Chapter 5

Binomial Coefficients

$\{1, 2, 3, 4\}$ how many ways to choose pairs

$\{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\}$

$\binom{4}{2} = 6$ "four choose two"

$\binom{n}{k}$ is a binomial coefficient

$$\binom{n}{k} = \frac{m}{l}$$

$$m = \underset{\text{first}}{n}\ \underset{\text{second}}{(n-1)} \cdots \underset{k\text{th}}{(n-k+1)}$$

number of choices

$l = k!$ # of permutations as they are equivalent n is the upper index, k the lower

can let upper index also be a real or complex, r

$$\binom{r}{k} = \begin{cases} \dfrac{r(r-1)\cdots(r+k-1)}{k!} & k \in \mathbb{Z},\ k \geq 0 \\ 0 & k < 0 \end{cases}$$

$$= \frac{r^{\underline{k}}}{k!} \quad \text{which is } = p(r)$$

an r^{th} degree polynomial!

Ex. $\binom{-1}{3} = (-1)(-2)(-3) / 3 \cdot 2 \cdot 1 = -1$

$r = 1/2$, $= -1$ are important cases.

can generaliz k as well, but later, also must have explicit restrictions.

$\binom{n}{n} = 1$ except when $n < 0$!

Have seen Pascal's (1623-1662) or Tartaglia's (1500-1557) triangle based on these. Memorize 1st 3 columns

$$\binom{r}{0} = 1, \quad \binom{r}{1} = r, \quad \binom{r}{2} = \frac{r(r-1)}{2}$$

Pascal's Δ: can detect many relations!

look at 56, 28, 36, 120, 126, 210:

Table 155 Pascal's triangle.

n	$\binom{n}{0}$	$\binom{n}{1}$	$\binom{n}{2}$	$\binom{n}{3}$	$\binom{n}{4}$	$\binom{n}{5}$	$\binom{n}{6}$	$\binom{n}{7}$	$\binom{n}{8}$	$\binom{n}{9}$	$\binom{n}{10}$
0	1										
1	1	1									
2	1	2	1								
3	1	3	3	1							
4	1	4	6	4	1						
5	1	5	10	10	5	1					
6	1	6	15	20	15	6	1				
7	1	7	21	35	35	21	7	1			
8	1	8	28	56	70	56	28	8	1		
9	1	9	36	84	126	126	84	36	9	1	
10	1	10	45	120	210	252	210	120	45	10	1

Alternating products around the hexagon:

$$56\cdot 36\cdot 210 = 28\cdot 120\cdot 126 = 423360$$

Note: This works with all such hexagons!

Some Simple Idenities

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!} \qquad n \geq k \geq 0 \in \mathbb{Z}$$

let $k \to n-k$

$$\binom{n}{n-k} = \frac{n!}{(n-k)!\,(n-n-k)!} = \binom{n}{k}$$

Pascal's Δ reads the same r to l as l to r!

Also makes combinatorial sense

What about $\binom{-1}{k} \stackrel{?}{=} \binom{-1}{-1-k}$?

No : $k=0$ $\quad 1 \stackrel{?}{=} 0$, $\binom{-1}{k} = \dfrac{(-1)(-2)\cdots(-k)}{k!}$

$= (-1)^k$

but $-1-k < 0$ when $k \geq 0$ so r.h.s. $= 0$

If $k < 0$ then l.h.s $= 0$ but $\binom{-1}{-1-k} = (-1)^{-1-k}$

So this is always false and symmetry must be used carefully! Symmetry does work when $k<0$ or $k>n$ because both sides are zero.

Absorption Identity

$$\binom{r}{k} = \frac{r}{k}\binom{r-1}{k-1} \qquad k \neq 0 \in \mathbb{Z},$$

when $k<0$ both sides $=0$. Similarly

$$k\binom{r}{k} = r\binom{r-1}{k-1} \qquad k \in \mathbb{Z} \text{ even } k=0.$$

$$\underset{(a)}{(r-k)\binom{r}{k}} = (r-k)\binom{r}{r-k} \quad \text{by symmetry}$$

$$= r\binom{r-1}{r-k-1} \quad \text{by absorption}$$

$$= \underset{(b)}{r\binom{r-1}{k}} \quad \text{by symmetry}$$

When does this hold in r? Symmetry needs $r-1 \in \mathbb{Z}^+$, but the two end pts. (a) & (b) are polynomials in r of degree $k+1$, their difference is as well, and since they agree (difference zero) at more than $k+1$ points the difference is identically zero & they are equal for all r. "Polynomial Argument"

Consider $\binom{r}{k} = \binom{r-1}{k} + \binom{r-1}{k-1}$ $k \in \mathbb{Z}$

when $r \in \mathbb{Z}^+$ in $P\Delta$ every element can be derived by add two from the previous row (we use this to write $P\Delta$) If true the by "Polynomial Argument" must be true for all r!

With $r \in \mathbb{Z}^+$ use combinatorics to explain things

$\binom{r}{k}$ "r choose k" one is identified then $\binom{r-1}{k}$ exclude that one,

and $\binom{r-1}{k-1}$ have that one. So this partitions the choices by whether or not they have a particular element, this is true always then by polynomial.

Recall $k\binom{v}{k} = v\binom{v-1}{k-1}$ &

$(v-k)\binom{v}{k} = v\binom{v-1}{k}$ add together

$$(v-k)\binom{v}{k} + k\binom{v}{k} = v\binom{v-1}{k} + v\binom{v-1}{k-1}$$

$$v\binom{v}{k} = v\left[\binom{v-1}{k} + \binom{v-1}{k-1}\right] \quad \square$$

Directly: $\binom{v-1}{k} + \binom{v-1}{k-1} = \frac{(v-1)^{\underline{k}}}{k!} + \frac{(v-1)^{\underline{k-1}}}{(k-1)!}$

$$= \frac{(v-1)^{\underline{k-1}}(v-k)}{k!} + \frac{(v-1)^{\underline{k-1}}\,k}{k!}$$

$$= \frac{(v-1)^{\underline{k-1}}}{k!}(v-k+k) = \frac{v(v-1)^{\underline{k-1}}}{k!} = \frac{v^{\underline{k}}}{k!} = \binom{v}{k}$$

Consider $\binom{5}{3} = \binom{4}{3} + \binom{4}{2}$

$$= \binom{4}{3} + \binom{3}{2} + \binom{3}{1}$$

$$= \binom{4}{3} + \binom{3}{2} + \binom{2}{1} + \binom{2}{0}$$

$$= \binom{4}{3} + \binom{3}{2} + \binom{2}{1} + \binom{1}{0} + \binom{1}{-1} = 0$$

, perhaps

$$\sum_{k\le n} \binom{r+k}{k} = \binom{r}{0} + \binom{r+1}{1} + \cdots + \binom{r+n}{n}$$

$$= \binom{r+n+1}{n} \quad n \in \mathbb{Z}$$

(a)

note: when $k<0$ terms vanish

New backwards

$$\binom{5}{3} = \binom{4}{3} + \binom{4}{2}$$

$$= \binom{3}{3} + \binom{3}{2} + \binom{4}{2}$$

$$= \binom{2}{3} + \binom{2}{2} + \binom{3}{2} + \binom{4}{2}$$

$$= \binom{1}{3} + \binom{1}{2} + \binom{2}{2} + \binom{3}{2} + \binom{4}{2}$$

$$= \cancel{\binom{0}{3}}^{\,0} + \binom{0}{2} + \binom{1}{2} + \binom{2}{2} + \binom{3}{2} + \binom{4}{2}$$

$$\sum_{0 \le k \le n} \binom{k}{m} = \binom{0}{m} + \binom{1}{m} + \cdots + \binom{n}{m} = \binom{n+1}{m+1}$$

(b)

"summation on the upper index," comb. interp.

choosing $m+1$ tickets from $n+1$ numbered 0 to n, $\binom{k}{m}$ ways when largest ticket $= k$

(b) $\Rightarrow$ (a), $r, n \in \mathbb{Z}^+$, use m instead of r

$$\sum_{k \le n} \binom{m+k}{k} = \sum_{-m \le k \le n} \binom{m+k}{k} = \sum_{-m \le k \le n} \binom{m+k}{m}$$

assures $m+k \ge 0$

symmetry OK

$$= \sum_{0 \le k \le m+n} \binom{k}{m} = \binom{m+n+1}{m+1} = \binom{m+n+1}{n}$$

(b) symmetry

Note (6) with $m=1$

$$\binom{0}{1}+\binom{1}{1}+\binom{2}{1}+\cdots+\binom{n}{1} = 0+1+\cdots+n = \binom{n+1}{2} = \frac{(n+1)n}{2}$$

$$\sum_{0\le k\le n} k^{\underline{m}} = \frac{(n+1)^{\underline{m+1}}}{m+1} \quad m,n \in \mathbb{Z}^+, \text{ now}$$

divide both sides by $m!$ & the addition form:

$$\binom{v}{k} = \binom{v-1}{k} + \binom{v-1}{k-1} \Longrightarrow$$

$$\binom{x+1}{m} - \binom{x}{m} = \binom{x}{m-1} = \Delta\left[\binom{x}{m}\right], \text{ so}$$

by finite calculus

$$\sum \binom{x}{m}\delta x = \binom{x}{m+1} + C$$

Binomial Theorem: $(x+y)^0 = 1x^0y^0$

$(x+y)^1 = 1x^1y^0 + 1x^0y^1$

$(x+y)^2 = 1x^2y^0 + 2x^1y^1 + 1x^0y^2$

$(x+y)^3 = 1x^3y^0 + 3x^2y^1 + 3x^1y^2 + 1x^0y^3$

$(x+y)^4 = 1x^4y^0 + 4x^3y^1 + 6x^2y^2 + 4x^1y^3 + 1x^0y^4$

Combinatorial: $\overbrace{(x+y)(x+y)\cdots(x+y)}^{n\text{-times}}$

coefficient of x^ky^{n-k} is $\binom{n}{k}$,

define $x^0=1 \ \forall x$, $x=0$ also,

this allows the binomial theorem to work when $x=0, y=0$, and/or $x=-y$

$$(x+y)^r = \sum_k \binom{r}{k} x^k y^{r-k} \qquad r \in \mathbb{Z} \geq 0 \text{ or } |x/y| < 1$$

When $r \in \mathbb{Z}^+$ this is finite, but this also hold when $r < 0$ or is $\in \mathbb{R}$ or $\mathbb{C}$.

If $x=y=1$ $\quad 2^n = \sum_k \binom{n}{k} = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n}$

If $x=-1, y=1$ $\quad 0^n = \sum_k (-1)^k \binom{n}{k} =$

$$\binom{n}{0} - \binom{n}{1} + \cdots + (-1)^n \binom{n}{n}, \qquad n \in \mathbb{Z} \geq 0$$

This last shows the rows in P's Δ sum up to zero w/alternating signs, except the top row.

When $r \in \mathbb{Z}$, usually $y=1$, $z \in \mathbb{C}$

$$(1+z)^r = \sum_k \binom{r}{k} z^k, \quad |z| < 1$$

Can't use the polynomial argument in this case since this may not be a polynomial, but

$$f(z) = \frac{f(0)}{0!} z^0 + \frac{f'(0)}{1!} z^1 + \frac{f''(0)}{2!} z^2 + \cdots \qquad \text{Taylor}$$

$$= \sum_{0 \leq k} \frac{f^{(k)}(0)}{k!} z^k$$

$f(z) = (1+z)^r \quad f^{(k)}(z) = r^{\underline{k}} (1+z)^{r-k}, \; z=0 \Rightarrow$

Sum converges because $\binom{r}{k} = O(k^{-1-r})$

Values when upper index is negative

Table 164 Pascal's triangle, extended upward.

n	$\binom{n}{0}$	$\binom{n}{1}$	$\binom{n}{2}$	$\binom{n}{3}$	$\binom{n}{4}$	$\binom{n}{5}$	$\binom{n}{6}$	$\binom{n}{7}$	$\binom{n}{8}$	$\binom{n}{9}$	$\binom{n}{10}$
−4	1	−4	10	−20	35	−56	84	−120	165	−220	286
−3	1	−3	6	−10	15	−21	28	−36	45	−55	66
−2	1	−2	3	−4	5	−6	7	−8	9	−10	11
−1	1	−1	1	−1	1	−1	1	−1	1	−1	1
0	1	0	0	0	0	0	0	0	0	0	0

$\binom{0}{0} = \binom{-1}{0} + \binom{-1}{-1}$ and $\binom{-1}{-1} = 0 \Rightarrow \binom{-1}{0} = 1$

$\binom{0}{1} = \binom{-1}{1} + \binom{-1}{0}$ and $\binom{0}{1} = 0 \Rightarrow \binom{-1}{1} = -1$

they look familiar but w.o. the negatives:

$$\binom{r}{k} = (-1)^k \binom{r-k-1}{k} \qquad k \in \mathbb{Z}, \text{ and}$$

$$r^{\underline{k}} = r(r-1)\cdots(r-k+1) = (-1)^k(-r)(1-r)\cdots(k-1-r)$$
$$= (-1)^k (k-r-1)^{\underline{k}} \quad \text{when } k \geq 0$$

both sides $= 0$ when $k < 0$, this is "negating the upper index" or "upper negation", a check

$$\binom{r}{k} = (-1)^k \binom{k-r-1}{k} = (-1)^k \cdot (-1)^k \binom{k-(k-r-1)-1}{k}$$
$$= (-1)^{2k} \binom{r}{k} = \binom{r}{k}, \text{ symmetrically:}$$

$$(-1)^m \binom{-n-1}{m} = (-1)^n \binom{-m-1}{n} \qquad m, n \in \mathbb{Z} \geq 0$$

both sides $= \binom{m+n}{n}$

Can also use upper negation on:

$$\sum_{k\le m} \binom{r}{k}(-1)^k = \binom{r}{0} - \binom{r}{1} + \binom{r}{2} + \cdots + (-1)^m\binom{r}{m}$$
$$= (-1)^m \binom{r-1}{m} \quad m \in \mathbb{Z}$$

$$\sum_{k\le m} \binom{r}{k}(-1)^k = \sum_{k\le m} \binom{k-r-1}{k} = \binom{-r+m}{m}$$

upper negation

$$= (-1)^m \binom{r-1}{m}^{(a)} \quad ⊛$$

this gives the partial sum of P's Δ with alternating signs: $r=5$, $m=2$

$$1-5+10 = 6 = (-1)^2\binom{5-1}{2},$$

In ⊛ if $m \ge r$ we get the whole row $= 0$ when $r \in \mathbb{Z}^+$, how about

$$\sum_{k\le m} \binom{n}{k} = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{m} \quad ?$$

there is no closed form for this!! BTW (b) does column partial sums. Note

$$\sum_{k\le m} \binom{r}{k}\left(\frac{r}{2}-k\right) = \frac{m+1}{2}\binom{r}{m+1}, \; m \in \mathbb{Z}$$

average w.r.t. distance from row center

Proof by induction: $m = 0$

$$\sum_{k\le m} \binom{r}{k}\left(\frac{r}{2}-k\right) = \binom{r}{0}\cdot\left(\frac{r}{2}\right) = \frac{r}{2} \qquad (\text{lhs})$$

$$\left(\frac{1}{2}\right)\binom{r}{1} = \frac{r}{2} \qquad (\text{rhs}) \quad \checkmark$$

Assume $\displaystyle\sum_{k\le m} \binom{r}{k}\left(\frac{r}{2}-k\right) = \frac{m+1}{2}\binom{r}{m+1}$

$$\sum_{k\le m+1} \binom{r}{k}\left(\frac{r}{2}-k\right) = \frac{m+1}{2}\binom{r}{m+1} + \binom{r}{m+1}\left(\frac{r}{2}-m-1\right)$$

$$= \binom{r}{m+1}\left(\frac{r}{2}-\frac{(m+1)}{2}\right) = \frac{r-m-1}{2}\cdot\frac{r^{\underline{m+1}}}{(m+1)!}\cdot\frac{m+2}{m+2}$$

$r^{\underline{m+2}}$

$$= \frac{m+2}{2}\cdot\frac{r^{\underline{m+2}}}{(m+2)} = \frac{m+2}{2}\binom{r}{m+2} \quad \checkmark$$

$(m+2)!$

The fact of intractibility seem similar to

$$\int_{-\infty}^{\alpha} x e^{-x^2}\,dx = -\frac{1}{2}e^{-\alpha^2}, \quad \int_{-\infty}^{\alpha} e^{-x^2}\,dx = ?$$

Consider $m \in \mathbb{Z}$

$$\sum_{k\le m}\binom{m+r}{k}x^k y^{m-k} = \sum_{k\le m}\binom{-r}{k}(-x)^k(x+y)^{m-k}$$

Proof by induction: when $m<0$ both sides $=0$, $m=0 \Rightarrow =1$, let S_m be the lhs

$$S_m = \sum_{k\le m}\left[\binom{m-1+r}{k}+\binom{m-1+r}{k-1}\right]x^k y^{m-k}$$

$$\sum_{k\le m} \binom{m-1+r}{k} x^k y^{m-k} = y\, S_{m-1} + \binom{m-1+r}{m} x^m \; +$$

$$\sum_{k\le m} \binom{m-1-r}{k-1} x^k y^{m-k} = x\, S_{m-1} \quad \text{, so } m>0 \Rightarrow$$

$$S_m = (x+y) S_{m-1} + \binom{-r}{m} (-x)^m, \text{ this}$$

is also satisfied by rhs. $\square$

Simpler Proof: when $r \in \mathbb{Z}$, $0 \ge r \ge -m$, both lhs & rhs $= (x+y)^{m+r} y^{-r}$ by binomial theorem. They agree at $m+1$ values of r so by the polynomial argument, they are poly's of degree m & so! Set $x=-1$, $y=1$:

$$\sum_{k\le m} \binom{m+r}{k} (-1)^k = \binom{-r}{m} \quad m \ge 0 \in \mathbb{Z}$$

similar to the alternating row sum.

set $x=y=1$, $r=m+1$

$$\underbrace{\sum_{k\le m} \binom{2m+1}{k}}_{\text{sums } \frac{1}{2} \text{ of the row of P's } \Delta \text{ with } 2m+1} = \sum_{k\le m} \binom{m+k}{k} 2^{m-k}$$

$$= \frac{1}{2}\left(2^{2m+1}\right) = 2^{2m} \quad \text{so!}$$

$$\sum_{k\le m} \binom{m+k}{k} 2^{-k} = 2^m \quad m \ge 0 \in \mathbb{Z}$$

check $m=2$ $\binom{2}{0}+\frac{1}{2}\binom{3}{1}+\frac{1}{4}\binom{4}{2}=1+\frac{3}{2}+\frac{6}{4}=4$!

Products of Binomial Coefficients:

$$\binom{r}{m}\binom{m}{k}=\binom{r}{k}\binom{r-k}{m-k} \quad m,k\in\mathbb{Z}$$

$k=1$ is the absorption identity.

Assume $r,m,k\in\mathbb{Z}$, $r\geq m\geq k\geq 0$:

$$\binom{r}{m}\binom{m}{k}=\frac{r!}{m!\,(r-m)!}\,\frac{m!}{k!\,(m-k)!}=\frac{r!}{k!\,(m-k)!\,(r-m)!}$$

$$=\frac{r!}{k!\,(r-k)!}\cdot\frac{(r-k)!}{(m-k)!\,(r-m)!}=\binom{r}{k}\binom{r-k}{m-k}$$

if $m<k$ or $k<0$ both sides vanish, POLY!

Trinomial Theorem:

$$(x+y+z)^n=\sum_{\substack{0\leq a,b,c\leq n\\ a+b+c=n}}\frac{(a+b+c)!}{a!\,b!\,c!}\,x^a y^b z^c$$

$$=\sum_{\substack{0\leq a,b,c\leq n\\ a+b+c=n}}\binom{a+b+c}{b+c}\binom{b+c}{c}\,x^a y^b z^c$$

so $\binom{r}{m}\binom{m}{k}$ is a trinomial coef.

Write $\binom{a+b+c}{a,b,c}=\frac{(a+b+c)!}{a!\,b!\,c!}$ trinom. coef!

$$\binom{a_1+a_2+\cdots+a_m}{a_1,\ldots,a_m}=\frac{(a_1+\cdots+a_m)!}{a_1!\cdots a_m!}=\binom{a_1+\cdots+a_m}{a_2+\cdots+a_m}\cdots\binom{a_{m-1}+a_m}{a_m}$$

Table 169 Sums of products of binomial coefficients.

$\sum_k \binom{r}{m+k}\binom{s}{n-k} = \binom{r+s}{m+n}$,	integers m, n.	(5.22)
$\sum_k \binom{l}{m+k}\binom{s}{n+k} = \binom{l+s}{l-m+n}$,	integer $l \geqslant 0$, integers m, n.	(5.23)
$\sum_k \binom{l}{m+k}\binom{s+k}{n}(-1)^k = (-1)^{l+m}\binom{s-m}{n-l}$,	integer $l \geqslant 0$, integers m, n.	(5.24)
$\sum_{k\leqslant l} \binom{l-k}{m}\binom{s}{k-n}(-1)^k = (-1)^{l+m}\binom{s-m-1}{l-m-n}$,	integers $l, m, n \geqslant 0$.	(5.25)
$\sum_{0\leqslant k\leqslant l} \binom{l-k}{m}\binom{q+k}{n} = \binom{l+q+1}{m+n+1}$,	integers $l, m \geqslant 0$, integers $n \geqslant q \geqslant 0$.	(5.26)

5.22 is the most memorable, Vandermonde's convolution and all the others can be derived from it, replace k with $k-m$ & n by $n-m$ and assume $m=0$:

$$\sum_k \binom{r}{k}\binom{s}{n-k} = \binom{r+s}{n} \quad r, s, n \in \mathbb{Z}, \quad r, s \geqslant 0$$

ways to choose $n-k$

ways to choose k

(5.23) $\binom{l}{m+k} \Rightarrow \binom{l}{l-m-k}$ & then 5.22 holds.

(5.24) Proof by induction on l:

$l=0$ all terms vanish except when $k=-m$

lhs $\binom{0}{0}\binom{s-m}{n}(-1)^m = (-1)^m\binom{s-m}{n}$ rhs ✓

Assume true for $l < l_{max}$

$\binom{l}{m+k} = \binom{l-1}{m+k} + \binom{l-1}{m+k-1}$ via addition, so =

$$\sum_k \binom{l-1}{m+k}\binom{s+k}{n}(-1)^k + \sum_k \binom{l-1}{m+k-1}\binom{s+k}{n}(-1)^k =$$

$$= (-1)^{l-1+m}\binom{s-m}{n-l+1} + (-1)^{l+m}\binom{s-m+1}{n-l+1}$$

$$= (-1)^{l+m}\Bigg[\underbrace{\binom{s-m+1}{n-l+1} - \binom{s-m}{n-l+1}}_{\binom{s-m}{n-l} \text{ via addition.}}\Bigg]$$

Sum w/products of more than two binomial coefs.

$$\sum_k \binom{m-r+s}{k}\binom{n+r-s}{n-k}\binom{r+k}{m+n} = \binom{r}{m}\binom{s}{n} \quad m,n \in \mathbb{Z}$$

$$\sum_k \binom{a+b}{a+k}\binom{b+c}{b+k}\binom{c+a}{c+k}(-1)^k = \frac{(a+b+c)!}{a!\,b!\,c!} \quad a,b,c \in \mathbb{Z}_{\geq 0}$$

Simpler $\sum_k \binom{a+b}{a+k}\binom{b+a}{b+k}(-1)^k = \frac{(a+b)!}{a!\,b!}$, less simple

$$\sum_k (-1)^k \binom{a+b}{a+k}\binom{b+c}{b+k}\binom{c+d}{c+k}\binom{d+a}{d+k} \Big/ \binom{2a+2b+2c+2d}{a+b+c+d+k} =$$

$$\frac{(a+b+c+d)!\,(a+b+c)!\,(a+b+d)!\,(a+c+d)!\,(b+c+d)!}{(2a+2b+2c+2d)!\,(a+c)!\,(b+d)!\;a!\,b!\,c!\,d!},$$

also a very complicated one follows:

$$\sum_{k_{ij}} (-1)^{\sum_{i<j} k_{ij}} \left(\prod_{1\le i<j<n} \binom{a_i+a_j}{a_j+k_{ij}} \right) \left(\prod_{1\le j<n} \binom{a_j+a_n}{a_n+\sum_{i<j} k_{ij} - \sum_{i>j} k_{ji}} \right)$$

$$= \binom{a_1+a_2+\cdots+a_n}{a_1, a_2, \ldots, a_n} \qquad a_1, \ldots, a_n \geq 0 \in \mathbb{Z}$$

(sum over $\binom{n-1}{2}$ indices, above are special cases, for $n=4$ w/ a,b,c,d / i,j,k:

$$\sum_{i,j,k} (-1)^{i+j+k} \binom{a+b}{b+i} \binom{a+c}{c+j} \binom{b+c}{c+k} \binom{a+d}{d-i-j} \binom{b+d}{d+i-k} \binom{c+d}{d+j+k}$$

$$= \frac{(a+b+c+d)!}{a!\,b!\,c!\,d!} \qquad a,b,c,d \geq 0 \in \mathbb{Z}$$

lhs is coef. of $z_1^0 z_2^0 \cdots z_n^0$ in $\displaystyle\prod_{1\le i\ne j\le n} \left(1 - \frac{z_i}{z_j}\right)^{a_i}$

rhs was proven later, and lastly

$$\sum_{j,k} (-1)^{j+k} \binom{j+k}{k+l} \binom{r}{j} \binom{n}{k} \binom{s+n-j-k}{m-j} =$$

$$(-1)^l \binom{n+r}{n+l} \binom{s-r}{m-n-l} \qquad l,m,n \in \mathbb{Z} \quad n \geq 0$$

Basic Practice: Sum of Ratios

$$\sum_{k=0}^{m} \binom{m}{k} \Big/ \binom{n}{k} \qquad n \geq m \geq 0 \in \mathbb{Z}$$

recall: $\binom{n}{m}\binom{m}{k} = \binom{n}{k}\binom{n-k}{m-k}$, then
(top)

$$\frac{\binom{n}{m}\binom{m}{k}}{\binom{n}{k}\binom{n}{m}} = \frac{\binom{n}{k}\binom{n-k}{m-k}}{\binom{n}{k}\binom{n}{m}} \implies$$

$\binom{m}{k}\binom{n}{k} = \binom{n-k}{m-k}/\binom{n}{m}$ so we instead use

$$\sum_{k=0}^{m} \binom{n-k}{m-k}/\binom{n}{m} = \binom{n}{m}^{-1} \sum_{k=0}^{m} \binom{n-k}{m-k}$$

↑ does not have k in it

can consider $k \geq 0$ as $k > m \Rightarrow$ term is zero

$$\sum_{k \geq 0} \binom{n-k}{m-k} = \sum_{m-k \geq 0} \binom{n-(m-k)}{m-(m-k)} =$$

let $m-k \Leftarrow k$

$$\sum_{k \leq m} \binom{n-m+k}{k} = \binom{(n-m)+m+1}{m} = \binom{n+1}{m},$$

by (a)

so $$\sum_{k=0}^{m} \binom{m}{k}/\binom{n}{k} = \binom{n+1}{m}/\binom{n}{m} =$$

$$\frac{(n+1)!}{m!\,(n+1-m)!} \cdot \frac{m!\,(n-m)!}{n!} = \frac{n+1}{n+1-m},$$ works,

as long as $n \notin \{0, 1, \dots, m-1\}$, n can be real as well, now check, $m=2$, $n=4$

$$\binom{2}{0}/\binom{4}{0} + \binom{2}{1}/\binom{4}{1} + \binom{2}{2}/\binom{4}{2} = 1 + \frac{2}{4} + \frac{1}{6} = \frac{5}{3} = \frac{4+1}{4+1-2} = \frac{5}{3}$$ ✓

A useful table for problems & memory

Table 174 The top ten binomial coefficient identities.

$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$,	integers $n \geqslant k \geqslant 0$.		*factorial expansion*
$\binom{n}{k} = \binom{n}{n-k}$,	integer $n \geqslant 0$, integer k.		*symmetry*
$\binom{r}{k} = \frac{r}{k}\binom{r-1}{k-1}$,	integer $k \neq 0$.		*absorption/extraction*
$\binom{r}{k} = \binom{r-1}{k} + \binom{r-1}{k-1}$,	integer k.		*addition/induction*
$\binom{r}{k} = (-1)^k \binom{k-r-1}{k}$,	integer k.		*upper negation*
$\binom{r}{m}\binom{m}{k} = \binom{r}{k}\binom{r-k}{m-k}$,	integers m, k.		*trinomial revision*
$\sum_k \binom{r}{k} x^k y^{r-k} = (x+y)^r$,	integer $r \geqslant 0$, or $\lvert x/y \rvert < 1$.		*binomial theorem*
$\sum_{k \leqslant n} \binom{r+k}{k} = \binom{r+n+1}{n}$,	integer n.		*parallel summation*
$\sum_{0 \leqslant k \leqslant n} \binom{k}{m} = \binom{n+1}{m+1}$,	integers $m, n \geqslant 0$.		*upper summation*
$\sum_k \binom{r}{k}\binom{s}{n-k} = \binom{r+s}{n}$,	integer n.		*Vandermonde convolution*

Problem 2 (from sorting): Expected # of saved transfers

$$T = \sum_{r=0}^{n} r \, \frac{(m-r-1)C_{(m-n-1)}}{{}_mC_n} \quad \text{(Knuth)}$$

$$= \sum_{k=0}^{n} k \binom{m-k-1}{m-n-1} \Big/ \binom{m}{n}$$

no k so →

$$S = \sum_{k=0}^{n} k \binom{m-k-1}{m-n-1}$$

this is a problem, so ...

if $m-k$, not k then

$$(m-k)\binom{m-k-1}{m-n-1} = (m-n)\binom{m-k}{m-n}$$

write k as $m-(m-k)$ $\Longrightarrow$

$$\sum_{k=0}^{n} k\binom{m-k-1}{m-n-1} = \sum_{k=0}^{n} m\binom{m-k-1}{m-n-1}$$

$$-\sum_{k=0}^{n}(m-k)\binom{m-k-1}{m-n-1} =$$

$$m\sum_{k=0}^{n}\binom{m-k-1}{m-n-1} - (m-n)\sum_{k=0}^{n}\binom{m-k}{m-n}$$

$$\sum_{k=0}^{n}\binom{m-k}{m-n} = \sum_{0\le m-k\le n}\binom{m-(m-k)}{m-n} =$$

$$\sum_{m-n\le k\le m}\binom{k}{m-n} = \sum_{0\le k\le m}\binom{k}{m-n} \overset{(b)}{=} \binom{m+1}{m-n+1}$$

(neg k's $=0$)

other is the same (why?)

$$= m\binom{m}{m-n} - (m-n)\binom{m+1}{m-n+1} =$$

$$\binom{m}{m-n}\left[m-(m-n)\left(\frac{m+1}{m-n+1}\right)\right] = \binom{m}{m-n}\frac{n}{m-n+1}$$

$$T = S/\binom{m}{n} = \frac{n}{m-n+1}\,\frac{\binom{m}{m-n}}{\binom{m}{n}}\Big\}\text{ same} = \frac{n}{m-n+1}$$

Check $m=4, n=2$

$$T = \frac{1}{\binom{4}{2}}\left[0\cdot\binom{3}{1} + 1\cdot\binom{2}{1} + 2\binom{1}{1}\right]$$

$$= \frac{1}{6}\left[0 + 2 + 2\right] = \frac{2}{3} = \frac{2}{4-2+1} = \frac{2}{3} \checkmark$$

Problem 3: $Q_n = \sum_{k\le 2^n} \binom{2^n-k}{k}(-1)^k, \quad n\in\mathbb{Z}\ge 0$

small cases

n			Q_n
0	$\binom{1}{0}$	$= 1$	1
1	$\binom{2}{0}-\binom{1}{1}$	$= 1-1$	0
2	$\binom{4}{0}-\binom{3}{1}+\binom{2}{2}$	$= 1-3+1$	-1
3	$\binom{8}{0}-\binom{7}{1}+\binom{6}{2}-\binom{5}{3}+\binom{4}{4}$	$= 1-7+15-10+1$	0

Seems to be a pattern? n always as 2^n, so try $m\ (=2^n)$:

$$R_m = \sum_{k\le m}\binom{m-k}{k}(-1)^k \qquad m\in\mathbb{Z}\ge 0$$

$Q_n = R_{2^n}$, so we generalise

m	0	1	2	3	4	5	6	7	8	9	10
R_m	1	1	0	-1	-1	0	1	1	0	-1	-1

$$R_m = \sum_{k\le m}\binom{m-k}{k}(-1)^k = \sum\left[\binom{m-1-k}{k}+\binom{m-1-k}{k-1}\right](-1)^k$$

$=$

$$= \sum_{k \le m-1} \binom{m-1-k}{k} (-1)^k + \sum_{k+1 \le m} \binom{m-2-k}{k} (-1)^{k+1}$$

$$= R_{m-1} + \binom{-1}{m} (-1)^m - \sum_{k \le m-2} \binom{m-2-k}{k} (-1)^k - \binom{-1}{m-1}(-1)^{m-1}$$

last term $_{k+1=m}$ sum #

$\binom{-1}{m} = (-1)^m$ so we finally get:

$$R_m = R_{m-1} + (-1)^{2m} - R_{m-2} - (-1)^{2(m-1)}$$

cancel out

$$= R_{m-1} - R_{m-2}$$

$m \bmod 6$	0	1	2	3	4	5	, proof
R_m	1	1	0	-1	-1	0	

$$R_m = (\overbrace{R_{m-2} - R_{m-3}}^{R_{m-1}}) - R_{m-2} = -R_{m-3}$$

so 3 initial values determine it

$$Q_n = R_{2^n} = \begin{cases} R_1 = 1 & n = 0 \\ R_2 = 0 & n \text{ odd} \\ R_4 = -1 & n > 0 \text{ even} \end{cases}$$

$Q_{1000000} = R_4 = -1.$

Problem 4: $\sum_{k=0}^{n} k \binom{m-k-1}{m-n-1}$ $m > n \ge 0 \in \mathbb{Z}$

(this is from problem 2)

$$\sum_{0\le k\le n} \binom{k}{1}\binom{m-k-1}{m-n-1}, \text{ use } (5.26)$$

$$= \sum_{0\le k\le m-1} \binom{k}{1}\binom{m-k-1}{m-n-1} \quad \text{terms } n<k\le m-1 \text{ vanish}$$

$$(l, m, n, q) \leftarrow (m-1, n-1, 1, 0)$$

$$= \binom{m}{m-n+1} = \frac{n}{m-n+1}\binom{m}{m-n}$$

Set $m=n=1$, $q=0$ in (5.26):

$$\sum_{0\le k\le l}(l-k)k = l\sum_{0\le k\le l} k - \sum_{0\le k\le l} k^2 = \binom{l+1}{3}$$

$$\sum_{0\le k\le l} k^2 = l\left(\frac{l(l+1)}{2}\right) - \binom{l+1}{3}$$

Problem 5: $\sum_k \binom{n}{k}\binom{s}{k} k \qquad n\in\mathbb{Z}_{\ge 0}$

try to use (5.23), first absorption

$$= \sum_k \binom{n}{k}\binom{s-1}{k-1} s = s\sum_k \binom{n}{k}\binom{s-1}{k-1} = s\binom{n+s+1}{n-1}$$

constant.

Problem 6:

$$\sum_{k\ge 0}\binom{n+k}{2k}\binom{2k}{k}\frac{(-1)^k}{k+1} \qquad n\in\mathbb{Z}_{\ge 0}$$

recall $\binom{r}{m}\binom{m}{k} = \binom{r}{k}\binom{r-k}{m-k} \qquad m,k\in\mathbb{Z}$

$$\text{So} = \sum_{k\ge 0} \binom{n+k}{k}\binom{n}{k}\frac{(-1)^k}{k+1} = \sum_k \binom{n+k}{k}\binom{n+1}{k+1}\frac{(-1)^k}{n+1}$$

absorption

$$= \frac{1}{n+1}\sum_k \binom{n+k}{k}\binom{n+1}{k+1}(-1)^k \text{, two tries}$$

Symmetry: $= \frac{1}{n+1}\sum_k \binom{n+k}{n}\binom{n+1}{k+1}(-1)^k$, now plug into (5.24) $(l,m,n,s) \leftarrow (n+1, 1, n, n)$

$$\frac{1}{n+1}\sum_k \binom{n+k}{n}\binom{n+1}{k+1}(-1)^k = \frac{1}{n+1}(-1)^n\binom{n-1}{-1} = 0$$

try $n=2$: $\binom{2}{0}\binom{2}{0}\frac{1}{1} - \binom{3}{1}\binom{2}{1}\frac{1}{2} + \binom{4}{2}\binom{2}{2}\frac{1}{3}$
$= 1 - 3\cdot 2\,\frac{1}{2} + 1\cdot 6\cdot\frac{1}{3} = 0$ ✓

try upper negation of $\binom{n+k}{k}$

$$= \frac{1}{n+1}\sum \binom{-n-1}{k}\binom{n+1}{k+1} \quad \text{now use}$$

(5.23) $(l,m,n,s) \leftarrow (n+1, 1, 0, -n-1)$

$$= \frac{1}{n+1}\binom{0}{n} = \begin{cases} 0 & n>0 \\ 1 & n=0 \end{cases}$$

$= [n=1]$ mistake in symmetry

$\binom{n+k}{k} \to \binom{n+k}{n}$ when $k<-n$ causes problems (no negatives allowed)

Problem 7: $\sum_{k\geq 0} \binom{n+k}{m+2k}\binom{2k}{k}\frac{(-1)^k}{k+1} \quad m,n \in \mathbb{Z}^+$

note, $m=0$ is problem 6, so try to eliminate m,

need to replace $\binom{n+k}{m+2k}$ using (5.26) via

$(l,m,n,q,k) \leftarrow (n+k+1, 2k, m-1, 0, j)$

$$= \sum_{k\geq 0} \sum_{0\leq j\leq n+k-1} \binom{n+k-1-j}{2k}\binom{j}{m-1}\binom{2k}{k}\frac{(-1)^k}{k+1}$$

$$= \sum_{j\geq 0} \binom{j}{m-1} \sum_{k\geq j-n+1\geq 0} \binom{n+k-1-j}{2k}\binom{2k}{k}\frac{(-1)^k}{k+1}$$ $\leftarrow$ can apply prob. 6 but..

this condition is superfluous since when violated terms vanish, so

$$= \sum_{j\geq 0} \binom{j}{m-1} \sum_{k\geq 0} \binom{n+k-1-j}{2k}\binom{2k}{k}\frac{(-1)^k}{k+1} =$$

$$= \sum_{j\geq 0} \binom{j}{m-1} [n-1-j=0] = \binom{n-1}{m-1}$$

Problem 8: $S_m = \sum_{k\geq 0} \binom{n+k}{2k}\binom{2k}{k}\frac{(-1)^k}{k+1+m}$ $\quad \underline{m,n \in \mathbb{Z} \geq 0}$

Use $\frac{r+1}{r+1-m} = \sum_{j=0}^{m} \binom{m}{j}\binom{r}{j}^{-1}$

$m \in \mathbb{Z} \geq 0 \quad r \notin \mathbb{Z}_m \quad r \leftarrow -k-2 \quad \left(\frac{k+1}{k+1+m}\right)$

$$S_m = \sum_{k\geq 0} \binom{n+k}{k}\binom{n}{k}\frac{(-1)^k}{k+1} \sum_{j\geq 0} \binom{m}{j}\binom{-k-2}{j}^{-1}$$

we absorb $(k+1)^{-1}$ into $\binom{n}{k}$

$$S_m = \frac{m!\,n!}{(m+n+1)!}\sum_{j\ge 0}(-1)^j\binom{m+n+1}{n+1+j}\sum_k\binom{n+1+j}{k+j+1}\binom{-n-1}{k}$$

$$= \frac{m!\,n!}{(m+n+1)!}\sum_{j\ge 0}(-1)^j\binom{m+n+1}{n+1+j}\binom{j}{n}$$ to use

(5.24) to sum over all j, which gives zero.

$-S_m$ is sum for $j<0$, $j \leftarrow -k-1$, $k\ge 0$:

$$S_m = \frac{m!\,n!}{(m+n+1)!}\sum_{k\ge 0}(-1)^k\binom{m+n+1}{n-k}\binom{-k-1}{n}$$

$$= \frac{m!\,n!}{(m+n+1)!}\sum_{k\le n}(-1)^{n-k}\binom{m+n+1}{k}\binom{k-n-1}{n}$$

$$= \frac{m!\,n!}{(m+n+1)!}\sum_{k\le n}(-1)^k\binom{m+n+1}{k}\binom{2n-k}{n}$$

$\leftarrow$ change to $k\le 2n$ + (5.25)

$$S_m = (-1)^n\frac{m!\,n!}{(m+n+1)!}\binom{m}{n}$$

$$= (-1)^n m^{\underline{n}}\, m^{\underline{-n-1}}$$, try $n=2$

$$S_m = \frac{1}{m+1} + \frac{6}{m+2} + \frac{6}{m+3} = \frac{m(m-1)}{(m+1)(m+2)(m+3)}$$

can also use polynomial argument so this holds for all m.

Duplication Formula:

$$r^{\underline{k}}\left(r-\tfrac{1}{2}\right)^{\underline{k}} = (2r)^{\underline{2k}}\, 2^{-2k} \qquad k \in \mathbb{Z} \geq 0$$

$$r\left(r-\tfrac{1}{2}\right)(r-1)\left(r-\tfrac{3}{2}\right)\cdots(r-k+1)\left(r-k+\tfrac{1}{2}\right) =$$

$$\frac{(2r)(2r-1)\cdots(2r-2k+1)}{2\quad 2\quad\cdots\quad 2} \quad \checkmark \, ,$$

now divide both sides by $(k!)^2$ to get

$$\frac{r^{\underline{k}}}{k!}\cdot\frac{\left(r-\frac{1}{2}\right)^{\underline{k}}}{k!} = \binom{r}{k}\binom{r-\frac{1}{2}}{k} \quad \text{lhs}$$

$$\frac{(2r)^{\underline{2k}}}{2k!}\cdot\frac{2k!}{k!\cdot k!}\cdot 2^{-2k} = \binom{2r}{2k}\binom{2k}{k}2^{-2k}$$

Set $k = r = n \in \mathbb{Z}$:

$$\binom{n}{n}\binom{n-1/2}{n} = \binom{2n}{2n}\binom{2n}{n}2^{-2n}$$

$$\binom{n-1/2}{n} = \binom{2n}{n}2^{-2n}$$, negate upper ind.

$$\circledast \quad \binom{-1/2}{n} = \left(\frac{-1}{4}\right)^n\binom{2n}{n} \qquad \text{for } n=4$$

$$\binom{-1/2}{4} = \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\left(-\frac{7}{2}\right)}{4!} = \left(\frac{-1}{2}\right)^4\cdot\frac{1\cdot3\cdot5\cdot7}{1\cdot2\cdot3\cdot4}$$

$$= \left(-\frac{1}{4}\right)^4\cdot\frac{1\cdot3\cdot5\cdot7\cdot2\cdot4\cdot6\cdot8}{1\cdot2\cdot3\cdot4\cdot1\cdot2\cdot3\cdot4} = \left(\frac{-1}{4}\right)^4\binom{8}{4}$$

Let $n = 2v$

$$\sum_k \binom{n}{2k}\binom{2k}{k} 2^{-2k} = \sum_k \binom{n/2}{k}\binom{(n-1)/2}{k}$$

via (5.23)

$$= \binom{n-\frac{1}{2}}{\lfloor \frac{n}{2} \rfloor} \qquad n \in \mathbb{Z}_{\geq 0} \quad \text{, also (5.27)}$$

$$\sum_k \binom{-1/2}{k}\binom{-1/2}{n-k} = \binom{-1}{n} = (-1)^n \quad n \in \mathbb{Z}_{\geq 0}$$

$$\binom{-1/2}{k}\binom{-1/2}{n-k} = \underbrace{\left(\frac{-1}{4}\right)^k \binom{2k}{k}}\underbrace{\left(\frac{-1}{4}\right)^{n-k}\binom{2(n-k)}{n-k}}_{\text{by } \circledast}$$

$$= \left(\frac{-1}{4}\right)^n \binom{2k}{k}\binom{2n-2k}{n-k} \text{, so}$$

$$\sum_k \binom{2k}{k}\binom{2n-2k}{n-k} = (-4)^n(-1)^n = 4^n \qquad n \in \mathbb{Z}_{\geq 0}$$

$$\binom{0}{0}\binom{6}{3} + \binom{2}{1}\binom{4}{2} + \binom{4}{2}\binom{2}{1} + \binom{6}{3}\binom{0}{0} =$$

$$1 \cdot 20 + 2 \cdot 6 + 6 \cdot 2 + 20 \cdot 1 = 64 = 4^3$$

Strategy $\binom{2k}{k} \rightarrow \binom{n-1/2}{k}$ may simplify

Differences: $\Delta f(x) = f(x+1) - f(x)$

$$\Delta^2 f(x) = \Delta f(x+1) - \Delta f(x) = f(x+2) - 2f(x+1) + f(x)$$

$\Delta^3 f(x) = f(x+3) - 3f(x+2) + 3f(x+1) - f(x)$

$\Delta^4 f(x) = f(x+4) - 4f(x+3) + 6f(x+2) - 4f(x+1) + f(x)$

and in general:

$$\Delta^n f(x) = \sum_k \binom{n}{k} (-1)^{n-k} (f(x+k))$$

$n \geq 0 \in \mathbb{Z}$

recall $Ef(x) = f(x+1)$, so $\Delta = E - I$,

$$\Delta^n = (E-I)^n = \sum_k \binom{n}{k} E^k (-1)^{n-k}, \text{ so}$$

$$\Delta^n f(x) = \sum_k \binom{n}{k} (-1)^{n-k} f(x+k) \quad (**)$$

let $f(x) = (x-1)^{\underline{-1}} = \frac{1}{x}$, $\Delta f(x) = -(x-1)^{\underline{-2}}$

$\Delta^2 f(x) = (-1)(-2)(x-1)^{\underline{-3}}$ and in general

$$\Delta^n \left((x-1)^{\underline{-1}} \right) = (-1)^n (x-1)^{\underline{-n-1}}$$

$$= (-1)^n \frac{n!}{x(x+1)\cdots(x+n)} \quad (**) \Rightarrow$$

$$\sum_k \binom{n}{k} \frac{(-1)^k}{x+k} = \frac{n!}{x(x+1)\cdots(x+n)} = x^{-1} \Big/ \binom{x+n}{n}$$

$x \notin \{0, -1, \ldots, -n\}$

$$\frac{1}{x} - \frac{4}{x+1} + \frac{6}{x+2} - \frac{4}{x+3} + \frac{1}{x+4}$$

$$= \frac{4!}{x(x+1)(x+2)(x+3)(x+4)} = 1 \Big/ \left[x \binom{x+4}{4} \right]$$

partial fraction expansion

If $f(x)$ is a degree d polynomial then

$$\Delta^d f(x) = \text{constant} \ \&$$
$$\Delta^n f(x) = 0, \quad n > d$$

$$f(x) = \sum_{i=0}^{d} a_i x^i \ \& \ \sum_{i=0}^{d} b_i x^{\underline{i}}$$

It turns out $a_d = b_d$ & $a_0 = b_0$, let

$c_k = k!\, b_k$, then

$$f(x) = c_d \binom{x}{d} + c_{d-1}\binom{x}{d-1} + \cdots + c_1\binom{x}{1} + c_0\binom{x}{0}$$

this expansion is the Newton Series of $f(x)$

Recall: $\Delta\left(\binom{x}{k}\right) = \binom{x}{k-1}$ so by induction

$$\Delta^n f(x) = c_d\binom{x}{d-n} + c_{d-1}\binom{x}{d-n-1} + \cdots + c_0\binom{x}{-n}$$

at $x = 0$, then $c_k\binom{x}{k-n} = 0$ except

if it's $\binom{x}{0} = \binom{0}{0} = 1$ when $k-n = 0$, so

$$\Delta^n f(0) = \begin{cases} c_n & \text{if } n \le d \\ 0 & \text{if } n > d \end{cases}$$, and so

$$f(x) = \Delta^d f(0)\binom{x}{d} + \Delta^{d-1} f(0)\binom{x}{d-1} + \cdots$$
$$+ \Delta f(0)\binom{x}{1} + f(0)\binom{x}{0}$$, Newton

$f(x)=x^3$, $f(0)=0$, $f(1)=1$, $f(2)=8$, $f(3)=27$

$\Delta f(0) = 1-0 = 1$
$\Delta f(1) = 8-1 = 7$
$\Delta f(2) = 27-8 = 19$

$\Delta^2 f(0) = 7-1 = 6$
$\Delta^2 f(1) = 19-7 = 12$
$\Delta^3 f(0) = 12-6 = 6$

So $x^3 = 6\binom{x}{3} + 6\binom{x}{2} + 1\binom{x}{1} + 0\binom{x}{0}$

Using $(**)$ we rewrite $\Delta f^n(0) = c_n$

$$\sum_k \binom{n}{k}(-1)^k\left(c_0\binom{k}{0} + c_1\binom{k}{1} + \cdots +\right) = (-1)^n c_n$$

$n \in \mathbb{Z} \geq 0$

this holds for $\langle c_0, c_1, \ldots \rangle$ arbitrary and is finite for $k \geq 0$, in particular

$$\sum_k \binom{n}{k}(-1)^k\left(a_0 + a_1 k + \cdots + a_n k^n\right) = (-1)^n n!\, a_n$$

this is because $a_0 + a_1 k + \cdots + a_n k^n$ can be written via Newton as

$$c_0\binom{k}{0} + c_1\binom{k}{1} + \cdots + c_n\binom{k}{n}, \quad c_n = a_n n!$$

Consider: $\displaystyle\sum_k \binom{n}{k}\binom{r-sk}{n}(-1)^k = s^n \quad n \in \mathbb{Z} \geq 0$

but $f(k) = \binom{r-sk}{n} = \frac{1}{n!}(-1)s^n k^n + \cdots + (-1)^n s_n \binom{k}{n} + \cdots$

this is a special case of the previous.

Can also make sense of infinite Newton series

$$f(x) = f(0)\binom{x}{0} + \Delta f(0)\binom{x}{1} + \Delta^2 f(0)\binom{x}{2} + \cdots \qquad (\dagger)$$

Note, if rhs converges for $x \in \mathbb{Z}_{\geq 0}$ other than it interpolates $f(x)$.

$f(x) = \sin(\pi x)$, $f(x) = 0$, $x \in \mathbb{Z}$ so the rhs is identically zero.

Taylor Series:

$$g(a+x) = \frac{g(a)}{0!}x^0 + \frac{g'(a)}{1!}x^1 + \frac{g''(a)}{2!}x^2 + \cdots$$

Newton Series

$$f(x) = g(a+x) = \frac{g(a)}{0!}x^{\underline{0}} + \frac{\Delta g(a)}{1!}x^{\underline{1}} + \frac{\Delta^2 g(a)}{2!}x^{\underline{2}} + \cdots$$

this is the same as $(\dagger)$ since $\Delta^n f(0) = \Delta^n g(a)$

Both T & N are finite when f is a poly. or at $x=0$. Also N is finite when $x \in \mathbb{Z}^+$.

Set $g(x) = (1+z)^x$, $z \in \mathbb{C}$ is fixed, $|z|<1$

$$\Delta g(x) = (1+z)^{x+1} - (1+z)^x = \underset{\text{constant}}{z}(1+z)^x \text{, so}$$

$$g(a+x) = \sum_n \binom{x}{n}\Delta^n g(a) = (1+z)^a \sum_n \binom{x}{n} z^n$$

converges to $(1+z)^{a+x}$ $\forall x$.

Stirling used NS to generalize the factorial:

$x! = \sum_n S_n \binom{x}{n}$ is true for $x = 0, 1, 2, \cdots$

but it diverges for $x \notin \mathbb{Z}^+$

$\ln(x!) = \sum_n s_n \binom{x}{n}$, $\Delta \ln(x!) = \ln(x+1)$

$$s_n = \Delta^n(\ln x!)\big|_{x=0} = \Delta^{n-1}(\ln(x+1))\big|_{x=0}$$

$$= \sum_k \binom{n-1}{k}(-1)^{n-1-k} \ln(k+1) \quad \text{by } (**)$$

$s_0 = s_1 = 0$, $s_2 = \ln 2$, $s_3 = \ln 3 - 2\ln 2$, $s_4 = \ln 4 - 3\ln 3 + 3\ln 2 = \ln 32/27$. This series converges for all $x > -1$ & gave him $t!$

Inversion: A special case for Newton series just derived can be rewritten as

$$g(n) = \sum_k \binom{n}{k}(-1)^k f(k) \Leftrightarrow \sum_k \binom{n}{k}(-1)^k g(k)$$

this is the "inversion formula" that seems like Möbius.

Proof: Let $g(n) = \sum_k \binom{n}{k}(-1)^k f(k)$ $\forall n \geq 0$

$$\sum_k \binom{n}{k}(-1)^k g(k) =$$

$$\sum_k \binom{n}{k}(-1)^k \sum_j \binom{k}{j}(-1)^j f(j) \qquad \text{exchange } k \rightleftarrows j$$

$$= \sum_j f(j) \sum_k \binom{n}{k}(-1)^{k+j}\binom{k}{j} \qquad k = n-j$$

$$= \sum_j f(j) \sum_k \binom{n}{j}(-1)^{k+j}\binom{n-j}{k-j} \qquad k = k+j$$

$$= \sum_j f(j)\binom{n}{j}\sum_k (-1)^k\binom{n-j}{k}$$

$$= \sum_j f(j)\binom{n}{j}[n-j=0] = f(n)\binom{n}{n} = f(n)$$

this proof works backwards as well & f & g are symmetric too.

Football Victory Problem: n fans throw up their hats, & fans recover one hat randomly

$h(n,k)$ = # ways k fans get back their own hats

$n=4$, A B C D, there are $4! = 24$ possibilities

Example w/24 possible perms.

ABCD	4	BACD	2	CABD	1	DABC	0
ABDC	2	BADC	0	CADB	0	DACB	1
ACBD	2	BCAD	1	CBAD	2	DBAC	1
ACDB	1	BCDA	0	CBDA	1	DBCA	2
ADBC	1	BDAC	0	CDAB	0	DCAB	0
ADCB	2	BDCA	1	CDBA	0	DCBA	0

Therefore $h(4,4) = 1$; $h(4,3) = 0$; $h(4,2) = 6$; $h(4,1) = 8$; $h(4,0) = 9$.

$h(n,k)$ = # ways to choose k lucky hatowners · # ways to rearrange $n-k$ so none go to their owner

$$h(n,k) = \binom{n}{k} h(n-k,0) = \binom{n}{k}(n-k)¡$$

$h(n-k,0) = (n-k)¡$ "subfactorial"

derangements, moves every item in a permutation

We need a closed form for $n¡$

$$n! = \sum_k h(n,k)$$

hat arrangements

total # of permutations

$$= \sum_k \binom{n}{k}(n-k)¡ = \sum_k \binom{n}{k} k¡ \quad @$$

symmetry holds

This allows us to compute $h(n,k)$:

n	$h(n,0)$	$h(n,1)$	$h(n,2)$	$h(n,3)$	$h(n,4)$	$h(n,5)$	$h(n,6)$
0	1						
1	0	1					
2	1	0	1				
3	2	3	0	1			
4	9	8	6	0	1		
5	44	45	20	10	0	1	
6	265	264	135	40	15	0	1

Row $n=4$: $h(4,4)=1$ & $h(4,3)=0$ are obvious

$$h(4,2) = \binom{4}{2} h(2,0) = 6\cdot 1$$
$$h(4,1) = \binom{4}{1} h(3,0) = 4\cdot 2$$

$$4! = h(4,0)+h(4,1)+h(4,2)+h(4,3)+h(4,4)$$
$$= h(4,0) + 8 + 6 + 0 + 1$$
$$24 = h(4,0) + 15 \implies h(4,0) = 9$$

Solve ⊙ $\quad n! = \sum_k \binom{n}{k} k!$

$g(n) = n! \quad f(k) = (-1)^k k!$ & inversion to

$$n! = (-1)^n \sum_k \binom{n}{k} (-1)^k k!$$

$$= \sum_{0\le k\le n} \frac{n!}{k!(n-k)!} \cdot k! \, (-1)^{n+k}$$

$$= \underbrace{n! \sum_{0\le k\le n} \frac{(-1)^k}{k!}}_{} \longrightarrow e^{-1} \text{ quickly}$$

$$n! \sum_{k\ge n} \frac{(-1)^k}{k!} = \frac{(-1)^{n+1}}{n+1} \sum_{k\ge 0} (-1)^k \frac{(n+1)!}{(k+n+1)!} =$$

$$= \frac{(-1)^{n+1}}{n+1}\underbrace{\left(1 - \frac{1}{n+2} + \frac{1}{(n+2)(n+3)} - \cdots\right)}_{1 \;\le\; \;\le\; 1-\frac{1}{n+2}}$$

$\frac{1}{n+1}$. alternating

So $\doteq n¡ - \frac{n!}{e} \sim \frac{1}{n} \le \frac{1}{n+2}$ but $n¡$ is $\in \mathbb{Z}^+$

so $n¡ = \left\lfloor \frac{n!}{e} + \frac{1}{2} \right\rfloor + [n=0]$ $n!=1$

rounded to nearest $\mathbb{Z}$

since $\frac{n¡}{n!} \approx e^{-1} \approx 0.367$, if the hats are thrown way up, as n gets bigger the chance that all hats are misplaced is $\sim 37\%$!!¡¡

Stirling expansion that didn't converge had $S_k = k¡$

Note: 1, 3, 6, 10, 15 ... below the 0-diagonal are the triangular numbers, $h(n, n-2)$

$$h(n, n-2) = \binom{n}{n-2} 2¡ = \binom{n}{n-2} = \binom{n}{2}$$

Also numbers in the first two columns differ by ± 1:

$$h(n,0) - h(n,1) = n¡ - n(n-1)¡$$

last term survives

$$= \left(n! \sum_{0 \le k \le n} \frac{(-1)^k}{k!}\right) - \left(n(n-1)! \sum_{0 \le k \le n-1} \frac{(-1)^k}{k!}\right)$$

$$= \frac{n!}{n!}(-1)^n = (-1)^n$$

So $ni = n(n-1)i + (-1)^n$

Now invert $\sum_k \binom{n}{k} \frac{(-1)^k}{x+k} = \frac{1}{x}\binom{x+n}{n}^{-1}$, so

$$\frac{x}{x+n} = \sum_{k\geq 0} \binom{n}{k}(-1)^k \binom{x+k}{k}^{-1}$$

Generating Functions:

$$\langle a_0, a_1, a_2, \dots \rangle \Rightarrow A(z) = a_0 + a_1 z + \cdots = \sum_{k\geq 0} a_k z^k$$

Power Series representation, we choose $z \in \mathbb{C}$ normally so the above is analytic or holomorphic

$[z^n]\,A(z) = a_n$ (the coefficient of z^n in $A(z)$)

$$A(z) \;—\; \langle a_0, a_1, \dots \rangle$$
$$B(z) \;—\; \langle b_0, b_1, \dots \rangle$$
, then product

$$\begin{aligned} A(z)\,B(z) &= (a_0 + a_1 z + a_2 z^2 + \cdots)(b_0 + b_1 z + b_2 z^2 + \cdots) \\ &= a_0 b_0 + (a_0 b_1 + a_1 b_0) z + (a_0 b_2 + a_1 b_1 + a_2 b_0) z^2 \cdots \end{aligned}$$

$$[z^n]\,A(z)B(z) = a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0 = \sum_{k=0}^{n} a_k b_{n-k}$$

$$c_n = [z^n]\,A(z)B(z) = \underbrace{\sum_{k=0}^{n} a_k b_{n-k}}_{\text{convolution of } \langle a_n \rangle, \langle b_n \rangle}$$

$$\left\langle \binom{r}{0}, \binom{r}{1}, \binom{r}{2}, \dots \right\rangle \rightarrow \sum_{k\geq 0} \binom{r}{k} z^k = (1+z)^r$$

$$(1+z)^s = \sum_{k\geq 0}^{0} \binom{s}{k} s^k$$

So $(1+z)^r(1+z)^s = (1+z)^{r+s}$, equate coefficient

$$\sum_{k=0}^{n} \binom{r}{k}\binom{s}{n-k} = \binom{r+s}{n} \quad \text{Vandermonde's conv. (5.27)}$$

Now $(1-z)^r = \sum_{k\geq 0} (-1)^k \binom{r}{k} z^k$

$(1-z)^r(1+z)^r = (1+z^2)^r$, equate coefs. of z^n

$$\sum_{k=0}^{n} \binom{r}{k}\binom{r}{n-k}(-1)^k = (-1)^{n/2}\binom{r}{n/2} \; [n \text{ even}]$$

try $n=3$ $\binom{r}{0}\binom{r}{3} - \binom{r}{1}\binom{r}{2} + \binom{r}{2}\binom{r}{1} - \binom{r}{3}\binom{r}{0} = 0$

$n=2$ $\binom{r}{0}\binom{r}{2} - \binom{r}{1}\binom{r}{1} + \binom{r}{0}\binom{r}{2} = 2\binom{r}{2} - r^2$

Note (5.30) is a special case of

$$= 2\,\frac{r(r-1)}{2} - r^2$$
$$= r^2 - r - r^2 = -r$$

$$(1-z)^{-(n+1)} = \sum_{k\geq 0} \binom{n+k}{n} z^n$$

$n \in \mathbb{Z} \geq 0$

$$z^n(1-z)^{-(n+1)} = \sum_{k\geq 0} \binom{k}{n} z^n \quad \text{(shifted by } n\text{)}$$

note: lower constant! Note

$(1-z)^{-n-1} \longrightarrow z^k$ coef. $= \binom{-n-1}{k}(-1)^k$ by

binomial theorem $\longrightarrow \binom{n+k}{n}$ by negation

When $n=0$ $\frac{1}{1-z} = 1+z+z^2\dots = \sum_{k\geq 0} z^k$

geometric series where $\langle 1, 1, 1, \dots \rangle$, & with convolution:

$$c_n = \sum_{k=0}^{n} a_n \quad \text{the } n \text{ partial sums}$$

If $A(z) \longrightarrow \langle a_0, a_1, \dots \rangle$, then $\frac{A(z)}{1-z} \to \langle c_0, c_1 \dots \rangle$

Derangement Application: $n! = \sum_k \binom{n}{k} (n-k)¡$

$$n! = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} (n-k)¡ \quad \text{divide by } n!$$

$$1 = \sum_{k=0}^{n} \frac{1}{k!} \frac{(n-k)¡}{(n-k)!}, \quad \langle \tfrac{1}{0!}, \tfrac{1}{1!}, \tfrac{1}{2!}, \dots \rangle - e^z$$

Let $D(z) = \sum_{k\geq 0} \frac{k¡}{k!} z^k$ via convolution:

$$\frac{1}{1-z} = e^z D(z) \quad \text{so } D(z) = \frac{e^{-z}}{1-z}$$

$$= \frac{1}{1-z}\left(\frac{1}{0!}z^0 - \frac{1}{1!}z^1 + \frac{1}{2!}z^2 - \frac{1}{3!}z^3 + \dots\right)$$

equating coefficients of z^n

$$\frac{n¡}{n!} = \sum_{k=0}^{n} \frac{(-1)^k}{k!}$$ which we derived earlier via inversion

$$\mathcal{B}_t(z) = \sum_{k\geq 0} (tk)^{\underline{k-1}} \frac{z^k}{k!}, \qquad \mathcal{E}_t(z) = \sum_{k\geq 0} (tk+1)^{k-1} \frac{z^k}{k!}$$

Generalized Bionomial Series

Generalized Exponential Series

Will prove (chapter 7):

$$\mathcal{B}_t(z)^{1-t} - \mathcal{B}_t(z)^{-t} = z, \qquad \mathcal{E}_t(z)^{-t} \ln \mathcal{E}_t(z) = z$$

When $t=0 \Rightarrow \mathcal{B}_0(z) = 1+z, \quad \mathcal{E}_0(z) = e^z$

$$\left.\begin{aligned} \mathcal{B}_t(z)^r &= \sum_{k\geq 0} \binom{tk+r}{k} \frac{r}{tk+r} z^k \\ \mathcal{E}_t(z)^r &= \sum_{k\geq 0} r \frac{(tk+r)^{k-1}}{k!} z^k \end{aligned}\right) \quad \text{A}$$

$$\left.\begin{aligned} \frac{\mathcal{B}_t(z)^r}{1-t+t\mathcal{B}_t(z)^{-1}} &= \sum_{k\geq 0} \binom{tk+r}{k} z^k \\ \frac{\mathcal{E}_t(z)^r}{1-zt\,\mathcal{E}_t(z)^t} &= \sum_{k\geq 0} \frac{(tk+r)^k}{k!} z^k \end{aligned}\right) \quad \text{B}$$

Caveat: when $tk+r=0$ each coef. of z^k is a poly. in r, e.g. constant term in $\mathcal{E}_t(z)^r = r(0+r)^{-1}$ and $=1$ when $r=0$ as well!?

(A) & (B) hold $\forall r$, so

$$\mathcal{B}_t(z)^r \cdot \frac{\mathcal{B}_t(z)^s}{1-t+t\mathcal{B}_t(z)^{-1}} = \sum_{k\geq 0} \binom{tk+r}{k}\frac{r}{tk+r} z^k \cdot \sum_{j\geq 0}\binom{tj+s}{j} z^j$$

$$= \sum_{n\geq 0} z^n \sum_{k\geq 0}\binom{tk+r}{k}\frac{r}{tk+r}\binom{t(n-k)+s}{n-k}$$

$$= \frac{\mathcal{B}_t(z)^{r+s}}{1-t+t\mathcal{B}_t(z)^{-1}} = \sum_{n\geq 0}\binom{tn+r+s}{n} z^n$$

equating coefs. of z^n:

$$\sum_k \binom{tk+r}{k}\frac{r}{tk+r}\binom{t(n-k)+s}{n-k} = \binom{tn+r+s}{n}$$

$$n \in \mathbb{Z}$$

When $t=0 \Longrightarrow$ Vandermonde's convolution

Holds $\forall\, r,s,t \in \mathbb{R}$, if $tk+r=0$, then we must assume the $tk+r$ on top and bottom as cancelling! We can get many more such identities by multiplying β's or Σ's together and equating coefs of z^n

Results:

Table 202 General convolution identities, valid for integer $n \geqslant 0$.

$$\sum_k \binom{tk+r}{k}\binom{tn-tk+s}{n-k}\frac{r}{tk+r} = \binom{tn+r+s}{n}. \tag{5.62}$$

$$\sum_k \binom{tk+r}{k}\binom{tn-tk+s}{n-k}\frac{r}{tk+r}\cdot\frac{s}{tn-tk+s} = \binom{tn+r+s}{n}\frac{r+s}{tn+r+s}. \tag{5.63}$$

$$\sum_k \binom{n}{k}(tk+r)^k(tn-tk+s)^{n-k}\frac{r}{tk+r} = (tn+r+s)^n. \tag{5.64}$$

$$\sum_k \binom{n}{k}(tk+r)^k(tn-tk+s)^{n-k}\frac{r}{tk+r}\cdot\frac{s}{tn-tk+s} = (tn+r+s)^n\frac{r+s}{tn+r+s}. \tag{5.65}$$

Set $t=1$, $\mathcal{B}_1(z) = \sum_{k\geq 0} z^k = \frac{1}{1-z}$

$$\mathcal{E}(z) = \sum_{k\geq 0} (k+1)^{k-1}\frac{z^k}{k!} = 1 + z + \tfrac{3}{2}z^2 + \tfrac{8}{3}z^3 + \tfrac{125}{24}z^4.$$

which satisfies $\mathcal{E}(z) = e^{z\mathcal{E}(z)}$, $\left(\mathcal{E}(\ln z) = z^{z^{z^{\cdot^{\cdot^{\cdot}}}}}\right)$

Special cases of $t=2$ or $t=-1$

On next page coefs. of $\mathcal{B}_2(z)$, $\binom{2n}{n}\frac{1}{n+1}$ are the Catalan numbers (# of parenthesizations)

n	0	1	2	3	4	5	6	7	8	9	10
C_n	1	1	2	5	14	42	132	429	1430	4862	16796

$$\mathcal{B}_2(z) = \sum_k \binom{2k}{k} \frac{z^k}{1+k}$$

$$= \sum_k \binom{2k+1}{k} \frac{z^k}{1+2k} = \frac{1-\sqrt{1-4z}}{2z}. \qquad (5.68)$$

$$\mathcal{B}_{-1}(z) = \sum_k \binom{1-k}{k} \frac{z^k}{1-k}$$

$$= \sum_k \binom{2k-1}{k} \frac{(-z)^k}{1-2k} = \frac{1+\sqrt{1+4z}}{2}. \qquad (5.69)$$

$$\mathcal{B}_2(z)^r = \sum_k \binom{2k+r}{k} \frac{r}{2k+r} z^k. \qquad (5.70)$$

$$\mathcal{B}_{-1}(z)^r = \sum_k \binom{r-k}{k} \frac{r}{r-k} z^k. \qquad (5.71)$$

$$\frac{\mathcal{B}_2(z)^r}{\sqrt{1-4z}} = \sum_k \binom{2k+r}{k} z^k. \qquad (5.72)$$

$$\frac{\mathcal{B}_{-1}(z)^{r+1}}{\sqrt{1+4z}} = \sum_k \binom{r-k}{k} z^k. \qquad (5.73)$$

The coefficients $\binom{2n}{n}\frac{1}{n+1}$ of $\mathcal{B}_2(z)$ are called the *Catalan numbers* C_n, because Eugène Catalan wrote an influential paper about them in the 1830s [52]. The sequence begins as follows:

n	0	1	2	3	4	5	6	7	8	9	10
C_n	1	1	2	5	14	42	132	429	1430	4862	16796

Also $\mathcal{B}_{-1}(z)$ has alternating Catalan #s so

$$\mathcal{B}_{-1}(z) = 1 + z\,\mathcal{B}_2(-z) \quad , \quad \mathcal{B}_{-1}(z) = \mathcal{B}_2(-z)^{-1}$$

Derive something based on (5.72) & (5.73)

$$\frac{\mathcal{B}_{-1}(z)^{n+1} - (-z)^{n+1}\mathcal{B}_2(-z)^{n+1}}{\sqrt{1+4z}} = \sum_{k\le n} \binom{n-k}{k} z^k$$

this works because the coef. of z^n in

$$[z^k] \frac{(-z)^{n+1} \beta_2(-z)^{n+1}}{\sqrt{1+4z}} = (-1)^{n+1}[z^{k-n-1}] \frac{\beta_2(-z)^{n+1}}{\sqrt{1+4z}}$$

$$= (-1)^{n+1}(-1)^{k-n-1}[z^{k-n-1}] \frac{\beta_2(z)^{n+1}}{\sqrt{1+4z}}$$

$$= (-1)^k \binom{2(k-n-1)+n+1}{k-n-1}$$

$$= (-1)^k \binom{2k-n-1}{k-n-1} = (-1)^k \binom{2k-n-1}{k}$$

$$= \binom{n-k}{k} = [z^k] \frac{\beta_{-1}(z)^{n+1}}{\sqrt{1+4z}}$$

Use 5.68 & 5.69 to get

$$\sum_{k \le n} \binom{n-k}{k} z^k = (1+4z)^{-1/2}\left[(A^+(z))^{n+1} - (A^-(z))^{n+1}\right]$$

$$A^{\pm}(z) = \left(\frac{1 \pm \sqrt{1+4z}}{2}\right)$$

special case $z=-1$ was in problem 3, $\frac{1}{2}(1 \pm \sqrt{3})$ are the 6^{th} roots of unity.

$$\sum_{k \le n} \binom{n-k}{k}(-1)^k \text{ is period w/period} = 6$$

w/ (5.70) & (5.71) we get

$$\sum_{k < n} \binom{n-k}{k} \frac{n}{n-k} z^k = (A^+(z))^n + (A^-(z))^n$$

$$n \in \mathbb{Z}^+$$

Hypergeometric Functions

A unifying principle for binomial coefs. summation comes from studying

Hypergeometric Series (Euler, Gauß, Riemann...) is a power series in z w/ $m+n$ parameters

$$F\left({a_1, \ldots, a_m \atop b_1, \ldots, b_n} \middle| z\right) = \sum_{k \geq 0} \frac{a_1^{\overline{k}} \cdots a_m^{\overline{k}}}{b_1^{\overline{k}} \cdots b_n^{\overline{k}}} \frac{z^k}{k!}$$

(note all rising powers), note

$b_i \notin \mathbb{Z}^- \cup \{0\}$ $b_i \in \mathbb{Z}^+$ to avoid division by zero

a's : upper parameters
b's : lower parameters
z — argument

Alternative Notation

$F(a_1, \ldots a_m; b_1, \ldots, b_n; z)$, ${}_mF_n$, F

Special Cases

$$F\left(\middle| z\right) = \sum_{k \geq 0} \frac{z^k}{k!} = e^z \text{ (no factorial powers)}$$

When $m \cdot n = 0$ can add dummy 1's:

$F\left({1 \atop 1} \middle| z\right) = e^z$, try $m=1, a_1 = 1, n = 0$:

$F\left({1,1 \atop 1} \middle| z\right) = \sum_{k \geq 0} 1^{\overline{k}} \cdot \frac{z^k}{k!}$, recall that

$1^{\overline{k}} = k!$, so $= \sum_{k \geq 0} z^k = \frac{1}{1-z}$

$F(a_1, \ldots, a_m; b_1, \ldots, b_n; z)$ is called hypergeometric because $F(1,1;1;z)$ is the geometric series.

$$F\left({a,1 \atop 1} \middle| z\right) = \sum_{k \geq 0} a^{\overline{k}} \frac{z^k}{k!} = \sum \binom{a+k-1}{k} z^k$$

note $a^{\overline{k}} = (a+k-1)^{\underline{k}}$ $\quad \frac{(a+k-1)^{\underline{k}}}{k!} = \binom{a+k-1}{k}$

$= \frac{1}{(1-z)^a}$ using upper neg. & binom. theorem

Let $a \leftarrow -a \qquad z \leftarrow -z$

$$F\left({-a,1 \atop 1} \middle| -z\right) = (1+z)^a$$

negative upper param. $\Rightarrow$ finite sum since $(-a)^{\overline{k}} = 0$ if $k > a \geq 0$, $a \in \mathbb{Z}$.

$m=0, n=1$ famous series:

$$F\left({1 \atop b,1}\middle|z\right) = \sum_{k\geq 0} \frac{(b-1)!}{(b-1+k)!}\frac{z^k}{k!} =$$

$\frac{1}{b^{\overline{k}}}$

$$\overbrace{I_{b-1}(2\sqrt{z})\frac{(b-1)!}{z^{(b-1)/2}}}$$, modified Bessel function of order $b-1$, $b=1 \Rightarrow$

$$F\left({1 \atop 1,1}\middle|z\right) = I_0(2\sqrt{z}) = \sum_{k\geq 0}\frac{z^k}{(k!)^2}$$

Special case $m=n=1$, "confluent" H.g. series

$$F\left({a \atop b}\middle|z\right) = \sum_{k\geq 0}\frac{a^{\overline{k}}}{b^{\overline{k}}}\frac{z^k}{k!} = M(a,b,z)$$

introduced by Ernst Kummer, important in some engineering applications.

"No convergence discussion": this is a formal series in the symbol z, and we can add/subtract/multiply, divide/differentiate/integrate/compose w/o convergence, e.g.

$$F\left({1,1,1 \atop 1}\middle|z\right) = \sum_{k\geq 0} k!\, z^k$$ diverges except when $z=0$, but is useful.

"The" Hypergeometric Series:

$$F\left({a,b \atop c}\middle| z\right) = \sum_{k\geq 0} \frac{a^{\overline{k}} b^{\overline{k}}}{c^{\overline{k}}} \frac{z^k}{k!}$$ called

"Gaussian hyper-geometric" due to early results proved by Gauß, Euler, Pfaff. Special

$$\ln(1+z) = z\, F\left({1,1 \atop 2}\middle| -z\right) = z \sum \frac{k!\,k!}{(k+1)!} \frac{(-z)^k}{k!}$$

$$= \sum_{k\geq 0} \frac{(-1)^k z^{k+1}}{k+1} = z - \frac{z^2}{2} + \frac{z^3}{3} - \cdots$$

$z^{-1}\ln(1+z)$ is hypergeometric, but $\ln(1+z)$ is not because $F(\cdots \mid 0) = 1$.

What series are hyper-geometric?

Rewrite: $F\left({a_1, \ldots, a_m \atop b_1, \ldots, b_n}\middle| z\right) = \sum_{k\geq 0} t_k$

$$t_k = \frac{a_1^{\overline{k}} \cdots a_m^{\overline{k}}}{b_1^{\overline{k}} \cdots b_n^{\overline{k}}} \frac{z^k}{k!} \quad \text{with } t_0 = 1$$

the ratio of terms is

$$\frac{t_{k+1}}{t_k} = \frac{a_1^{\overline{k+1}} \cdots a_m^{\overline{k+1}}}{a_1^{\overline{k}} \cdots a_m^{\overline{k}}} \cdot \frac{b_1^{\overline{k}} \cdots b_n^{\overline{k}}}{b_1^{\overline{k+1}} \cdots b_n^{\overline{k+1}}} \frac{k!}{(k+1)!} \frac{z^{k+1}}{z^k}$$

$$= \frac{(k+a_1)\cdots(k+a_m)\, z}{(k+b_1)\cdots(k+b_n)(k+1)}$$

which is a rational func. in k.

So the a's are the zeros of the numerator polynomial & the b's the zeros of the denominator polynomial, $(k+1)$ is always needed in the denominator, and can add it with $(k+1)\cdot(k+1)^{-1}$, we call the constant factor z.

Hypergeometric Series:

1st term: 1

$$\frac{t_{k+1}}{t_k} = \text{rational function in } k$$

Consider $$\frac{t_{k+1}}{t_k} = \frac{k^2+7k+10}{4k^2+1} = \frac{(k+2)(k+5)}{4(k+\frac{i}{2})(k-\frac{i}{2})}$$

$$= \frac{(k+2)(k+5)(k+1)(1/4)}{(k+\frac{i}{2})(k-\frac{i}{2})(k+1)}, \text{ so}$$

$$\sum_{k\geq 0} t_k = t_0 F\left(\begin{matrix}2, 5, 1\\ i/2, -i/2\end{matrix}\,\middle|\, \tfrac{1}{4}\right)$$

This is a general way to evaluate S as an infinite series Σt_k, t_{k+1}/t_k expressed as factored rational function:

$$S = t_0 F(a_1, \dots, a_m; b_1, \dots, b_n; z)$$

E.g. Gauß hypergeometric series:

$$F\left({a,b \atop c}\middle| z\right) = 1 + \frac{a}{1}\frac{b}{c}z\left(1+\frac{(a+1)(b+1)}{2\,(c+1)}z\left(1+\cdots\right.\right.$$

Recall: $$\sum_{k\le n}\binom{r+k}{k} = \binom{r+n+1}{n},\quad n\in\mathbb{Z}$$

must write this as series starting at $k=0$

$$\sum_{k\ge 0}\binom{r+n-k}{n-k} = \sum_{k\ge 0}\frac{(r+n-k)!}{r!\,(n-k)!} = \sum_{k\ge 0} t_k$$

finite!

$$\frac{t_{k+1}}{t_k} = \frac{(r+n-k-1)!}{r!\,(n-k-1)!}\cdot\frac{r!\,(n-k)!}{(r+n-k)!} = \frac{n-k}{r+n-k}$$

$$= \frac{(k+1)(k-n)(1)}{(k-n-r)(k+1)},\quad t_0 = \binom{r+n}{n},\text{ so}$$

$$\binom{r+n}{n} F\left({1,\,-n \atop -n-r}\middle| 1\right) = \binom{r+n+1}{n},$$

$$F\left({1,\,-n \atop -n-r}\middle| 1\right) = \binom{r+n+1}{n}\binom{r+n}{n}^{-1}$$

$$= \frac{(r+n+1)!}{n!\,(r+1)!}\cdot\frac{n!\,r!}{(r+n)!} = \frac{r+n+1}{r+1} \qquad (\dagger)$$

when $\binom{r+n}{n}\neq 0$

$$\sum_{k\le m}\binom{r}{k}(-1)^k = (-1)^m\binom{r-1}{m},\quad m\in\mathbb{Z}$$

Here $\frac{t_{k+1}}{t_k} = \frac{(k-m)}{(r-m+k+1)} = \frac{(k+1)(k-m)(1)}{(k-m+r+1)(k+1)}$

so (5.16), where this came from gives a closed-form for

$$F\left(\begin{matrix}-1, m\\ -m+r+1\end{matrix}\,\middle|\,1\right)$$

this is similar to (†)

$$m \to n \;;\; r+1 \to -r$$

Hypergeometrics are undefined when b's are zero or negative integer, such as (†). How can we rigorously deal w/this case?

$$\lim_{\varepsilon\to 0} F\left(\begin{matrix}1, -n\\ -n-r+\varepsilon\end{matrix}\,\middle|\,1\right) = ?$$

Also in (†)'s derivation we used

$$\binom{r+n-k}{n-k} = \frac{(r+n-k)!}{r!\,(n-k)!} \quad , \text{ this}$$

doesn't hold when r is a negative $\mathbb{Z}$, must again consider $\lim_{\varepsilon\to 0}$ w/$r+\varepsilon$, so we need a definition of the factorial that holds for all $\mathbb{C}$:

$$\frac{1}{z!} = \lim_{n\to\infty}\binom{n+z}{n} n^{-z}$$

Was discovered by Euler when 22. This exists $\forall z$, and is zero only when $z \in \mathbb{Z}^-$, also

$$z! = \int_0^\infty t^z \cdot e^{-t} dt \quad , \quad Re\{z\} > -1,$$

but $z! = z(z-1)!$ can be used to extend.

This is similar to Gamma function:

$$\Gamma(z+1) = z!$$
$$(-z)!\,\Gamma(z) = \pi / \sin \pi z$$

All these generalization can be used to define generalized factorial powers for $w, z \in \mathbb{C}$:

$$z^{\underline{w}} = \frac{z!}{(z-w)!} \quad ; \quad z^{\overline{w}} = \frac{\Gamma(z+w)}{\Gamma(z)},$$

when they are ∞/∞, must use limiting behavior, so

$$\binom{z}{w} = \lim_{\zeta \to z} \lim_{\omega \to w} \frac{\zeta!}{\omega!\,(\zeta-\omega)!}$$

Turn to Vandermonde's Convolution:

$$\sum_k \binom{r}{k}\binom{s}{n-k} = \binom{r+s}{n} \quad , \quad n \in \mathbb{Z}$$

$$t_k = \frac{r!}{(r-k)!\,k!} \; \frac{s!}{(s-n+k!)\,(n-k)!}$$

so $$\frac{t_{k+1}}{t_k} = \frac{k-r}{k+1}\,\frac{k-n}{k+s-n+1}\quad\left[(-1)^2 \text{ also}\right],$$

to $\rightarrow \binom{s}{n} F\left({-r,\ -n \atop s-n+1}\Big|\, 1\right) = \binom{r+s}{n}$, this can be used to find $F(a,b;c;z)\big|_{z=1}$ when $b \in \mathbb{Z}^-$.

Let's rewrite this

$$F\left({a,\ b \atop c}\Big|\, 1\right) = \frac{\Gamma(c-a-b)\,\Gamma(c)}{\Gamma(c-a)\ \Gamma(c-b)} \qquad b \in \mathbb{Z}^- \cup 0$$

or $\operatorname{Re}\{c\} > \operatorname{Re}\{a\} + \operatorname{Re}\{b\}$

There are other cases when $F\left({a,b \atop c}\big|1\right)$ doesn't conv. when $b=-n$

$$F\left({a,\ -n \atop c}\Big|\,1\right) = \frac{(c-a)^{\overline{n}}}{c^{\overline{n}}} = \frac{(a-c)^{\underline{n}}}{(-c)^{\underline{n}}} \qquad (*)$$

$n \in \mathbb{Z} \geq 0.$

Note (+) is just $a=1$ of $(*)$.

Recall Problem 1 before:

$\sum_{k\geq 0} \binom{m}{k}\Big/\binom{n}{k}$ is $F(1,-m;-n;1)$

In Problems 2 & 4 has $F(2,1-n;2-m;1)$ &

Problem 6 is $F(n+1,-n;2|1)$

Problem 3 is a special case of $\sum_k \binom{n-k}{k} z^k$

and relates to $F\left({1+2\lceil n/2\rceil, -n \atop 1/2} \middle| -z/4\right)$.

When we look at the coefficients of $(1-z)^r(1+z)^r$:

$$F\left({1-c-2n, -2n \atop c} \middle| 1\right) = (-1)^n \frac{(2n)!}{n!} \frac{(c-1)!}{(c+n-1)!}$$

$$n \in \mathbb{Z}_{\geq 0}$$

this is Kummer's formula when generalized to $\mathbb{C}$:

$$F\left({a, b \atop 1+b-a} \middle| -1\right) = \frac{(b/2)!}{b!} \frac{(b-a)^{\underline{b/2}}}{},$$

compare these two, replace $c \rightarrow 1-2n-a$, then they are the same $\Longleftrightarrow$

$$(-1)^n \frac{(2n!)}{n!} = \lim_{b \to -2n} \frac{(b/2)!}{b!} = \lim_{x \to -n} \frac{x!}{(2x)!}$$

when $n \in \mathbb{Z}^+$. with $n = -3$, should have:

$$\frac{(-6!)}{3!} = \lim_{x \to -3} \frac{x!}{(2x)!}, \quad (-3)! \ \& \ (-6)! \text{ are } \infty$$

imagine $(-3)! = (-3)(-4)(-5)(-6)!$, but

$$\lim_{x \to -3} \frac{x!}{(2x)!} \neq (-3)(-4)(-5)$$

$$= (-4)(-5)(-6) \quad \text{by}$$

Correct approach is to use $(-z!)\,\Gamma(z) = \frac{\pi}{\sin \pi z}$

$$\frac{(-n-\epsilon)!}{(-2n-2\epsilon)!}\frac{\Gamma(n+\epsilon)}{\Gamma(2n+2\epsilon)} = \frac{\sin(2n+2\epsilon)\pi}{\sin(n+\epsilon)\pi} = \frac{\sin 2x\pi}{\sin x\pi}$$

$x = n+\epsilon$

$\sin(2x) = 2\sin x\cos x \Rightarrow$

$$= \frac{2\sin\pi x\cos\pi x}{\sin\pi x} = 2\cos(n+\epsilon)\pi, \text{ so}$$

we get $\displaystyle\lim_{\epsilon\to 0}\frac{(-n-\epsilon)!}{(-2n-2\epsilon)!} = 2(-1)^n\frac{\Gamma(2n)}{\Gamma(n)} =$

$$2(-1)^n\frac{(2n-1)!}{(n-1)!} = (-1)^n\frac{(2n)!}{n!}$$

$$\sum_k\binom{a+b}{a+k}\binom{b+c}{b+k}\binom{c+a}{c+k}(-1)^k = \frac{(a+b+c)!}{a!\,b!\,c!} \quad \text{can}$$

be rewritten as

$$F\left({1-a-2n,\ 1-b-2n,\ -2n \atop a,\ b}\middle|\,1\right) = (-1)^n\frac{(2n)!}{n!}\cdot\frac{(a+b+2n-1)^{\overline{n}}}{a^{\overline{n}}\,b^{\overline{n}}}$$

when this is extended to $\mathbb{C}$, Dixon's Formula:

$$F\left({a,\ b,\ -n \atop c,\ a+b-c-n+1}\middle|\,1\right) = \frac{(c/2)!}{c!}\frac{(c-a)^{\underline{c/2}}(c-b)^{\underline{c/2}}}{(c-a-b)^{\underline{c/2}}}$$

$$Re\{a\} + Re\{b\} < 1 + Re\{\tfrac{c}{2}\}$$

Also $\displaystyle\sum_k\binom{m-r+s}{k}\binom{n+r-s}{n-k}\binom{r+k}{m+n} = \binom{r}{m}\binom{s}{n}$ $\quad m,n\in\mathbb{Z}$

produces Saalschütz's identity:

$$F\left({a,\ b,\ n \atop c,\ a+b-c-n+1}\middle|\,1\right) = \frac{(c-a)^{\overline{n}}(c-b)^{\overline{n}}}{c^{\overline{n}}(c-a-b)^{\overline{n}}} \qquad n \geq 0 \in \mathbb{Z}$$

$$= \frac{(a-c)^{\underline{n}}(b-c)^{\underline{n}}}{(-c)^{\underline{n}}(a+b-c)^{\underline{n}}}$$

this evaluates $F\left({\_,\_,\_ \atop \_,\_}\middle|\,1\right)$ provided an a is nonpositive $\mathbb{Z}$, $b_1+b_2 = a_1+a_2+a_3+1$.

Problem 8 reduces to

$$\frac{1}{1+x}\, F\left({x+1,\ n+1,\ -n \atop 1,\ x+2}\middle|\,1\right) = (-1)^n x^{\underline{n}} x^{\underline{-n-1}}$$

special case of SI w/ $\ell = 1$.

Problem 7: $F\left({n+1,\ m-n,\ 1,\ \frac{1}{2} \atop \frac{1}{2}n+1,\ \frac{m+1}{2},\ 2}\middle|\,1\right) = \frac{n}{n} \qquad n \geq m > 0 \in \mathbb{Z}$

this is 1st case w/ 3 lower params., but lhs can be replaced with

$$F\left({n,\ n-m,\ -\frac{1}{2} \atop \frac{1}{2}n,\ \frac{m-1}{2}}\middle|\,1\right) - 1 \quad \text{using an exercise.}$$

The convolution identities are not all hypergeometric when $t=1$. (5.62)

$$\sum_k \binom{tk+r}{k}\binom{tn-tk+s}{n-k}\frac{r}{tk+r} = \binom{tn+r+s}{n}$$

when t is small

$$F\left({\tfrac{1}{2}r,\ \tfrac{1}{2}r+\tfrac{1}{2},\ -n,\ -n-s \atop r+1,\ -n-\tfrac{1}{2}s,\ -n-\tfrac{1}{2}s-\tfrac{1}{2}} \middle| 1 \right) = \binom{r+s+2n}{n}\binom{s+2n}{n}^{-1}$$

$$F\left({\tfrac{1}{3}r,\ \tfrac{r+1}{3},\ \tfrac{r+2}{3},\ -n,\ -n-\tfrac{1}{2}s,\ -n-\tfrac{1}{2}s+\tfrac{1}{2} \atop \tfrac{r+1}{2},\ \tfrac{r+2}{2},\ -n-\tfrac{1}{3}s,\ -n-\tfrac{1}{3}s+\tfrac{1}{3},\ -n-\tfrac{1}{3}s+\tfrac{2}{3}} \middle| 1 \right) = \binom{r+s+3n}{n}\binom{s+3n}{n}^{-1}$$

this gives Problem 7 $(r,s,n) \to (1, m-2n-1, n-m)$

Unexpected sum

$$\sum_{k\le m} \binom{m+k}{k} 2^{-k} = 2^m \iff$$

$$\sum_{k\ge 0} \binom{2m-k}{m-k} 2^k = 2^{2m}$$

term ratio is $\dfrac{2(k-m)}{(k-2m)}$ so $z=2$

$$\binom{2m}{m} F\left({1,\ -m \atop -2m} \middle| 2 \right) = 2^{2m} \qquad m \in \mathbb{Z}_{\ge 0}$$

not allowed

what about limiting cases?

$$\lim_{\varepsilon \to 0} F\left({-1+\varepsilon,\ -3 \atop -2+\varepsilon} \middle| 1 \right) =$$

$$= \lim_{\varepsilon \to 0} \left(1 + \frac{(-1+\varepsilon)(-3)}{(-2+\varepsilon)\ 1!} + \frac{(-1+\varepsilon)(-3)(-2)}{(-2+\varepsilon)(-1+\varepsilon)\ 2!} + \frac{(-1+\varepsilon)(\varepsilon)(1+\varepsilon)(-3)(-2)(-1)}{(-2+\varepsilon)(-1+\varepsilon)\ \varepsilon\ 3!} \right)$$

$$= 1 + \tfrac{3}{2} + 0 + \tfrac{1}{2} = 0 \text{, so}$$

$$\lim_{\varepsilon \to 0} \left({-1, -3 \atop -2+\varepsilon} \middle| 1 \right) = \lim_{\varepsilon \to 0} \left(1 + \frac{(-1)(-3)}{(-2+\varepsilon)1!} + 0 + 0 \right)$$

$$= 1 - \tfrac{3}{2} + 0 + 0 = -\tfrac{1}{2}$$

also $\binom{-1}{-1} = 0 = \lim_{\varepsilon \to 0} \binom{-1+\varepsilon}{-1} \neq \lim_{\varepsilon \to 0} \binom{-1+\varepsilon}{-1+\varepsilon} = 1$

So $\binom{2m}{m} \lim_{\varepsilon \to 0} F\left({1, -m \atop -2m+\varepsilon} \middle| 2 \right) = 2^{2m}$

holds.