

CHAPTER 3

Boolean Algebra

1. Boolean Functions

1.1. Boolean Functions.

DEFINITIONS 1.1.1.

1. A **Boolean variable** is a variable that may take on values only from the set $B = \{0, 1\}$.
2. A **Boolean function of degree n or of order n** is a function with domain $B^n = \{(x_1, x_2, \dots, x_n) | x_i \in B\}$ and codomain B . In other words, Boolean functions of degree n are functions of the form $F : B^n \rightarrow B$.
3. Two Boolean functions, F and G are **equivalent** if

$$F(x_1, x_2, x_3, \dots, x_n) = G(x_1, x_2, x_3, \dots, x_n)$$

for every ordered n -tuple $(x_1, x_2, x_3, \dots, x_n) \in B^n$.

Discussion

We have already encountered Boolean variables, namely, propositional variables, which must have value 1 (“true”) or 0 (“false”). Indeed, the propositional calculus is the motivating concept for the material in this chapter, and we shall refer to it frequently.

1.2. Example 1.2.1.

EXAMPLE 1.2.1. A Boolean function of degree 2, $F(x, y) : B^2 \rightarrow B$, may be defined by a chart. For example, this function may be defined as follows:

x	y	$F(x, y)$
1	1	1
1	0	0
0	1	0
0	0	0

Discussion

There are several ways we may define a Boolean function. A table of values is one way. After we define addition, multiplication, and other operations on B , we may also use these operations to define functions.

Notice a Boolean function of two variables must assign to each of the four ordered pairs a value from B . This means there are $2^4 = 16$ different Boolean functions of order 2.

EXERCISE 1.2.1. *Use a table of values to define all 16 Boolean functions of order 2.*

EXERCISE 1.2.2. *How many Boolean functions of degree n are there?*

1.3. Binary Operations.

DEFINITIONS 1.3.1. *Let $x, y \in B$.*

1. **Addition** is defined by the table

x	y	$x + y$
1	1	1
1	0	1
0	1	1
0	0	0

2. **Multiplication** is defined by the table

x	y	$x \cdot y$
1	1	1
1	0	0
0	1	0
0	0	0

3. The **complement** is defined by the table

x	\bar{x}
1	0
0	1

Discussion

If you think of the 1 as “true” and the 0 as “false”, as we used in *Logic*, you should notice that Boolean addition corresponds to the logical “or”, Boolean multiplication corresponds to the logical “and”, and complementation corresponds to the logical “not”. In fact, many authors use the notation \vee , \wedge , and \neg for $+$, \cdot , and $\bar{}$, respectively. The notation $\bar{x} = x'$ is another common alternative for complementation. We

will use this alternative on the discussion board and it may be used in homework. When there would be no confusion, we drop the \cdot when denoting a Boolean product, just as is done in algebra.

Notice that Boolean addition defined here on $\{0, 1\}$ is NOT the same as the addition on the set of integers modulo 2.

1.4. Example 1.4.1.

EXAMPLE 1.4.1. *The following functions are equivalent.*

- (1) $F(x, y) = x + y$
 (2) $G(x, y) = xy + \bar{x}y + x\bar{y}$

PROOF. We show the two functions have the same values for every possible ordered pair in B^2 .

x	y	xy	\bar{x}	\bar{y}	$\bar{x}y$	$x\bar{y}$	$xy + \bar{x}y + x\bar{y}$	$x + y$
1	1	1	0	0	0	0	1	1
1	0	0	0	1	0	1	1	1
0	1	0	1	0	1	0	1	1
0	0	0	1	1	0	0	0	0

□

Discussion

Example 1.4.1 gives an example of equivalent functions that are defined quite differently, although both representations are in terms of the algebra we have defined on $\{0, 1\}$.

1.5. Boolean Identities. Below is a table of the Boolean Identities you should know.

Identity	Name
$\overline{\overline{x}} = x$	Law of Double Complement
$x + x = x$ and $x \cdot x = x$	Idempotent Laws
$x + 0 = x$ and $x \cdot 1 = x$	Identity Laws
$x + 1 = 1$ and $x \cdot 0 = 0$	Dominance Laws
$x + y = y + x$ $xy = yx$	Commutative Laws
$x + (y + z) = (x + y) + z$ $x(yz) = (xy)z$	Associative Laws
$x + yz = (x + y)(x + z)$ $x(y + z) = xy + xz$	Distributive Laws
$\overline{x \cdot y} = \overline{x} + \overline{y}$ $\overline{x + y} = \overline{x} \cdot \overline{y}$	DeMorgan's Laws

Discussion

The distributive law for addition over multiplication and the DeMorgan's Laws may seem somewhat unusual to you at this stage, since they have no counterpart in the operations of addition and multiplication used in our familiar number systems. Indeed, you should avoid any analogies with ordinary arithmetic and, instead, use the propositional calculus as your model whenever you feel the need to use a familiar setting in which to exemplify these or any other properties you might encounter.

EXERCISE 1.5.1. *Verify the distributive law $x + yz = (x + y)(x + z)$.*

1.6. Dual.

DEFINITION 1.6.1. *The **dual** of a Boolean expression is the expression one obtains by interchanging addition and multiplication and interchanging 0's and 1's. The dual of the function F is denoted F^d .*

THEOREM 1.6.1 (Duality Principle). *If F and G are Boolean functions such that $F = G$, then $F^d = G^d$.*

Discussion

EXAMPLE 1.6.3. *The dual of $x\overline{y} + \overline{x}z$ is $(x + \overline{y}) \cdot (\overline{x} + z)$.*

Notice that we have implicitly assumed an order of operations for a Boolean expression: unless grouping is present to indicate otherwise, complements are evaluated first, then products, and then sums. This order conforms to the convention we established earlier for the order of operations in the predicate calculus. As the example above shows, you must be careful to preserve the correct order of operation when taking the dual of an expression, using parentheses wherever necessary.