SPHERICAL REPRESENTATIONS OF SHAPE USING PARAMETRIZATIONS WITH MINIMAL DISTORTION

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ABSTRACT

Spherical harmonics are commonly used in the construction of multi-resolution representations of complex spherical shapes such as brain surface meshes. A key step in generating such representations for a spherical mesh is to construct a one-to-one map onto a sphere. A parametrization inevitably introduces local distortions such as stretching and compression, so that some regions can be severely undersampled or oversampled. As such, parametrizations with minimal metric distortion are desirable because they yield more accurate representations. In this paper, we use a spherical parametrization method that explicitly minimizes the overall distortion penalizing stretching and compression symmetrically. The optimization process uses a hierarchical and iterative scheme. We show numerically that the proposed parametrizations lead to accurate spherical representations of various brain structures.

Index Terms— Spherical harmonics, parametrization, brain mapping, shape representation.

1. INTRODUCTION

Spherical harmonics [1] have been widely used to represent contours of brain structures (e.g. [2, 3, 4]) and more general surfaces (e.g. [5]), typically given by triangular meshes. If a spherical surface is convex or near convex, radial projection of the vertices relative to its centroid is a simple and effective way of constructing a parametrization over a standard sphere. For more complex surfaces with bends and folds such as those of brain structures, constructing a one-to-one parametrization is a critical step for creating accurate representations. Several methods have been proposed in recent years, especially for applications to medical imaging (e.g. [2, 3, 4]). Parametrization methods differ in the choice of optimality criterion, as well as in the optimization process. For a surface presented as a triangular mesh, the parametrization determines corresponding vertices on the sphere and thus a sampling of the domain. To obtain an accurate representation, we propose the construction of parametrizations that minimize the overall metric distortion such as those due to stretching or compression. Since metric properties of the mapping are determined by its action on the first fundamental form [6], the distortion metrics can be categorized based on this fact. For example, conformal parametrizations preserve angles (e.g., [2]). However, local areas can be stretched and compressed significantly due to local scaling [7] and thus the parametrization can introduce large metric distortions. Thus, a conformal parametrization often requires a much larger number of fundamental harmonics to produce a reconstruction with error comparable to one with more controlled distortion [4]. To address this type of problems, several methods have been proposed for the construction of parametrizations that are as close as possible to local isometries. For example, [4] employs a criterion that involves a combination of measures of area and angle distortion. However, in practice, it can be difficult to construct an optimal or near-optimal parametrization with this method.

In this paper, we use a parametrization technique that directly maps a given spherical mesh to a sphere[7, 8]. For high-resolution meshes, with sufficiently small triangles, the mapping can be well approximated by a piecewise linear mapping. One each triangle, we use the eigenvalues of the associated Jacobian matrices to quantify the local metric distortion. To construct a nearly optimal parametrization, we use an iterative coarse-to-fine process based on a hierarchy of meshes. We apply the method to brain surfaces and show numerically that the parametrizations lead to representations with a fairly small number of spherical harmonics with small reconstruction error.

The rest of the paper is organized as follows. In section 2, we describe the optimality criterion for spherical parametrizations [8]. Section 3 offers a brief review of spherical harmonics and Section 4 presents experimental results on spherical surface representations and reconstructions. Section 5 concludes the paper with a summary and brief discussion.

2. PARAMETRIZATIONS WITH MINIMAL DISTORTION

Let $\mathbb{S}^2$ denote the unit sphere centered at the origin in $\mathbb{R}^3$ and $\phi\colon \mathbb{S}^2 \to M$ a parametrization of the surface M over $\mathbb{S}^2$ (see Fig. 1). Given a spherical triangulation of the unit sphere, let Δ_{S_i} denote the ith triangle of $\mathbb{S}^2$ and Δ_{M_i} its image under ϕ.

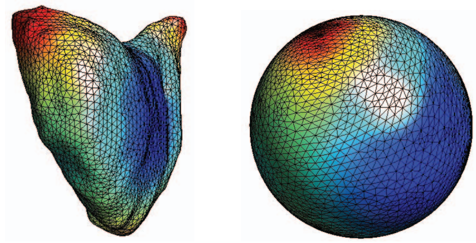


Fig. 1. A parametrization from spherical surface to the unit sphere, where the correspondences are color coded.

The inverse map is denoted $\phi^{-1}\colon M \to S^2$. Assuming that the "triangles" Δ_{M_i} and Δ_{S_i} are sufficiently small, the mapping can be approximated as follows. First, we replace the triangles Δ_{S_i} and Δ_{M_i} with the planar triangles with the same vertices. Second, we approximate ϕ by a mapping that is linear on each triangle, so that ϕ maps each Δ_{S_i} linearly onto Δ_{M_i}. Thus, both $\mathbb{S}^2$ and M are replaced with triangle meshes and the mapping is piecewise linear. Under this assumption, the local stretching and compression of the mapping on each triangle can be characterized by the eigenvalues of the linear mapping, which is an approximation of the 3×2 Jacobian matrix of ϕ. Following [7, 8], we compute the eigenvalues γ_i and Γ_i of $J_{\phi_i^{-1}}$, where $0 < \gamma_i \leq \Gamma_i$. If the mapping is a local isometry, $\gamma_i = \Gamma_i = 1$. If there is significant stretching, Γ_i will be much larger than 1. Similarly, γ_i much smaller than 1 reflects a significant compression. The eigenvalues of J_{ϕ_i} are $1/\gamma_i$ and $1/\Gamma_i$. The cost function used in [7] is the average value of

$$\frac{1}{\gamma_i^2} + \frac{1}{\Gamma_i^2}$$

over M. Clearly, the smaller eigenvalue γ_i dominates this expression, especially when γ_i is small relative to Γ_i. Consequently, compression caused by ϕ_i^{-1} is penalized much more than stretching. Here we use the following cost function, proposed in [8], given by the average value of

$$\log^2 \gamma_i + \log^2 \Gamma_i$$

over M. There are several desirable properties satisfied by this measure of distortion. First, it is symmetric in penalizing compression and stretching by the same factor[1]. Second, the error metric is invariant as to ϕ and ϕ^{-1}. Additionally, this error function is smoother and thus an iterative optimization is more effective. A potential problem with the above approximation is that γ_i and Γ_i can underestimate the stretching and compression of the spherical triangles especially when their aspect ratio is large. The problem can be alleviated by either further subdividing the traingles to improve their aspect ratios [7] or by using the exponential map to estimate the mapping and the eigenvalues [8].

[1]In applications where an asymmetric error metric is preferred, a weight can be used.

A direct minimization of the cost function can be computationally expensive and even ineffective as the number of vertices is large. In addition, a valid parametrization must be given to initialize the process. Following [7, 8], we use a hierarchical, coarse-to-fine scheme [9]. First the given mesh is simplified until it becomes a tetrahedron. A spherical parametrization is constructed and then vertices are inserted on the sphere based on the barycentric coordinates of the vertices that already have been mapped. When a vertex is inserted, a local optimization is performed by exhaustively evaluating all valid locations at a given resolution. An iterative optimization is performed by perturbing all vertices inserted.

3. SPHERICAL HARMONICS

Given a parametrization of a triangular mesh, computing its representations and multi-resolution reconstruction via spherical harmonics are routine tasks. Spherical harmonics define orthonormal basis functions, which are characterized by two parameters: $l \geq 0$ is the band index, and m, $-l \leq m \leq l$, which is the order of the associated Legendre function $P_l^{(m)}$ given by

$$P_l^{(m)}(x) = \frac{(-1)^m}{2^l l!}(1-x^2)^{(m/2)}\frac{d^{l+m}}{dx^{l+m}}(x^2-1)^l.$$

Note that $P_l^{(m)}(x)$ is defined on $[-1, 1]$. The corresponding spherical harmonic is given by

$$Y_l^{(m)}(\theta,\phi) = \sqrt{\frac{(2l+1)}{4\pi}\frac{(l-m)!}{(l+m)!}}P_l^{(m)}(cos\theta)e^{im\phi}.$$

For a fixed l, there are $2l+1$ spherical harmonics of different orders. Numerically, $Y_l^{(-m)}(\theta,\phi)$, where $m > 0$, can be computed using

$$Y_l^{(-m)}(\theta,\phi) = (-1)^m\ \overline{Y_l^{(m)}(\theta,\phi)},$$

where $\overline{Y_l^{(m)}(\theta,\phi)}$ is the conjugate of $Y_l^{(m)}(\theta,\phi)$.

For a mesh M in $\mathbb{R}^3$ with N vertices, given vertex (x_i, y_i, z_i), its corresponding location on the unit sphere in spherical coordinate (θ_i, ϕ_i), and a maximum spherical harmonic degree l_{max}, let B be the spherical harmonic matrix at (θ_i, ϕ_i), ordered by l first and within each band ordered by m. Explicitly, we have

$$B = \begin{bmatrix} Y_0^{(0)}(\theta_1,\phi_1) & Y_1^{(-1)}(\theta_1,\phi_1) & \cdots & Y_{l_{max}}^{(l_{max})}(\theta_1,\phi_1) \\ \vdots & \vdots & \ddots & \vdots \\ Y_0^{(0)}(\theta_N,\phi_N) & Y_1^{(-1)}(\theta_N,\phi_N) & \cdots & Y_{l_{max}}^{(l_{max})}(\theta_N,\phi_N) \end{bmatrix}.$$

The size of B is $N \times (l_{max}+1)^2$, where $(l_{max}+1)^2$ is given by $\sum_{l=0}^{l_{max}}(2\times l+1)$. Given B and the vertices, computing the

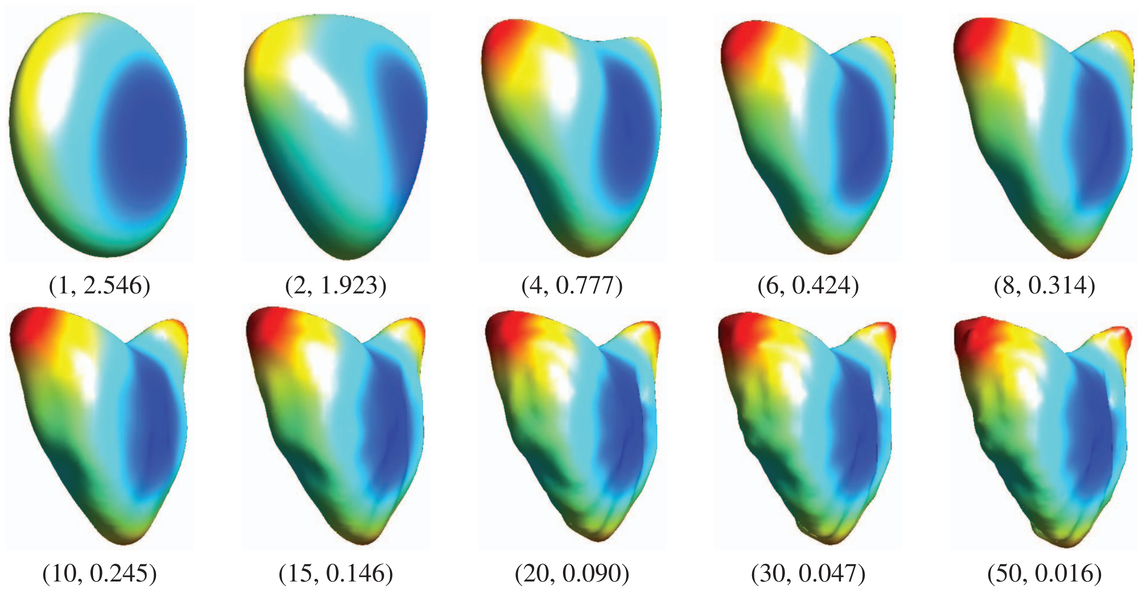


Fig. 2. Reconstructed putamen with different l_{max} (the first number) and the corresponding RMS reconstruction error in mm (the second number).

spherical representation is reduced to a standard least square problem, given by

$$C = B^T X \backslash B^T B,$$

where X is an $N \times 3$ data matrix, where row i corresponds to vertex (x_i, y_i, z_i). The reconstructed vertices of the mesh are given by BC.

4. EXPERIMENTAL RESULTS

We have tested the parametrization and reconstruction method on the Harvard brain data set[2]. The full dataset consists of MRI scans of 18 subjects with a slice of 1.5mm; the pixel size within each slice varies from 0.837mm $\times$ 0.837mm to 1.00mm$\times$$1.00$mm, depending on the subject. We first convert the volume data for a selected brain structure to a mesh using the Marching Cubes Algorithm. Then the resulting triangular mesh is parametrized using the proposed parametrization.

Figure 2 shows reconstructions of the surface of a putamen using the parametrization shown in Fig. 1. The voxel size is 0.837mm $\times$ 0.837mm $\times$ 1.50mm and the mesh extracted has $3,154$ vertices and $6,304$ triangles. We use the L^2 stretch efficiency (defined as the root mean of $\frac{1}{\gamma_i^2} + \frac{1}{\Gamma_i^2}$ over the mesh triangles [7]) to measure the effectiveness of the obtained parametrization; L^2 stretch efficiency is 0.948, indicating the parametrization is close to being isometric (whose L^2 stretch efficiency is 1). In the experiment, we vary l_{max} from 1 to 50. Fig, 2 shows the reconstructed meshes along with the root-mean-square (RMS) errors of the reconstructions compared to the original mesh. For $l_{max} = 10$, the reconstruction error is 0.245mm, which is small compared to the voxel size; when $l_{max} = 20$, the reconstruction error is 0.09mm, which only perturbs vertices slightly. As the number of spherical harmonics grows to 441, we achieve a compression rate of 7:1 with a very small loss in accuracy.

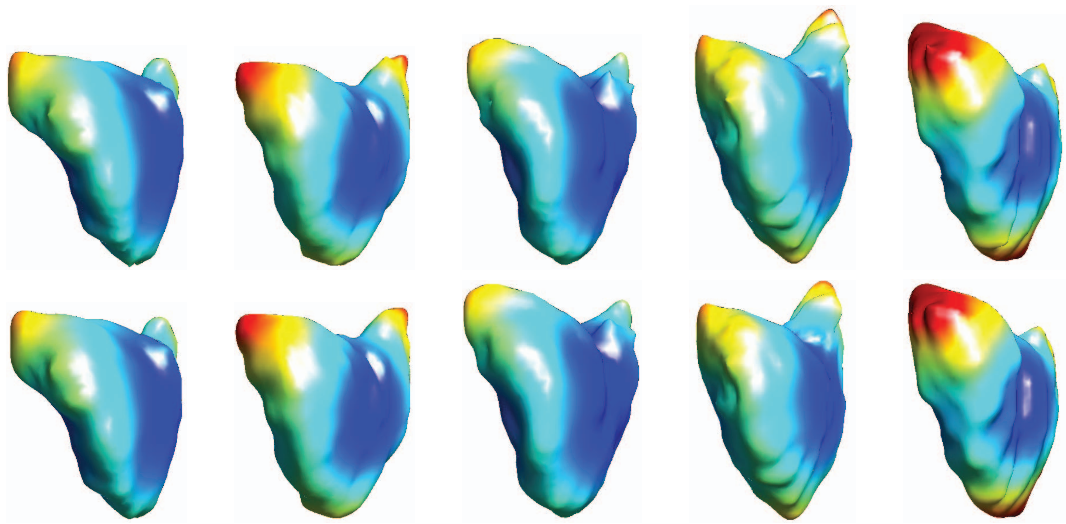
We applied the parametrization and spherical reconstruction method to putaminal surfaces of different subjects. Figure 3 shows several of the original meshes with the corresponding reconstructions. Similar to the example shown in Fig. 2, the L^2 stretch efficiencies for these meshes are also around 0.95, demonstrating the effectiveness of the method and the optimization process. As shown by the RMS reconstruction error, the proposed method consistently provides accurate spherical representations. Compared to the RMS error from [4], where the smallest is about 1.00mm, the error with the present method is much smaller. However, the study uses a different dataset and further experiments need to be done for a fair comparison.

We have applied the parametrization and spherical reconstruction to all the brain surfaces in the data set and obtained similar performance. For example, on a third ventricle surface and a left caudate surface, with $l_{max} = 30$, the reconstruction error is 0.024mm and 0.053mm respectively. These examples show the proposed method is effective and applicable in minimizing the cost function and the parametrizations yield accurate multi-resolution representations.

5. CONCLUSION

We presented a parametrization method for computing representations of genus zero surfaces with complex geometries.

[2] Available at http://www.cma.mgh.harvard.edu/ibsr/.



(30, 0.039) (30, 0.043) (30, 0.044) (30, 0.072) (30, 0.058)

Fig. 3. Selected putamen surfaces (top row) and the corresponding spherical reconstruction surfaces with $l_{max} = 30$, where the RMS reconstruction error (in mm) is shown as the second number. The number of vertices from left to right is 2334, 2900, 2218, 2970, and 3580 respectively.

By using a criterion that symmetrically penalizes both stretching and compression, we obtain spherical parametrizations that lead to accurate multi-resolution representations of spherical shapes, as shown numerically in experiments with brain surfaces.

Clearly, the proposed technique is not limited to brain surfaces and can be used to generate effective spherical representations for other surfaces of genus zero. The coefficients of a decomposition into spherical harmonics also can be used as feature vectors that characterize the underlying meshes and thus applicable to the classification, analysis, and retrieval of surfaces. These applications will be investigated further.

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