Projective shape manifolds and coplanarity of landmark configurations. A nonparametric approach.

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Abstract

This is a paper in 2D projective shape statistical analysis, with an application to face analysis. We test nonparametric methodology for an analysis of shapes of almost planar configurations of landmarks on real scenes from their regular camera pictures. Projective shapes are regarded as points on projective shape manifolds. Using large sample and nonparametric bootstrap methodology for intrinsic total variance on manifolds, we derive tests for coplanarity of a configuration of landmarks, and apply it our results to a BBC image data set for face recognition, that was previous analyzed using planar projective shape.

Keywords pinhole camera images, high level image analysis, projective shape, total variance, asymptotic distributions on manifolds, nonparametric bootstrap.

AMS subject classification Primary 62H11 Secondary 62H10, 62H35

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1 Introduction

Advances in statistical analysis of projective shape have been slowed down due to overemphasized importance of similarity shape in image analysis that ignored basic principles of image acquisition. Progress was also hampered by lack of a geometric model for the space of projective shapes, and ultimately by insufficient dialogue between researchers in geometry, computer vision and statistical shape analysis.

For reasons presented above, projective shapes have been studied only recently, and except for one concrete 3D example due to Sughatadasa(2006), to be found in Liu et al.(2007), the literature was bound to linear or planar projective shape analyzes. Examples of 2D projective shape analysis can be found in Maybank (1994), Mardia et. al. (1996), Goodall and Mardia (1999), Patrangenaru (2001), Lee et. al. (2004), Paige et. al. (2005), Mardia and Patrangenaru (2005), Kent and Mardia (2006, 2007) and Munk et. al. (2007).

In this paper, we study the shape of a 2D configuration from its 2D images in pictures of this configuration, without requiring any restriction for the camera positioning vs the scene pictured. A non-planar configuration of landmarks may often seem to be 2D, depending on the way the scene was pictured and on the distance between the scene pictured and the camera location. A test for coplanarity of $k \geq 4$ landmarks is derived here, extending a similar test for $k = 4$ due to Patrangenaru (1999).

Once the configuration passed the coplanarity test, we use a nonparametric statistical methodology to estimate its 2D projective shape, based on Efron's bootstrap. In this paper, a 2D projective shape is regarded as a random object on a projective shape space. Since typically samples of images are small, in order to estimate the mean projective shape we use nonparametric bootstrap for the studentized sample mean projective shape on a manifold, as shown in Patrangenaru et al. (2008).

A summary by sections follows. Section 2 is devoted to a recollection of basic geometry facts needed further in the paper, such as projective invariants, projective frames, and projective coordinates from Patrangenaru (2001).

In Section 3 we introduce projective shapes of configurations of points in $\mathbb{R}^m$. We will represent a projective shape of such a configuration as a point in $(\mathbb{R}^m)^{k-4}$ using a registration modulo a projective frame. Then we derive the asymptotic distribution of its total sample variance, and the corresponding pivotal bootstrap distribution, needed in the estimation of the total population variance.

Since projective shapes are identified via projective frames with multivariate axial data, in section 4 we refer to the multivariate axial distributions via a representation of the projective shape space $P\Sigma_m^k$ as product of $k-m-2$ copies of $\mathbb{R}P^m$. This space is provided with a Riemannian structure that is locally flat around the support of the distributuions considered here as in Patrangenaru (2001). In Theorem 4.1 an asymptotic result is derived for the sampling distribution of the total intrinsic variance of a random k-ad. Based on this result we derive confidence intervals the total intrinsic population variance oof the projective shape, as well as for affine coordinates of the marginal axial distributions. A. Bhattacharya (2008) derived a similar test for the total extrinsic variance of a distribution on an embedded manifold. Section 5 id dedicated to a face analysis example, where we test the planarity of eight anatomic landmarks selected on the face of an individual.

2 Projective Geometry for Ideal Pinhole Camera Image Acquisition

Projective geometry governs the physics of ideal pinhole camera image acquisition from a 2D flat scene to the 2D camera film. It also provides a justification for the reconstruction of a 2D configuration from monocular retinal images, since classical similarity shape is often meaningless in computer vision and in pattern recognition. In this section we review some of the basics of projective geometry that are useful in understanding of image formation and 2D scene retrieval from ideal pinhole camera images.

2.1 Basics of Projective Geometry

Consider a real vector space V. Two vectors $x, y \in V\backslash\{0_V\}$ are equivalent if they differ by a scalar multiple. The equivalence class of $x \in V\backslash\{0_V\}$ is labeled $[x]$, and the set of all such equivalence classes is the *projective space* $P(V)$ associated with V, $P(V) = \{[x], x \in V\backslash O_V\}$. The real projective space in m dimensions, $\mathbb{R}P^m$, is $P(\mathbb{R}^{m+1})$. Another notation for a *projective point* $p = [x] \in \mathbb{R}P^m$, equivalence class of $x = (x^0, \dots, x^m) \in \mathbb{R}^{m+1}$, is $p = [x^0 : x^1 : \dots : x^m]$ features the *homogeneous coordinates* $(x^0, \dots, x^m)$ of p, which are determined up to a multiplicative constant. A projective point p admits also a *spherical representation* , when thought of as a pair of antipodal points on the m dimensional unit sphere, $p = \{x, -x\}, x = (x^0, x^1, \dots, x^m), (x^0)^2 + \cdots + (x^m)^2 = 1$. A d - dimensional *projective subspace* of $\mathbb{R}P^m$ is a projective space $P(V)$, where V is a $(d+1)$-dimensional vector subspace of $\mathbb{R}^{m+1}$. A codimension one projective subspace of $\mathbb{R}P^m$ is also called *hyperplane.* The *linear span* of a subset D of $\mathbb{R}P^m$ is the smallest projective subspace of $\mathbb{R}P^m$ containing D. We say that k points in $\mathbb{R}P^m$ are in *general position* if their linear span is $\mathbb{R}P^m$. If k points in $\mathbb{R}P^m$ are in general position, then $k \geq m+2$.

The numerical space $\mathbb{R}^m$ can be *embedded* in $\mathbb{R}P^m$, preserving collinearity. An example of such an *affine* embedding is

$$h((u^1, ..., u^m)) = [1 : u^1 : \ \ ... \ \ : u^m] = [\tilde{u}], \tag{2.1}$$

where $\tilde{u} = (1, u^1, \dots, u^m)^T$, and in general, an affine embedding is given for any $A \in Gl(m+1, \mathbb{R})$, by $h_A(u) = [A\tilde{u}]$. The complement of the range of the embedding h in (2.1) is the hyperplane $\mathbb{R}P^{m-1}$, set of points $[x^1 : \cdots : x^m : 0] \in \mathbb{R}P^m$, which has volume measure 0 with respect to the Riemannian volume associated with the Riemannian structure on $\mathbb{R}P^m$ induced by the standard metric on S^m.

The *inhomogeneous (affine) coordinates* $(u^1, \dots, u^m)$ of a point $p = [x^0 : x^1 : \cdots : x^m] \in \mathbb{R}P^m \backslash \mathbb{R}P^{m-1}$ are given by

$$u^j = \frac{x^j}{x^0}, \forall j = 1, \dots, m. \tag{2.2}$$

Consider now the linear transformation from $\mathbb{R}^{m'+1}$ to $\mathbb{R}^{m+1}$ defined by the matrix $B \in M(m+1, m'+1; \mathbb{R})$ and its kernel $K = \{x \in \mathbb{R}^{m'+1}, Bx = 0\}$. The *projective map* $\beta : \mathbb{R}P^{m'} \backslash P(K) \to \mathbb{R}P^m$, *associated with* B is defined by $\beta([x]) = [Bx]$. In particular, a *projective transformation* β of $\mathbb{R}P^m$ is the projective map associated with a nonsingular matrix $B \in GL(m+1, \mathbb{R})$ and its action on $\mathbb{R}P^m$:

$$\beta([x^1 : \cdots : x^{m+1}]) = [B(x^1, \dots, x^{m+1})^T]. \tag{2.3}$$

In affine coordinates (inverse of the affine embedding (2.1)), the projective transformation (2.3) is given by $v = f(u)$, with

$$v^j = \frac{a_0^j + \sum_{i=1}^m a_i^j u^i}{a_0^0 + \sum_{i=1}^m a_i^0 u^i}, \forall j = 1, \dots, m \tag{2.4}$$

where $\det B = \det((a_i^j)_{i,j=0,\ldots,m}) \neq 0$. An *affine* transformation of $\mathbb{R}P^m, v = Au + b, A \in GL(m, \mathbb{R}), t \in \mathbb{R}^m$, is a particular case of projective transformation α, associated with the matrix $B \in GL(m+1, \mathbb{R})$, given by

$$B = \begin{pmatrix} 1 & 0_m^T \\ b & A \end{pmatrix}. \tag{2.5}$$

A *projective frame* in an m dimensional projective space (or *projective basis* in the computer vision literature, see e.g. Hartley (1993)) is an ordered set of $m+2$ projective points in general position. An example of projective frame in $\mathbb{R}P^m$ is the *standard projective frame* is $([e_1], \ldots, [e_{m+1}], [e_1 + ... + e_{m+1}])$.

In projective shape analysis it is preferable to employ coordinates invariant with respect to the group PGL(m) of projective transformations. A projective transformation takes a projective frame to a projective frame, and its action on $\mathbb{R}P^m$ is determined by its action on a projective frame, therefore if we define the *projective coordinate(s)* of a point $p \in \mathbb{R}P^m$ w.r.t. a projective frame $\pi = (p_1, \ldots, p_{m+2})$ as being given by

$$p^\pi = \beta^{-1}(p), \tag{2.6}$$

where $\beta \in PGL(m)$ is a projective transformation taking the standard projective frame to π, these coordinates have automatically the invariance property.

REMARK 2.1. *Assume $u, u_1, \ldots, u_{m+2}$ are points in $\mathbb{R}^m$, such that $\pi = ([\tilde{u}_1], \ldots, [\tilde{u}_{m+2}])$ is a projective frame. If we consider the $(m+1) \times (m+1)$ matrix $U_m = [\tilde{u}_1^T, \ldots, \tilde{u}_{m+1}^T]$, the projective coordinates of $p = [\tilde{u}]$ w.r.t. π are given by*

$$p^\pi = [y^1(u) : \cdots : y^{m+1}(u)], \tag{2.7}$$

where

$$v(u) = U_m^{-1}\tilde{u}^T \tag{2.8}$$

and

$$y^j(u) = \frac{v^j(u)}{v^j(u_{m+2})}, \forall j = 1, \ldots, m+1. \tag{2.9}$$

Note that in our notation, the superscripts are reserved for the components of a point whereas the subscripts are for the labels of points. The projective coordinate(s) of x are given by the point $[z^1(x) : \cdots : z^{m+1}(x)] \in \mathbb{R}P^m$. In affine coordinates, the projective coordinate of a point $\tilde{x}$ can be obtained as follows. Consider a projective transformation β of $\mathbb{R}^m$ that takes the $m+2$-tuple $(x_1, \ldots, x_{m+2}) \in \mathbb{R}^m$ of affine coordinates in a projective frame to the standard projective frame (of $\mathbb{R}^m$), $(0, e_1, \ldots, e_m, 1_m^T)$. Then inhomogeneous coordinates of $[\tilde{x}]$ are given by $y = \beta(x)$.

3 Projective Shape

DEFINITION 3.1. *Two configurations of points in $\mathbb{R}^m$ have the same the projective shape if they differ by a projective transformation of $\mathbb{R}^m$.*

Projective transformations of $\mathbb{R}^m$ have a pseudo-group structure under composition (the domain of definition of the composition of two such maps is smaller than the maximal domain of a projective transformation in $\mathbb{R}^m$). A *projective shape* of a k-ad (configuration of k landmarks or labelled points) is the orbit of that k-ad under projective transformations with respect to the diagonal action

$$\alpha_k(p_1, \ldots, p_k) = (\alpha(p_1), \ldots, \alpha, (p_k)). \tag{3.1}$$

From (2.4), if two k-ads $(u_1, \ldots, u_k), (v_1, \ldots, v_k)$ in $\mathbb{R}^m$ have the same projective shape, there is a projective transformation α given by (2.4) with $\alpha(u_j) = v_j, j = 1, \ldots, k.$

Patrangenaru (1999, 2001) considered the set $G(k, m)$ of k-ads $(p_1, ..., p_k), k > m+2$ for which $\pi = (p_1, ..., p_{m+2})$ is a projective frame. $PGL(m)$ acts on $G(k, m)$ and the projective shape space $P\Sigma_m^k$, is the quotient $G(k, m)/PGL(m)$. Using the projective coordinates $(p_{m+3}^\pi, \ldots, p_k^\pi)$ given by (2.6) one can show that $P\Sigma_m^k$ is a manifold diffeomorphic with $(\mathbb{R}P^m)^{k-m-2}$. The projective frame representation has two useful features: firstly, the projective shape space has a manifold structure, thus allowing to use the asymptotic theory for means on manifolds in Bhattacharya and Patrangenaru (2003, 2005), an secondly, it can be extended to infinite dimensional projective shape spaces, such as projective shapes of curves, as shown in Munk et al. (2007). This approach has the advantage of being inductive in the sense that each new landmark of a configuration adds an extra marginal axial coordinate, thus allowing to detect its overall contribution to the variability of the configuration as well as correlation to the other landmarks. The effect of change of projective coordinates, due to projective frame selection, can be understood via a group of projective transformations, but is beyond the scope of this paper.

4 Nonparametric estimation and testing for total intrinsic variance and for the planarity of a 3D k-ad

The local chart around the support of the distribution Q is given in inhomogeneous affine coordinates in the projective frame representation of the projective shape of a k-ad in $\mathbb{R}^m$ recorded in random digital images. Assume the registered coordinates are $(x_1, \dots, x_k), k > m+2$ and $((x_1, \dots, x_{m+2})$ yields a projective frame. To represent the projective shape of this k-ad with respect to this frame one may use equations (2.9), (2.8), or, alternately one may (i) solve for a the linear system of (2.4) for the pairs of points $(u = x_1, v = 0), (u = x_2, v = e_1), \dots, (u = x_{m+1}, v = e_m), (u = x_{m+2}, v = e_1 + \dots + e_m)$, and, (ii) for the values of a obtained at step (i), compute from (2.4) the affine coordinates $v_1, \dots, v_{k-m-2}$ corresponding respectively to $x_{m+3}, \dots, x_k$.

We will consider an intrinsic analysis on the projective shape space, for a Riemannian metric which is flat around the support of the distribution Q. Here Q is the probability measure on the projective shape space, associated with a random projective shape $Y = P\sigma(\mathbf{X})$ of a k-ad $\mathbf{X} = (X_1, \dots, X_k)$ in $\mathbb{R}^m$. The support of Q contained in the domain of our affine chart, is represented by the the random vector $W = (V_1, \dots, V_{k-m-2})$ in $(\mathbb{R}^m)^{k-m-2}$ with the total (intrinsic) variance

$$t\Sigma_I = \sum_{a=1}^{k-m-2} Tr\Sigma_a = E({d_g}^2(X, \mu_I)) = Tr(Cov(W). \tag{4.1}$$

where d_g is the Riemannian distance, Σ_a is the covariance matrix of V_a, and μ_I is the extrinsic mean, and $Cov(W)$ is the covariance matrix of the vectorized form of W.

One may assume that the probability distribution Q of $P\sigma(\mathbf{X})$ has *small flat support* on $P\Sigma_m^k$ in the sense of Patrangenaru (2001).

THEOREM 4.1. *If Q has small flat support on M and has finite moments up to the fourth order, then $n^{1/2}(t\hat{\Sigma}_{I,n} - t\Sigma_I)$ converges in distribution to a random vector with a multivariate distribution $\mathcal{N}(0, Var({d_g}^2(X, \mu_I)))$.*

Proof. Let $W_1, \dots, W_n$ be independent identically distributed random vectors in $\mathbb{R}^{m(k-m-2)}$ representing the independent identically distributed random projective shapes $\mathbf{Y_1}, \dots, \mathbf{Y_n}$ with the distribution Q. Assume $E(W_1) = \mu$. Given the small flat support assumption,

$$
\begin{aligned}
t\hat{\Sigma}_{I,n} - t\Sigma_I = Tr(\frac{1}{n}\sum_{i=1}^{n}(W_i - \overline{W})(W_i - \overline{W})^T - TrE((W_1 - \mu)(W_1 - \mu)^T) = \\
= \frac{1}{n}\sum_{i=1}^{n} Tr((W_i - \overline{W})(W_i - \overline{W})^T) - TrE((W_1 - \mu)(W_1 - \mu)^T) = \\
= \frac{1}{n}\sum_{i=1}^{n} Tr((W_i - \mu)(W_i - \mu)^T) - TrE((W_1 - \mu)(W_1 - \mu)^T) = \\
= \frac{1}{n}\sum_{i=1}^{n} \|W_i - \mu\|^2 - E(\|W_1 - \mu\|^2).
\end{aligned}
\tag{4.2}
$$

From the Central Limit Theorem (applied here for a random sample from the probability distribution of distribution of $\|W_1 - \mu\|^2$, it follows that

$$
\sqrt{n}(t\hat{\Sigma}_{I,n} - t\Sigma_I) = \sqrt{n}(\overline{\|W - \mu\|^2} - E(\|W_1 - \mu\|^2) \to_d Z, \tag{4.3}
$$

where $Z \sim \mathcal{N}(0, Var(\|W_1 - \mu\|^2) = \mathcal{N}(0, Var({d_g}^2(X, \mu_I)))$, done ■

Under the assumptions of Theorem 4.1 if we set $S^2 = \frac{1}{n}\sum_{i=1}^{n}({d_g}^2(X_i, \overline{X}_I) - \overline{{d_g}^2(X, \overline{X}_I)})^2$, then we obtain

COROLLARY 4.1. *If Q has small flat support on M then*

$$
n^{\frac{1}{2}}(\frac{(t\hat{\Sigma}_{I,n} - t\Sigma_I)}{S}) \tag{4.4}
$$

converges in distribution to a $\mathcal{N}(0, 1)$ distributed random variable.

Further, we get

COROLLARY 4.2. *A $100(1-\alpha)\%$ large sample symmetric confidence interval for $t\Sigma_I$ is given by*

$$
(t\hat{\Sigma}_{I,n} - z_{\frac{\alpha}{2}}\frac{s}{\sqrt{n}}, t\hat{\Sigma}_{I,n} + z_{\frac{\alpha}{2}}\frac{s}{\sqrt{n}}) \tag{4.5}
$$

.

Note that for any affine coordinate U we have

LEMMA 4.1. *Assume $U_1, \dots, U_n$ are i.i.d.r.v.'s from a probability distribution Q with finite mean μ, variance σ^2, third and fourth moments about zero $\mu_{3,0}, \mu_{4,0}$, and let $\overline{U^a} = \frac{1}{n}\sum_{i=1}^{n} U_i^a$ be the sample estimator of $\mu_{a,0}$. Then for n large enough, $\sqrt{n}(\hat{\sigma}^2 - \sigma^2) \to_d W$ where $\hat{\sigma}^2 = \overline{U^2} - \overline{U}^2$ and*

$$W \sim \mathcal{N}(0, \sigma^2(6\mu^2 - \sigma^2) + 3\mu^4 - 4\mu_{3,0}\mu + \mu_{4,0}), \tag{4.6}$$

therefore if we assume $\sigma^2(6\mu^2 - \sigma^2) + 3\mu^4 - 4\mu_{3,0}\mu + \mu_{4,0} > 0.$ and studentize we obtain the following:

PROPOSITION 4.1. *Under the hypothesis of Lemma 4.1 if we set*

$$T = \frac{\sqrt{n}(\hat{\sigma}^2 - \sigma^2)}{(\widehat{\sigma^2}(6\overline{U}^2 - \widehat{\sigma^2}) + 3\overline{U}^4 - 4\overline{U^3}\,\overline{U} + \overline{U^4})^{\frac{1}{2}}}, \tag{4.7}$$

then T has asymptotically a $\mathcal{N}(0,1)$ distribution.

COROLLARY 4.3. *A large sample $100(1-\alpha)\%$ confidence interval for σ^2 is given by*

$$\begin{aligned}
&(\widehat{\sigma^2} - \frac{z_\beta}{\sqrt{n}}(\widehat{\sigma^2}(6\overline{u}^2 - \widehat{\sigma^2}) + 3\overline{u}^4 - 4\overline{u^3}\,\overline{u} + \overline{u^4})^{\frac{1}{2}},\\
&\widehat{\sigma^2} + \frac{z_\gamma}{\sqrt{n}}(\widehat{\sigma^2}(6\overline{u}^2 - \widehat{\sigma^2}) + 3\overline{u}^4 - 4\overline{u^3}\,\overline{u} + \overline{u^4})^{\frac{1}{2}}),
\end{aligned} \tag{4.8}$$

where $\beta + \gamma = \alpha$.

5 Example from face analysis

From the previous section we formulate the problem of planarity of a 3D scene as a hypothesis testing question :

$$H_0 : t\Sigma_I = 0 \mathrm{vs} H_1 : t\Sigma_I \neq 0, \tag{5.1}$$

where $t\Sigma_I$ is the total intrinsic variance of the random 2D projective shape of the k-ad under consideration (here $m = 2$). Rejecting H_0 means the k-ad is in general position (not planar). If the number of images is small, we may use a bootstrap confidence interval, derived from the bootstrap distribution of Corollary 4.1.

If we want to check the coplanarity of one of the landmarks with the projective frame, we use instead the confidence intervals in (4.8) or their bootstrap analogues for the variances of the affine coordinates corresponding to a given landmark.

A face recognition example based on a data set used in a live BBC program "Tomorrow's World" is given below. The example was introduced in Mardia and Patrangenaru (2005), where six landmarks (ends of eyes plus ends of lips) have been recorded from fourteen digital images of the same person (an actor posing in different disguises), in fourteen pictures. Face appearance in these pictures may be neither frontal or lateral: In this paper we use two additional

Figure 1: BBC data:14 views of an actor face.

landmarks(bridge of the nose and tip of the nose). The affine coordinates of the projective shapes of this 8-ad, with respect to the projective frame given by the "end of lips" and "outer ends of eyes" are displayed in figure 2 below. The

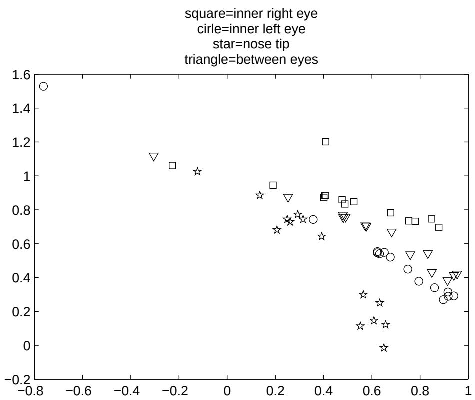

Figure 2: Affine coordinates of four projective representation of facial landmarks

results are as follows: observation 1 and 2 are the inner eye ends, 3 is the nose-bridge, 4 is nose tip.

The confidence intervals are as follows:

for $\alpha = 0.05$ $\gamma = 0.025, \beta = 0.025$:

C.I. for x1 =[0.00853129340099773, 0.148558674971251]

C.I. for x2 =[-0.0840617129477059, 0.432726727843371]

C.I. for x3 =[0.0184935709543756, 0.0837586203816484]

C.I. for x4 =[-0.00290403837745329, 0.212652481111851]

C.I. for y1 =[0.00211307898182853, 0.0327929403581654]

C.I. for y2 =[-0.0382290058234896, 0.229364025574857]

C.I. for y3 =[0.0656967552395945, 0.149609901393107]

C.I. for y4 =[0.01161789651198, 0.0691394183317944] for $\alpha = 0.05, \gamma = 0.045, \beta = 0.005$:

C.I. for $x1 = [-0.0134686026942806, 0.139107857780284]$

C.I. for x2 =[-0.165255061513897, 0.39784731336349]

C.I. for x3 =[0.00823968853543718, 0.0793537101121015]

C.I. for x4 =[-0.0367704220678841, 0.198104003255809]

C.I. for y1 =[-0.0027070765829481, 0.0307222756205941]

C.I. for y2 =[-0.0802709180592805, 0.211303466325924]

C.I. for y3 =[0.0525130444845735, 0.143946381906257]

C.I. for y4 =[0.00258061045344363, 0.0652571391577564]

If we select centered confidence intervals, at level $\alpha = 0.05$ we fail to reject the planarity of three of the four eye ends and the lip ends , and reject it for the other three landmarks. If we are more inclusive for the lower bound of the confidence intervals, by taking $\gamma = 0.045$ then we reject the tip of the nose as being in the plane determined by the lines of the eyes and of the lip ends, which is comes in agreement with our expectations.

We use corollary 4.1 to construct a 95% symmetric confidence interval for the total projective shape variance, from the 14 views in figure 1, to see if all of the 8 landmarks mentioned are in the same plane. We compute $S = 7.3174$ and $t\hat{\Sigma}_I = 7.5301$ to yield the interval [3.6970, 11.3632]. We observe that zero is not in the interval, hence, we reject the hypothesis that all 8 landmarks are in the same plane. Next, we exclude the bridge of the nose and tip of the nose and conduct the same procedure on the remaining 6 landmarks. We compute $S = 6.7872$ and $tSi\hat{g}ma_I = 3.0057$ to yield the interval [-0.5497, 6.5610]. We observe that zero is the interval, hence, we fail to reject the hypothesis that the 4 corners of the eyes and the ends of the mouth are in the same plane. Note that Mardia and Patrangenaru(2005) assumed that these six landmarks are coplanar in their analysis. Our results show that within a 95% confidence, their coplanarity assumption was correct, as expected.

References

[1] Balan, V. and Patrangenaru, V. (2006) Geometry of shape spaces. In *Proceedings of the 5th conference of the Balkan Society of Geometers*. (Balan, Vladimir (ed.)). Bucharest, Geometry Balkan Press. 28-33.

[2] Bhattacharya, A. (2008). Statistical analysis on manifolds: A nonparametric approach for inference on shape spaces', to appear in *Ann. Statist.*

[3] Bhattacharya, R.N. and Patrangenaru,V. (2002) Nonparametric estimation of location and dispersion on Riemannian manifolds, *Journal of Statistical Planning and Inference*, Volume **108**, Issues 1-2, pp. 23–35.

[4] Bhattacharya, R.N. and Patrangenaru, V (2003) Large sample theory of intrinsic and extrinsic sample means on manifolds-I, *Ann. Statist.* Vol. **31**, no. 1, 1–29.

[5] Bhattacharya, R.N. ; Patrangenaru V. (2005), Large sample theory of intrinsic and extrinsic sample means on manifolds- Part II, *Ann. Statist.*, Vol. **33**, No. 3, 1211–1245.

[6] Efron, B. (1979) Bootstrap methods: another look at the jackknife. it Ann. Statist. **7**, No. 1, 1–26.

[7] Efron, B. (1982). *The Jackknife, the Bootstrap and Other Resampling Plans*, CBMS-NSF Regional Conference Series in Applied Mathematics, **38**. SIAM.

[8] Goodall,C. and Mardia,K.V. (1999). Projective shape analysis, *J. Graphical & Computational Statist.*, **8**, 143–168.

[9] Hartley, R. I. ; Gupta R. ; and Chang T. (1992). Stereo from uncalibrated cameras, in *Proc. IEEE Conference on Computer Vision and Pattern Recognition*.

[10] Kent, J.T. (2007). Projective Shape Analysis, in *Summer 2007 Program on the Geometry and Statistics of Shape Spaces* July 7-13, 2007, Speaker Abstracts.

[11] Kent, J.T. and Mardia,K.V. (2006) A new representation for projective shape, in S. Barber, P.D. Baxter, K.V.Mardia, & R.E. Walls (Eds.), *Interdisciplinary Statistics and Bioinformatics*, pp. 75-78. Leeds, Leeds University Press. http://www.maths.leeds.ac.uk/lasr2006/proceedings/

[12] Lee, J. L. ; Paige, R. ; Patrangenaru, V. and Ruymgaart, F. (2004) Nonparametric density estimation on homogeneous spaces in high level image analysis, in R.G. Aykroyd, S. Barber, & K.V. Mardia (Eds.), *Bioinformatics, Images, and Wavelets, pp. 37-40*. Department of Statistics, University of Leeds. http://www.maths.leeds.ac.uk/Statistics/workshop/leeds2004/temp

[13] Liu, X.; Patrangenaru, V. and Sugathadasa, S.(2007) *Projective Shape Analysis for Noncalibrated Pinhole Camera Views. To the Memory of W.P. Dayawansa* , Florida State University-Department of Statistics, Technical Report M983.

[14] Mao, Y., Soatto, S., Kosecka, J. and Sastry, S.S. (2006). *An invitation to 3-D Vision*, Springer, New York.

[15] Mardia, K.V. and Patrangenaru, V (2005) Directions and projective shapes, *Ann. Statist.* Vol. **33**, No. 4., 1666–1699.

[16] Mardia K.V., Goodall C., Walder A.N. (1996) Distributions of projective invariants and model based machine vision *Adv. in App. Prob.* **28** 641–661.

[17] Maybank,S.J. (1995) Probabilistic analysis of the application of the cross ratio to model based vision: Misclassification. *Inter. J. of Computer Vision,* 14(3).

[18] Maybank, S.J. (1994). Classification based on the cross ratio. in *Applications of Invariance in Computer Vision.* Lecture Notes in Comput. Sci. 825 (J.L. Mundy, A. Zisserman and D. Forsyth, eds.) 433–472. Springer, Berlin.

[19] A. Munk, R. Paige, J. Pang, V. Patrangenaru and F. Ruymgaart (2007), The One and Multisample Problem for Functional Data with Applications to Projective Shape Analysis, to appear in *Journal of Multivariate Analysis.*

[20] Patrangenaru, V. (1999) Moving projective frames and spatial scene identification, in *Proceedings in Spatial-Temporal Modeling and Applications*, Edited by K. V. Mardia, R. G. Aykroyd and I.L. Dryden, Leeds University Press, p. 53–57.

[21] Patrangenaru,V. (2001). New large sample and bootstrap methods on shape spaces in high level analysis of natural images, *Commun. Statist.*, **30**, 1675–1693.