2. Connectivity

2.1. Connectivity.

DEFINITION 2.1.1.

(1) *A* **path** *in a graph* $G = (V, E)$ *is a sequence of vertices* $v_0, v_1, v_2, \ldots, v_n$ *such that* $\{v_{i-1}, v_i\}$ *is an edge of* G *for* $i = 1, ..., n$*. The edge* $\{v_{i-1}, v_i\}$ *is an* **edge of the path**.
(2) *A path with* n *edges is said to have* **length** n*.*
(3) *A path beginning and ending with same vertex (that is,* $v_0 = v_n$*) is a* **circuit**.
(4) *A path is* **simple** *if no vertex or edge is repeated, with the possible exception that the first vertex is the same as the last.*
(5) *A simple path that begins and ends with the same vertex is a* **simple circuit** *or a* **cycle**.

Discussion

This section is devoted to defining what it means for a graph to be connected and the theorems about connectivity.

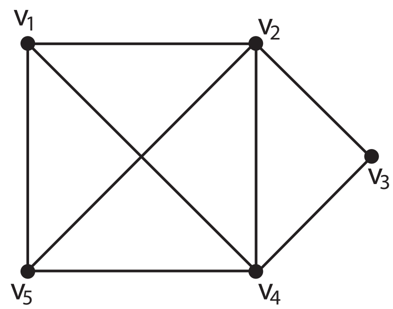
In the definition above we use a *vertex sequence* to define a path. We could also use an *edge sequence* to define a path as well. In fact, in a multigraph a path may not be well-defined by a vertex sequence. In this case an edge sequence must be used to clearly define a path.

A *circuit* must begin and end at the same vertex, but this is the only requirement for a circuit. A path that goes up one vertex and then right back is a circuit. Our definition of a simple path may be different than that found in some texts: some writers merely require that the same edge not be traversed more than once. In addition, our definition of a *simple* circuit does not include the circuit that goes up an edge and travels back by the same edge.

Some authors also allow the possibility of a path having length 0 (the path consists of a single vertex and no edges). We will require that paths have length at least 1. Notice that a path must also be finite. It is possible for a graph to have infinitely many vertices and/or edges and one could also imagine a kind of path with infinitely many edges but our definition of a path requires the path be finite.

2.2. Example 2.2.1.

EXAMPLE 2.2.1. *Let* G_1 *be the graph below.*



(1) v_1, v_4, v_2, v_3 *is a simple path of length 3 from* v_1 *to* v_3.
(2) $\{v_1, v_4\}, \{v_4, v_2\}, \{v_2, v_3\}$ *is the edge sequence that describes the same path in part 1*
(3) $v_1, v_5, v_4, v_1, v_2, v_3$ *is a path of length 5 from* v_1 *to* v_3.
(4) v_1, v_5, v_4, v_1 *is a simple circuit of length 3.*
(5) $v_1, v_2, v_3, v_4, v_2, v_5, v_1$ *is a circuit of length 6, but it is not simple.*

Discussion

This example gives a variety of paths and circuits. You can certainly come up with many more.

Exercise 2.2.1. *In this exercise consider two cycles different if they begin at a different vertex and/or if they traverse vertices in a different direction. Explain your answer:*

(a) *How many different cycles are there in the graph* K_4*?*
(b) *How many different circuits are there in the graph* K_4*?*

Exercise 2.2.2. *In this exercise consider two cycles different if they begin at a different vertex and/or if they traverse vertices in a different direction. How many different cycles are there in the graph* K_n *where* n *is some integer greater than 2.*

Exercise 2.2.3. *(Uses combinations from counting principles) In this exercise consider two cycles are the same if they begin at a different vertex and/or if they traverse vertices in a different direction, but they use the same vertices and edges. Explain your answer:*

(a) *How many different cycles are there in the graph* K_4*?*
(b) *How many different circuits are there in the graph* K_4*?*

EXERCISE 2.2.4. *(Uses combinations from counting principles) In this exercise consider two cycles are the same if they begin at a different vertex and/or if they traverse vertices in a different direction, but they use the same vertices and edges. How many different cycles are there in the graph K_n where n is some integer greater than 2.*

EXERCISE 2.2.5. *Prove a finite graph with all vertices of degree at least 2 contains a cycle.*

EXERCISE 2.2.6. *Prove a graph with n vertices and at least n edges contains a cycle for all positive integers n. You may use Exercise 2.2.5.*

2.3. Connectedness.

DEFINITION 2.3.1. *A simple graph is* **connected** *if there is a path between every pair of distinct vertices.*

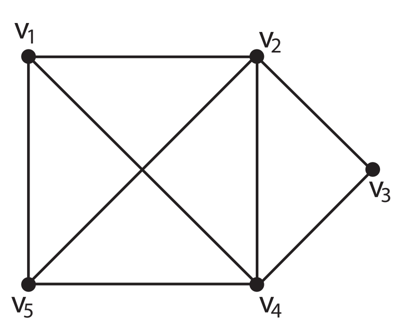
Discussion

When looking at a sketch of a graph just look to see if each vertex is connected to each of the other vertices by a path. If so, this graph would be connected. If it has two or more distinct pieces with no edge connecting them then it is *disconnected*.

2.4. Examples.

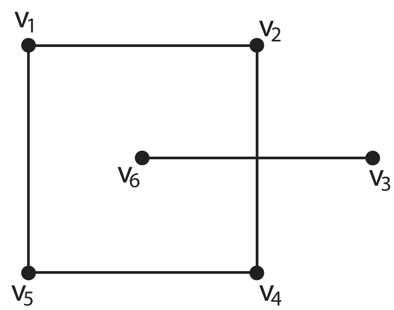
EXAMPLE 2.4.1.

This graph is connected

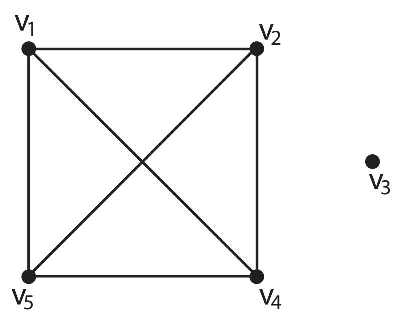


EXAMPLE 2.4.2.

This graph is not connected



EXAMPLE 2.4.3. *The following graph is also not connected. There is no edge between v_3 and any of the other vertices.*



2.5. Theorem 2.5.1.

THEOREM 2.5.1. *There is a simple path between every pair of distinct vertices in a connected graph.*

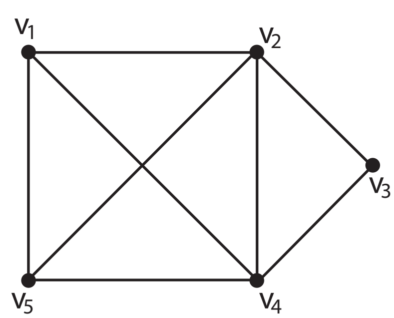
PROOF. Suppose u and v are arbitrary, distinct vertices in a connected graph, G. Because the graph is connected there is a path between u and v. Among all paths between u and v, choose a path $u = v_0, v_1, ..., v_n = v$ of shortest length. That is, there are no paths in G of length $< n$. Suppose this path contains a circuit starting and ending with, say, v_i. This circuit must use at least one edge of the path; hence, after removing the circuit we will have a path from u to v of length $< n$, contradicting the minimality of our initial path.

□

Discussion

Theorem 2.5.1 implies that if we need a path between two vertices in a connected graph we may use a simple path. This really simplifies (no pun intended) the types of paths we need to consider when examining properties of connected graphs. Certainly there are many paths that are not simple between any two vertices in a connected graph, but this theorem guarantees there are nicer paths to work with.

2.6. Example 2.6.1.



EXAMPLE 2.6.1. *The path $v_1, v_5, v_4, v_1, v_2, v_3$ is a path between v_1 and v_3. However, v_1, v_5, v_4, v_1 is a circuit. Remove the circuit (except the endpoint) to get from the original path v_1, v_2, v_3. This is still a path between v_1 and v_3, but this one is simple. There are no edges in this last path that are used more than one time.*

Discussion

In many examples it is possible to find more than one circuit that could be removed to create a simple path. Depending on which circuit is chosen there may be more than one simple path between two given vertices. Let us use the same graph in Example 2.6.1, but consider the path $v_1, v_2, v_5, v_1, v_4, v_2$. We could either remove the circuit v_1, v_2, v_5, v_1 or the circuit v_2, v_5, v_1, v_4, v_2. If we removed the first we would be left with v_1, v_4, v_2, while if we removed the latter we would get v_1, v_2. Both of these are parts of the original path between v_1 and v_2 that are simple.

2.7. Connected Component.

DEFINITION 2.7.1. *The maximally connected subgraphs of G are called the* **connected components** *or just the* **components**.

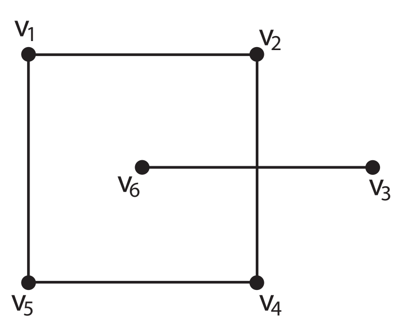
Discussion

Another way we could express the definition of a component of G is: A is a component of G if

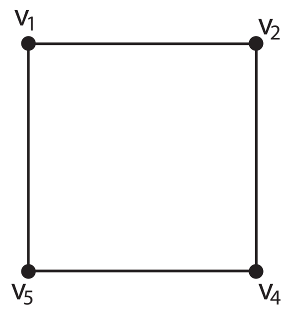
(1) A is a connected subgraph of G and
(2) if B is another subgraph of G containing A then either $B = A$ or B is disconnected.

2.8. Example 2.8.1.

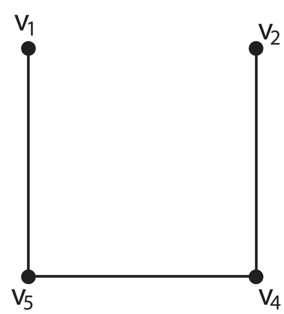
EXAMPLE 2.8.1. *In the graph below the vertices v_6 and v_3 are in one component while the vertices $v_1, v_2, v_4,$ and v_5 are in the other component.*



If we looked at just one of the components and consider it as a graph by itself, it would be a connected graph. If we try to add any more from the original graph, however, we no longer have a connected graph. This is what we mean by "largest". Here are pictures that may help in understanding the components of the graph in Example 2.8.1



Above is a connected component of the original graph.



Above is *not* a connected component of the original. We are missing an edge that should have been in the component.

2.9. Cut Vertex and Edge.

DEFINITION 2.9.1.

(1) If one can remove a vertex and all incident edges from a graph and produce a graph with more components than the original graph, then the vertex that was removed is called a **cut vertex** *or an* **articulation point**.

(2) If one can remove an edge from a graph and create more components than the original graph, then the edge that was removed is called a **cut edge** *or* **bridge**.

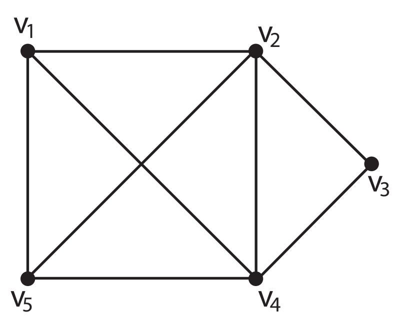
Note: When removing a vertex, you must remove all the edges with that vertex as an endpoint. When removing an edge we do *not* remove any of the vertices. Remember, edges depend on vertices, but vertices may stand alone.

EXERCISE 2.9.1. *Prove that every connected graph has at least two non-cut vertices. [Hint: Use the second principle of mathematical induction on the number of vertices.]*

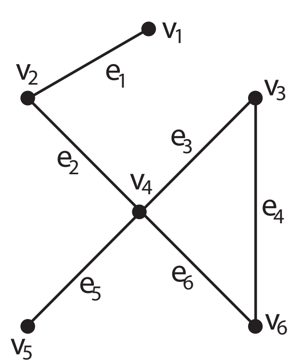
EXERCISE 2.9.2. *Prove that if a simple connected graph has exactly two non-cut vertices, then the graph is a simple path between these two non-cut vertices. [Hint: Use induction on the number of vertices and Exercise 2.9.1.]*

2.10. Examples.

EXAMPLE 2.10.1. *There are no cut vertices nor cut edges in the following graph.*



EXAMPLE 2.10.2. *v_2 and v_4 are cut vertices. e_1, e_2, and e_5 are cut edges in the following graph.*



Discussion

EXERCISE 2.10.1. *In each case, find how many cut edges and how many cut vertices there are for each integer n for which the graph is defined.*

(1) Star Network
(2) Cycle
(3) Complete Graphs.

2.11. Counting Edges.

THEOREM 2.11.1. *A connected graph with n vertices has at least $n-1$ edges.*

Discussion

Notice the Theorem states there are *at least* $n-1$ edges, not exactly $n-1$ edges. In a proof of this theorem we should be careful not to assume equality. Induction is the natural choice for a proof of this statement, but we need to be cautious of how we form the induction step.

Recall in the induction step we must show that a connected graph with $n+1$ vertices has at least n edges if we know every connected graph with n vertices has at least $n-1$ edges. It may seem like a good idea to begin with an arbitrary graph with n vertices and add a vertex and edge(s) to get one with $n+1$ vertices. However, the graph with $n+1$ vertices would depend on the one we started with. We want to make sure we have covered every possible connected graph with $n+1$ vertices, so we would have to prove *every* connected graph with $n+1$ vertices may be obtained this way to approach the proof this way. On the other hand, if we begin with an arbitrary graph with $n+1$ vertices and remove some vertex and adjacent edges to create a graph with n vertices the result may no longer be connected and we have to consider this possibility.

The proof of this theorem is a graded exercise.

2.12. Connectedness in Directed Graphs.

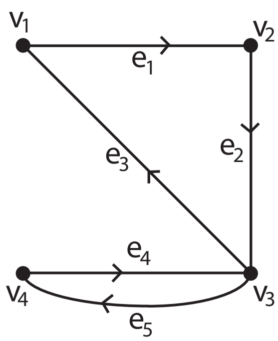
DEFINITION 2.12.1.
(1) A directed graph is **strongly connected** *if there is a directed path between every pair of vertices.*
(2) A directed graph is **weakly connected** *if the underlying undirected graph is connected.*

Discussion

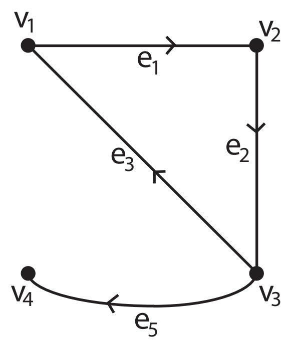
Recall that the underlying graph of a directed graph is the graph obtained by eliminating all the arrows. So the weakly connected means you can ignore the direction of the edges when looking for a path. Strongly directed means you must respect the direction when looking for a path between vertices. To relate this to something more familiar, if you are a pedestrian you do not have to worry about the direction of one way streets. This is not the case, however, if you are driving a car.

Exercise 2.12.1. *Are the following graphs strongly connected, weakly connected, both or neither?*

(a)



(b)



2.13. Paths and Isomorphism.

Theorem 2.13.1. *Let M be the adjacency matrix for the graph G. Then the (i, j)th entry of M^r is the number of paths of length r from vertex i to vertex j, where M^r is the standard matrix product of M by itself r times (not the Boolean product).*

Proof. The proof is by induction on the length of the path, r. Let p be the number of vertices in the graph (so the adjacency matrix is $p \times p$).

Basis: The adjacency matrix represents paths of length one by definition, so the basis step is true.

Induction Hypothesis: Assume each entry, say $m_{ij}^{[n]}$, in $M^n = [m_{ij}^{[n]}]$ equals the number of paths of length n from the i-th vertex to the j-th vertex.

Inductive Step: Prove each entry, say $m_{ij}^{[n+1]}$, in $M^{n+1} = [m_{ij}^{[n+1]}]$ equals the number of paths of length $n+1$ from the i-th vertex to the j-th vertex.

We begin by recalling $M^{n+1} = M^n \cdot M$ and by the definition of matrix multiplication the entry $m_{ij}^{[n+1]}$ in M^{n+1} is

$$m_{ij}^{[n+1]} = \sum_{k=1}^{p} m_{ik}^{[n]} \cdot m_{kj}$$

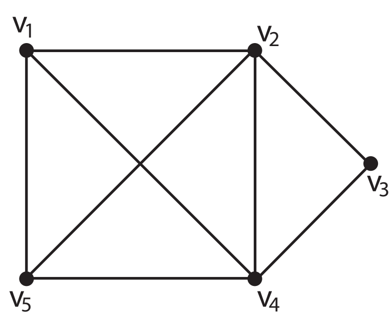
where $M^n = [m_{ij}^{[n]}]$ and $M = [m_{ij}]$.

By the induction hypothesis, $m_{ik}^{[n]}$ is the number of paths of length n between the i-th vertex and the k-th vertex, while m_{kj} is the number of paths of length 1 from the k-th vertex to the j-th vertex. Each of these paths may be combined to create paths of length $n+1$ from the i-th vertex to the j-th vertex. Using counting principles we see that the number of paths of length $n+1$ that go through the k-th vertex just before reaching the j-th vertex is $m_{ik}^{[n]} \cdot m_{kj}$.[(1)]

The above sum runs from $k = 1$ to $k = p$ which covers all the possible vertices in the graph. Therefore the sum counts all the paths of length $n+1$ from the i-th vertex to the j-the vertex.

□

2.14. Example 2.14.1.



EXAMPLE 2.14.1. *The adjacency matrix for the graph above is*

$$\begin{bmatrix} 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \end{bmatrix}$$

We get the following powers of M:

$$M^2 = \begin{bmatrix} 3 & 2 & 2 & 2 & 2 \\ 2 & 4 & 1 & 3 & 2 \\ 2 & 1 & 2 & 1 & 2 \\ 2 & 3 & 1 & 4 & 2 \\ 2 & 2 & 2 & 2 & 3 \end{bmatrix}$$

$$M^3 = \begin{bmatrix} 6 & 9 & 4 & 9 & 7 \\ 9 & 8 & 7 & 9 & 9 \\ 4 & 7 & 2 & 7 & 4 \\ 9 & 9 & 7 & 8 & 9 \\ 7 & 9 & 4 & 9 & 6 \end{bmatrix}$$

The last matrix tells us there are 4 paths of length 3 between vertices v_3 and v_1. Find them and convince yourself there are no more.

Discussion

If you recall that the adjacency matrix and all its powers are symmetric, you will cut your work in half when computing powers of the matrix.

EXERCISE 2.14.1. *Find the page(s) in the text that covers the counting principal(s) used in the sentence referenced as* (1) *in the proof of Theorem 2.13.1. Explain how the conclusion of this gives us the result of the sentence.*

2.15. Theorem 2.15.1.

THEOREM 2.15.1. *If G is a disconnected graph, then the compliment of G, $\overline{G}$, is connected.*

Discussion

The usual approach prove a graph is connected is to choose two arbitrary vertices and show there is a path between them. For the Theorem 2.15.1 we need to consider the two cases where the vertices are in different components of G and where the vertices are in the same component of G.

EXERCISE 2.15.1. *Prove Theorem 2.15.1.*

EXERCISE 2.15.2. *The compliment of a connected graph may or may not be connected. Find two graphs such that the compliment is (a) connected and (b) disconnected.*