

Theory of Computation

Prof. Michael Mascagni



Florida State University
Department of Computer Science

This course aims to cover ...

- ▶ the development of computability theory using an extremely simple abstract programming language,
- ▶ the various different formulations of computability and their equivalence

By the end of this course, you should be able to ...

- ▶ appreciate the existence of universal digital computers,
- ▶ understand there are well-defined functions that cannot be computed even by the universal computers,
- ▶ know that certain problems are truly harder than others,
- ▶ use various formalized computation models to solve your problems, and
- ▶ show that some problems are just too difficult for the models at hand.

Textbook

Martin Davis, Ron Sigal, and Elaine J. Weyuker. *Computability, Complexity, and Languages: Fundamentals of Theoretical Computer Science*, 2nd edition. February 1994, Morgan Kaufmann. ISBN: 0122063821.

- ▶ Written for people who may know programming, but from a mathematical view of the subjects. Enjoyably readable but very rigorous.
- ▶ “It is our purpose . . . to provide an introduction to the various aspects of theoretical computer science for undergraduate and graduate students that is sufficiently comprehensive that . . . research papers will become accessible to our readers.” (the authors)
- ▶ We will cover just one half of the materials in the book.

Outline of Lecture

- ▶ Review some preliminary materials.
- ▶ Define an abstract programming language \mathcal{S} that is extremely simple.
- ▶ Write some programs in \mathcal{S} .

Cartesian Product

- ▶ If S_1, S_2, \dots, S_n are given sets, then we write $S_1 \times S_2 \times \dots \times S_n$ for the set of all n -tuples (a_1, a_2, \dots, a_n) such that $a_1 \in S_1, a_2 \in S_2, \dots, a_n \in S_n$.
- ▶ $S_1 \times S_2 \times \dots \times S_n$ is called the *Cartesian product* of S_1, S_2, \dots, S_n .
- ▶ In case $S_1 = S_2 = \dots = S_n = S$ we write S^n for the Cartesian product $S_1 \times S_2 \times \dots \times S_n$.

Functions

- ▶ A function f is a set whose members are ordered pairs (i.e., 2-tuples) and has the special property

$$(a, b) \in f \text{ and } (a, c) \in f \text{ implies } b = c.$$

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- ▶ A *partial function* on a set S is a function whose domain is a subset of S . If a partial function on S has the domain S , then it is called a *total function*.

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- ▶ A *partial function* on a set S is a function whose domain is a subset of S . If a partial function on S has the domain S , then it is called a *total function*.
- ▶ We write $f(a) \downarrow$ and say that $f(a)$ is *defined* if a is in the domain of f ; if a is not in the domain of f , we write $f(a) \uparrow$ and say that $f(a)$ is *undefined*.

Examples of Functions

- ▶ Let f be the set of ordered pairs (n, n^2) for $n \in \mathbb{N}$. Then, for each $n \in \mathbb{N}$, $f(n) = n^2$. The domain of f is \mathbb{N} . The range of f is the set of perfect squares. f is a total function.

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- ▶ Assuming N is our universe, an example of a partial function on N is given by $g(n) = \sqrt{n}$. The domain of g is the set of perfect squares. The range of g is N . g is not a total function.

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- ▶ For a partial function f on a Cartesian product $S_1 \times S_2 \times \cdots \times S_n$, we write $f(a_1, \dots, a_n)$ rather than $f((a_1, \dots, a_n))$.

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- ▶ A partial function f on a set S^n is called an n -ary partial function on S , or a function of n variables on S . We use *unary* and *binary* for 1-ary and 2-ary, respectively.

Predicate

A *predicate*, or a *Boolean-valued function*, on a set S is a *total function* P on S such that for each $a \in S$, either

$$P(a) = \text{TRUE} \quad \text{or} \quad P(a) = \text{FALSE}$$

We also identify the truth value TRUE with number 1 and the truth value FALSE with number 0.

Logic Connectives

The three *logic connectives*, or *propositional connectives*, \sim , \vee , & are defined by the two tables below.

p	$\sim p$
0	1
1	0

p	q	$p \& q$	$p \vee q$
1	1	1	1
0	1	0	1
1	0	0	1
0	0	0	0

Characteristic Function

Given a predicate P on a set S , there is a corresponding subset R of S consisting of all elements $a \in S$ for which $P(a) = 1$. We write

$$R = \{a \in S \mid P(a)\}.$$

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Conversely, given a subset R of a given set S , the expression $x \in R$ defines a predicate P on S :

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$$\{x \in S \mid P(x) \& Q(x)\} = \{x \in S \mid P(x)\} \cap \{x \in S \mid Q(x)\},$$

$$\{x \in S \mid P(x) \vee Q(x)\} = \{x \in S \mid P(x)\} \cup \{x \in S \mid Q(x)\},$$

$$\{x \in S \mid \sim P(x)\} = S - \{x \in S \mid P(x)\}.$$

Bounded Existential Quantifier

Let $P(t, x_1, \dots, x_n)$ be a $(n + 1)$ -ary predicate. Let predicate $Q(y, x_1, \dots, x_n)$ be defined by

$$\begin{aligned} Q(y, x_1, \dots, x_n) &= P(0, x_1, \dots, x_n) \\ &\vee P(1, x_1, \dots, x_n) \\ &\vee \dots \\ &\vee P(y, x_1, \dots, x_n) \end{aligned}$$

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That is, $Q(y, x_1, \dots, x_n)$ is true if there is a value $t \leq y$ such that $P(t, x_1, \dots, x_n)$ is true. We write this predicate Q as

$$(\exists t)_{\leq y} P(t, x_1, \dots, x_n)$$

“ $(\exists t)_{\leq y}$ ” is called a *bounded existential quantifier*.

Bounded Universal Quantifier

Let $P(t, x_1, \dots, x_n)$ be a $(n + 1)$ -ary predicate. Let predicate $Q(y, x_1, \dots, x_n)$ be defined by

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That is, $Q(y, x_1, \dots, x_n)$ is true if for all value $t \leq y$ such that $P(t, x_1, \dots, x_n)$ is true. We write this predicate Q as

$$(\forall t)_{\leq y} P(t, x_1, \dots, x_n)$$

$(\forall t)_{\leq y}$ is called a *bounded universal quantifier*.

Proof by Contradiction

In a *proof by contradiction*, we begin by assuming the assertion we wish to prove is *false*. We then derive a contradiction based on this (faulty) assumption along with (faultless) logical reasoning. We then conclude that the original assertion must be true.

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However, the equation $2 = (m/n)^2$ can be rewritten as $m^2 = 2n^2$ which shows that m must be even. Let $m = 2k$, then $m^2 = (2k)^2 = 4k^2$. But this implies $n^2 = 2k^2$. Thus n is even. Now both m and n are even, which is a contradiction.

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We conclude that $2 = (m/n)^2$ has no solution $m, n \in \mathbb{N}$. □

Mathematical Induction

Given a predicate $P(x)$, and the assertion “ $P(n)$ is true for all $n \in \mathbb{N}$ ”, we can use mathematical induction to try to establish this assertion. One proceeds by proving a pair of auxiliary statements about $P(x)$, namely,

$$P(0)$$

and

$$\text{For all } n \in \mathbb{N}, P(n) \text{ implies } P(n + 1)$$

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$$\text{For all } n \in \mathbb{N}, P(n) \text{ implies } P(n + 1)$$

In the second statement above, $P(n)$ is called an induction hypothesis. If both statements above are proved to be true, one then concludes that

$$\text{For all } n \in \mathbb{N}, P(n)$$

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We expand $\sum_{i=0}^{n+1} (2i + 1)$ by its definition,

$$\begin{aligned} \sum_{i=0}^{n+1} (2i + 1) &= \sum_{i=0}^n (2i + 1) + 2(n + 1) + 1 \\ &= (n + 1)^2 + 2(n + 1) + 1 \quad (\text{by induction hypothesis}) \\ &= (n + 2)^2. \end{aligned}$$

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We conclude that for all $n \in \mathbb{N}$, $\sum_{i=0}^n (2i + 1) = (n + 1)^2$. \square

The Programming Language \mathcal{I}

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 - $V \leftarrow V + 1$ Increase by 1 the value of the variable V .
 - $V \leftarrow V - 1$ If the value of V is 0, leave it unchanged; otherwise decrease by 1 the value of V .
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- ▶ Labels: $A_1, B_1, C_1, D_1, E_1, A_2, B_2, C_2, D_2, E_2, A_3, \dots$
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- ▶ Exit label: E .
- ▶ All variables and labels are in the global scope.

Programming in \mathcal{I}

- ▶ A program is a list (i.e., a finite sequence) of instructions.
- ▶ *The output variable Y and the local variables Z_i initially have the value 0.*
- ▶ A program halts when there is no more instruction to execute.
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- ▶ A program halts when there is no more instruction to execute.
- ▶ A program also halts if an instruction labeled L is to be executed, but there is no instruction with that label.
- ▶ What does this program do?

```
[A]  X ← X - 1  
      Y ← Y + 1  
      IF X ≠ 0 GOTO A
```

A Bug?

- ▶ What does this program do?

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- ▶ The above program *computes* the function

$$f(x) = \begin{cases} 1 & \text{if } x = 0 \\ x & \text{otherwise.} \end{cases}$$

A Program That Computes $f(x) = x$

```
[A]  IF  $X \neq 0$  GOTO B  
      $Z \leftarrow Z + 1$   
     IF  $Z \neq 0$  GOTO E  
[B]   $X \leftarrow X - 1$   
      $Y \leftarrow Y + 1$   
      $Z \leftarrow Z + 1$   
     IF  $Z \neq 0$  GOTO A
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- ▶ What does Z actually do?

A Program That Computes $f(x) = x$

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$Z \leftarrow Z + 1$

IF $Z \neq 0$ GOTO E

[B] $X \leftarrow X - 1$

$Y \leftarrow Y + 1$

$Z \leftarrow Z + 1$

IF $Z \neq 0$ GOTO A

- ▶ What does Z actually do?
- ▶ What does the following do?

$Z \leftarrow Z + 1$

IF $Z \neq 0$ GOTO L

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$Y \leftarrow Y + 1$

$Z \leftarrow Z + 1$

IF $Z \neq 0$ GOTO A

- ▶ What does Z actually do?
- ▶ What does the following do?

$Z \leftarrow Z + 1$

IF $Z \neq 0$ GOTO L

- ▶ That is an unconditional goto!

GOTO L

A Macro for Unconditional GOTO

- ▶ Before macro expansion:

[A] IF $X \neq 0$ GOTO B
GOTO E

[B] $X \leftarrow X - 1$
 $Y \leftarrow Y + 1$
GOTO A

A Macro for Unconditional GOTO

- ▶ Before macro expansion:

[A] IF $X \neq 0$ GOTO B
GOTO E

[B] $X \leftarrow X - 1$
 $Y \leftarrow Y + 1$
GOTO A

- ▶ After macro expansion:

[A] IF $X \neq 0$ GOTO B
 $Z_1 \leftarrow Z_1 + 1$
IF $Z_1 \neq 0$ GOTO E

[B] $X \leftarrow X - 1$
 $Y \leftarrow Y + 1$
 $Z_2 \leftarrow Z_2 + 1$
IF $Z_2 \neq 0$ GOTO A

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 GOTO E

[B] $X \leftarrow X - 1$
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[B] $X \leftarrow X - 1$
 $Y \leftarrow Y + 1$
 $Z_2 \leftarrow Z_2 + 1$
 IF $Z_2 \neq 0$ GOTO A

- ▶ *Fresh local variables are always used during macro expansions.* 22 / 33

Copy The Value of Variable X to Variable Y

```
▶ [A]  IF  $X \neq 0$  GOTO  $B$   
      GOTO  $E$   
      [B]  $X \leftarrow X - 1$   
           $Y \leftarrow Y + 1$   
          GOTO  $A$ 
```

Copy The Value of Variable X to Variable Y

- ▶ [A] IF $X \neq 0$ GOTO B
GOTO E
- [B] $X \leftarrow X - 1$
 $Y \leftarrow Y + 1$
GOTO A
- ▶ Anything wrong?

Copy The Value of Variable X to Variable Y

▶ [A] IF $X \neq 0$ GOTO B
GOTO E

[B] $X \leftarrow X - 1$
 $Y \leftarrow Y + 1$
GOTO A

- ▶ Anything wrong?
- ▶ The value of X is “destroyed” while copied to Y !

Copy The Value of Variable X to Variable Y , Continued

- ▶ [A] IF $X \neq 0$ GOTO B
GOTO C
- [B] $X \leftarrow X - 1$
 $Y \leftarrow Y + 1$
 $Z \leftarrow Z + 1$
GOTO A
- [C] IF $Z \neq 0$ GOTO D
GOTO E
- [D] $Z \leftarrow Z - 1$
 $X \leftarrow X + 1$
GOTO C

Copy The Value of Variable X to Variable Y , Continued

- ▶ [A] IF $X \neq 0$ GOTO B
GOTO C
- [B] $X \leftarrow X - 1$
 $Y \leftarrow Y + 1$
 $Z \leftarrow Z + 1$
GOTO A
- [C] IF $Z \neq 0$ GOTO D
GOTO E
- [D] $Z \leftarrow Z - 1$
 $X \leftarrow X + 1$
GOTO C
- ▶ Anything wrong?

Copy The Value of Variable X to Variable Y , Continued

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▶ [A]  IF  $X \neq 0$  GOTO  $B$   
      GOTO  $C$   
[B]   $X \leftarrow X - 1$   
       $Y \leftarrow Y + 1$   
       $Z \leftarrow Z + 1$   
      GOTO  $A$   
[C]  IF  $Z \neq 0$  GOTO  $D$   
      GOTO  $E$   
[D]   $Z \leftarrow Z - 1$   
       $X \leftarrow X + 1$   
      GOTO  $C$ 
```

- ▶ Anything wrong?
- ▶ This program is correct only when Y and Z are initialized to the value 0. It cannot be used as a macro.

A Macro for $V \leftarrow V'$

- ▶ $V \leftarrow 0$
- [A] IF $V' \neq 0$ GOTO B
GOTO C
- [B] $V' \leftarrow V' - 1$
 $V \leftarrow V + 1$
 $Z \leftarrow Z + 1$
GOTO A
- [C] IF $Z \neq 0$ GOTO D
GOTO E
- [D] $Z \leftarrow Z - 1$
 $V' \leftarrow V' + 1$
GOTO C

A Macro for $V \leftarrow V'$

- ▶ $V \leftarrow 0$
 - [A] IF $V' \neq 0$ GOTO B
GOTO C
 - [B] $V' \leftarrow V' - 1$
 $V \leftarrow V + 1$
 $Z \leftarrow Z + 1$
GOTO A
 - [C] IF $Z \neq 0$ GOTO D
GOTO E
 - [D] $Z \leftarrow Z - 1$
 $V' \leftarrow V' + 1$
GOTO C
- ▶ Anything wrong?

A Macro for $V \leftarrow V'$

- ▶ $V \leftarrow 0$
- [A] IF $V' \neq 0$ GOTO B
 GOTO C
- [B] $V' \leftarrow V' - 1$
 $V \leftarrow V + 1$
 $Z \leftarrow Z + 1$
 GOTO A
- [C] IF $Z \neq 0$ GOTO D
 GOTO E
- [D] $Z \leftarrow Z - 1$
 $V' \leftarrow V' + 1$
 GOTO C
- ▶ Anything wrong?
- ▶ $V \leftarrow 0$ is not an instruction in \mathcal{I} .

A Macro for $V \leftarrow 0$

A Macro for $V \leftarrow 0$

```
[L]  V ← V - 1  
     IF V ≠ 0 GOTO L
```

A Program That Computes $f(x_1, x_2) = x_1 + x_2$

```
Y ← X1
Z ← X2
[B] IF Z ≠ 0 GOTO A
    GOTO E
[A] Z ← Z - 1
    Y ← Y + 1
    GOTO B
```

Note that Z is used to preserve the value of X_2 so that it will not be destroyed during the computation.

A Program That Computes $f(x_1, x_2) = x_1 \cdot x_2$

```
▶      Z2 ← X2
[B]    IF Z2 ≠ 0 GOTO A
      GOTO E
[A]    Z2 ← Z2 - 1
      Z1 ← X1 + Y
      Y ← Z1
      GOTO B
```


A Program That Computes $f(x_1, x_2) = x_1 \cdot x_2$

- ▶ $Z_2 \leftarrow X_2$
[B] IF $Z_2 \neq 0$ GOTO A
GOTO E
- [A] $Z_2 \leftarrow Z_2 - 1$
 $Z_1 \leftarrow X_1 + Y$
 $Y \leftarrow Z_1$
GOTO B
- ▶ OK!

A Shorter Program That Computes $f(x_1, x_2) = x_1 \cdot x_2$?

```
▶      Z2 ← X2
[B]    IF Z2 ≠ 0 GOTO A
      GOTO E
[A]    Z2 ← Z2 - 1
      Y ← X1 + Y
      GOTO B
```

A Shorter Program That Computes $f(x_1, x_2) = x_1 \cdot x_2$?

- ▶ $Z_2 \leftarrow X_2$
[B] IF $Z_2 \neq 0$ GOTO A
GOTO E
[A] $Z_2 \leftarrow Z_2 - 1$
 $Y \leftarrow X_1 + Y$
GOTO B
- ▶ *NO GOOD!*

A Shorter Program That Computes $f(x_1, x_2) = x_1 \cdot x_2$?

- ▶ $Z_2 \leftarrow X_2$
[B] IF $Z_2 \neq 0$ GOTO A
GOTO E
- [A] $Z_2 \leftarrow Z_2 - 1$
 $Y \leftarrow X_1 + Y$
GOTO B
- ▶ *NO GOOD!*
- ▶ Why?

- ▶ The macro for $f(x_1, x_2) = x_1 + x_2$

$Y \leftarrow X_1$

$Z \leftarrow X_2$

[B] IF $Z \neq 0$ GOTO A

GOTO E

[A] $Z \leftarrow Z - 1$

$Y \leftarrow Y + 1$

GOTO B

- ▶ The macro for $f(x_1, x_2) = x_1 + x_2$

$Y \leftarrow X_1$

$Z \leftarrow X_2$

[B] IF $Z \neq 0$ GOTO A
 GOTO E

[A] $Z \leftarrow Z - 1$
 $Y \leftarrow Y + 1$
 GOTO B

- ▶ Macro expanding $Y \leftarrow X_1 + Y$:

$Y \leftarrow X_1$

$Z \leftarrow Y$

[B] IF $Z \neq 0$ GOTO A
 GOTO E

[A] $Z \leftarrow Z - 1$
 $Y \leftarrow Y + 1$
 GOTO B

- ▶ The macro for $f(x_1, x_2) = x_1 + x_2$

$Y \leftarrow X_1$

$Z \leftarrow X_2$

[B] IF $Z \neq 0$ GOTO A
 GOTO E

[A] $Z \leftarrow Z - 1$
 $Y \leftarrow Y + 1$
 GOTO B

- ▶ Macro expanding $Y \leftarrow X_1 + Y$:

$Y \leftarrow X_1$

$Z \leftarrow Y$

[B] IF $Z \neq 0$ GOTO A
 GOTO E

[A] $Z \leftarrow Z - 1$
 $Y \leftarrow Y + 1$
 GOTO B

- ▶ The above actually computes $f(x_1, x_2) = 2 \cdot x_1$

A Program That Computes $f(x_1, x_2) = x_1 \cdot x_2$, Revisited

- ▶ Need to macro expand $Z_1 \leftarrow X_1 + Y$.
- ▶ After macro expansion:

```
      Z2 ← X2
[B]   IF Z2 ≠ 0 GOTO A
      GOTO E
[A]   Z2 ← Z2 - 1
      Z1 ← X1
      Z3 ← Y
[B2] IF Z3 ≠ 0 GOTO A2
      GOTO E2
[A2] Z3 ← Z3 - 1
      Z1 ← Z1 + 1
      GOTO B2
[E2] Y ← Z1
      GOTO B
```


Note on The Macro Expansion

- ▶ The output variable Y in the macro $f(x_1, x_2) = x_1 + x_2$ is now fresh variable Z_1 in the expanded form.
- ▶ The local variable Z in the macro $f(x_1, x_2) = x_1 + x_2$ is now fresh variable Z_3 in the expanded form (as variables Z_1 and Z_2 are already used).
- ▶ Fresh labels $A_2, B_2,$ and E_2 are used in the expanded form (as the original labels $A, B,$ and E are already used).
- ▶ The instruction GOTO E_2 only terminates the addition. The computation must continue to place following the addition. Hence, the instruction immediately following the addition is labeled E_2 .
- ▶ *Unlimited supply of fresh local variables and local labels!*
- ▶ More about macro expansion next week.

A Final Example

- ▶ What does this program compute?

$Y \leftarrow X_1$

$Z \leftarrow X_2$

[C] IF $Z \neq 0$ GOTO A
GOTO E

[A] IF $Y \neq 0$ GOTO B
GOTO A

[B] $Y \leftarrow Y - 1$
 $Z \leftarrow Z - 1$
GOTO C

A Final Example

- ▶ What does this program compute?

$Y \leftarrow X_1$

$Z \leftarrow X_2$

[C] IF $Z \neq 0$ GOTO A
GOTO E

[A] IF $Y \neq 0$ GOTO B
GOTO A

[B] $Y \leftarrow Y - 1$
 $Z \leftarrow Z - 1$
GOTO C

- ▶ If we begin with $X_1 = 5$ and $X_2 = 2, \dots$

A Final Example

- ▶ What does this program compute?

$Y \leftarrow X_1$

$Z \leftarrow X_2$

[C] IF $Z \neq 0$ GOTO A
GOTO E

[A] IF $Y \neq 0$ GOTO B
GOTO A

[B] $Y \leftarrow Y - 1$
 $Z \leftarrow Z - 1$
GOTO C

- ▶ If we begin with $X_1 = 5$ and $X_2 = 2, \dots$
- ▶ If we begin with $X_1 = 2$ and $X_2 = 5, \dots$

A Final Example

- ▶ What does this program compute?

$Y \leftarrow X_1$

$Z \leftarrow X_2$

[C] IF $Z \neq 0$ GOTO A
GOTO E

[A] IF $Y \neq 0$ GOTO B
GOTO A

[B] $Y \leftarrow Y - 1$
 $Z \leftarrow Z - 1$
GOTO C

- ▶ If we begin with $X_1 = 5$ and $X_2 = 2, \dots$
- ▶ If we begin with $X_1 = 2$ and $X_2 = 5, \dots$
- ▶ This program computes the following *partial function*

$$g(x_1, x_2) = \begin{cases} x_1 - x_2 & \text{if } x_1 \geq x_2 \\ \uparrow & \text{if } x_1 < x_2 \end{cases}$$