contains $\Delta$.

\[ PET_{\mathfrak{p}}(\mathfrak{p}, \mathfrak{p}(\cdot)) > 0 \] 3 $ \leq PET_{S_{0,0}}(\mathfrak{p}, \mathfrak{p}(\cdot)) > 0 \] 3. From the example this last prob.

\[ PET_{\mathfrak{p}}(\mathfrak{p}, \mathfrak{p}(\cdot)) > 0 \] 3 $ \leq \frac{\text{a vol}}{V(\mathfrak{p})} \quad \mathfrak{p} \not\in S_{0,0}$. So
\[ \lim_{\mathfrak{p} \to \infty} U(\mathfrak{p}) = 0. \]

Case 3: $\mathfrak{p} \in \Delta$, and assume it is regular in the sense of Poincaré, \[ \exists \text{ a sphere } S(\mathfrak{p}, \mathfrak{p}) \] which lies entirely in $\Delta$ and contains $\mathfrak{p}$.

We have first for $\mathfrak{p} \not\in \Delta$
\[ U(\mathfrak{p}) = PET_{\mathfrak{p}}(\mathfrak{p}, \mathfrak{p}(\cdot)) > 0 \] 3 $ \geq PET_{S(\mathfrak{p}, \mathfrak{p})}(\mathfrak{p}, \mathfrak{p}(\cdot)) > 0 \] 3.
\[ \frac{3}{\nu - \gamma_0} \quad \text{At } \gamma \to \gamma_0, \quad \frac{\nu}{\nu - \gamma} \to 1 \quad \text{and since } \nu(\gamma) \leq 1 \quad \text{we have } \lim_{\gamma \to \gamma_0} \nu(\gamma) = 1. \]

Hence we see that if \( \mathcal{A} \) is a bounded closed region, each point of the boundary of which is regular in the sense of Poincare, then \( \nu(\gamma) \) is the capacity potential of \( \mathcal{A} \). Remember

\[ \nu(\gamma) = \lim_{\gamma \to \gamma_0} \frac{1}{2\pi} \int_\gamma \frac{1}{|\rho - \gamma_1|} \varphi(\rho) d\rho. \]

\( \forall \rho \in \mathbb{R}^3 \quad 1 - h(\gamma, u) = \frac{1}{2\pi} \int_{\gamma_0} \frac{u h(\rho, u)}{|\rho - \gamma_0|} d\rho. \) We note this implies

\[ \lim_{\gamma_0 \to \gamma_0} \frac{1}{2\pi} \int_{\gamma_0} \frac{u h(\rho, u)}{|\rho - \gamma_0|} d\rho \quad \text{again } \mathcal{A} \subset \mathcal{S}(\gamma_0) \]

then \( h(\gamma, u) \geq E \{ e^{-u T_{\gamma_0}} \} \geq E \{ e^{-u T_{\gamma_0}} \} \) therefore for

\( \gamma \not\in \mathcal{S}(\gamma_0) \quad h(\gamma, u) \geq 1 - \frac{a}{\gamma_0} \) or \( 1 - h(\gamma, u) \leq \frac{a}{\gamma_0} \) and so

\[ \frac{u}{2\pi} \int_{\gamma_0} h(\rho, u) d\rho \leq a. \]

Consider now the family of set function \( \mu_1(B) \) defined by

\[ \mu_1(B) = \frac{u}{2\pi} \int_{\gamma_0} h(\rho, u) d\rho \quad \text{for each } \gamma_0 \]

\( \mu_1(B) \) is a non-negative completely additive set function on the Lebesque measurable sets in \( \mathbb{R}^3 \), moreover \( \mu_1(\mathbb{R}^3) \leq a \). We see that

\[ 1 - h(\gamma, u) = \frac{1}{\gamma_0 - \gamma_1} \mu_1(d\gamma), \] since the set functions \( \mu_1(.) \) are uniformly bounded in \( u \) we can select, by the Helly selection principle a sequence \( u_n \to \infty \) where \( \mu_1(B) \to \mu(B) \)

where \( \mu(B) \) is again a non-negative completely additive set.
function and \( \mu(\mathbb{R}^3) = a \). So for \( \vec{y} \neq \vec{0} \) we have
\[
V(\vec{y}) = \lim_{u_n \to \infty} (1 - h(\vec{y}, u_n)) = \int \frac{1}{|\vec{y}| - \gamma} \mu(d\vec{p}) .
\]
Aside: In \( \mathbb{R}^d \) \( d > 3 \). \( T_{S_{2\pi},2\pi}(0, \vec{r}(\cdot)) \). What is
\[
P \left( \lim_{x \to 0} \frac{T_{S^2}(0, \vec{r}(x))}{\log(|x|)} = \frac{2}{p_d^2} \right) = 1 \text{ where } p_d \text{ is the } 1^{st} \text{ positive zero of } \zeta(x), \ x = \frac{d}{2} - 2.