CHAPTER 2

Graphs

1. Introduction to Graphs and Graph Isomorphism

1.1. The Graph Menagerie.

Definition 1.1.1.

- *A* **simple graph** $G = (V, E)$ *consists of a set* V *of vertices and a set* E *of edges, represented by unordered pairs of elements of* V*.*
- *A* **multigraph** *consists of a set* V *of vertices, a set* E *of edges, and a function*

$$f : E \to \{\{u, v\} : u, v \in V \text{ and } u \neq v\}.$$

  If $e_1, e_2 \in E$ *are such that* $f(e_1) = f(e_2)$*, then we say* e_1 *and* e_2 *are* **multiple** *or* **parallel edges***.*
- *A* **pseudograph** *consists of a set* V *of vertices, a set* E *of edges, and a function* $f : E \to \{\{u, v\} : u, v \in V\}$*. If* $e \in E$ *is such that* $f(e) = \{u, u\} = \{u\}$*, then we say* e *is a* **loop***.*
- *A* **directed graph or digraph** $G = (V, E)$ *consists of a set* V *of vertices and a set* E *of directed edges, represented by* **ordered pairs** *of vertices.*

Discussion

In Section 1.1 we recall the definitions of the various types of graphs that were introduced in MAD 2104. In this section we will revisit some of the ways in which graphs can be represented and discuss in more detail the concept of a graph isomorphism.

1.2. Representing Graphs and Graph Isomorphism.

Definition 1.2.1. *The* **adjacency matrix***,* $A = [a_{ij}]$*, for a simple graph* $G = (V, E)$*, where* $V = \{v_1, v_2, ..., v_n\}$*, is defined by*

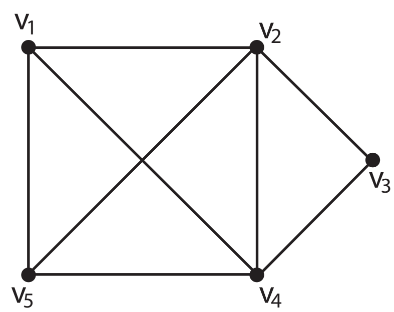
$$a_{ij} = \begin{cases} 1 & \text{if } \{v_i, v_j\} \text{ is an edge of } G, \\ 0 & \text{otherwise.} \end{cases}$$

Discussion

We introduce some alternate representations, which are extensions of connection matrices we have seen before, and learn to use them to help identify isomorphic graphs.

REMARKS 1.2.1. *Here are some properties of the adjacency matrix of an undirected graph.*

1. *The adjacency matrix is always symmetric.*
2. *The vertices must be ordered: and the adjacency matrix depends on the order chosen.*
3. *An adjacency matrix can be defined for multigraphs by defining a_{ij} to be the* **number** *of edges between vertices i and j.*
4. *If there is a natural order on the set of vertices we will use that order unless otherwise indicated.*



EXAMPLE 1.2.1. *An adjacency matrix for this graph is*

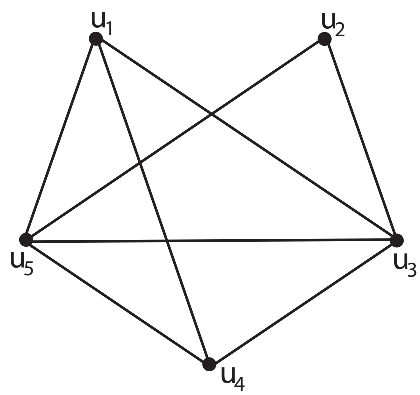
$$\begin{bmatrix} 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \end{bmatrix}$$

Discussion

As with connection matrices, an adjacency matrix can be constructed by using a table with the columns and rows labeled with the elements of the vertex set.

Here is another example

EXAMPLE 1.2.2. *The adjacency matrix for the graph*



is the matrix $M = \begin{bmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix}$

1.3. Incidence Matrices.

DEFINITION 1.3.1. *The* **incidence matrix**, $A = [a_{ij}]$, *for the undirected graph* $G = (V, E)$ *is defined by*

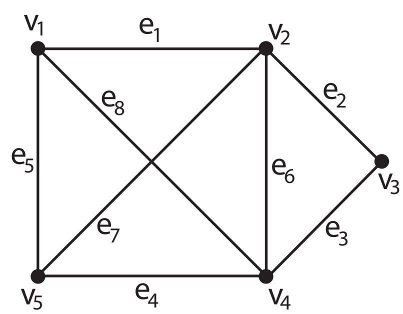
$$a_{ij} = \begin{cases} 1 & \text{if edge } j \text{ is incident with vertex } i \\ 0 & \text{otherwise.} \end{cases}$$

Discussion

The incidence matrix is another way to use matrices to represent a graph.

REMARKS 1.3.1.
(1) *This method requires the edges and vertices to be labeled and the matrix depends on the order in which they are written.*
(2) *Every column will have exactly two 1's.*
(3) *As with adjacency matrices, if there is a natural order for the vertices and edges that order will be used unless otherwise specified.*

1.4. Example 1.4.1.



EXAMPLE 1.4.1. *The incidence matrix for this graph is*

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \end{bmatrix}$$

Discussion

Again you can use a table to get the matrix. List all the vertices as the labels for the rows and all the edges for the labels of the columns.

1.5. Degree.

DEFINITION 1.5.1.

(1) *Let $G = (V, E)$ be an undirected graph.*
- *Two vertices $u, v \in V$ are* **adjacent** *or* **neighbors** *if there is an edge e between u and v.*
  - *The edge e* **connects** *u and v.*
  - *The vertices u and v are* **endpoints** *of e.*
- *The* **degree** *of a vertex v, denoted $deg(v)$, is the number of edges for which it is an endpoint. A loop contributes twice in an undirected graph.*
  - *If $deg(v) = 0$, then v is called* **isolated**.
  - *If $deg(v) = 1$, then v is called* **pendant**.

(2) *Let $G = (V, E)$ be a directed graph.*
- *Let (u, v) be an edge in G. Then u is an* **initial vertex** *and is* **adjacent to** *v. The vertex v is a* **terminal vertex** *and is* **adjacent from** *u.*
- *The* **in degree** *of a vertex v, denoted $deg^-(v)$ is the number of edges which terminate at v.*

- *Similarly, the* **out degree** *of v, denoted $deg^+(v)$, is the number of edges which initiate at v.*

Discussion

We now recall from MAD 2104 the terminology we use with undirected and directed graphs. Notice that a loop contributes two to the degree of a vertex.

1.6. The Handshaking Theorem.

THEOREM 1.6.1 (The Handshaking Theorem). *Let $G = (V, E)$ be an undirected graph. Then*

$$2|E| = \sum_{v \in V} deg(v)$$

PROOF. Each edge contributes twice to the sum of the degrees of all vertices. $\square$

Discussion

The handshaking theorem is one of the most basic and useful combinatorial formulas associated to a graph. It lets us conclude some facts about the numbers of vertices and the possible degrees of the vertices. Notice the immediate corollary.

COROLLARY 1.6.1.1. *The sum of the degrees of the vertices in any graph must be an even number.*

In other words, it is impossible to create a graph so that the sum of the degrees of its vertices is odd (try it!).

1.7. Example 1.7.1.

EXAMPLE 1.7.1. *Suppose a graph has 5 vertices. Can each vertex have degree 3? degree 4?*

- *The sum of the degrees of the vertices would be $3 \cdot 5$ if the graph has 5 vertices of degree 3. This is an odd number, though, so this is not possible by the handshaking Theorem.*
- *The sum of the degrees of the vertices if there are 5 vertices with degree 4 is 20. Since this is even it is possible for this to equal $2|E|$.*

Discussion

If the sum of the degrees of the vertices is an even number then the handshaking theorem is not contradicted. In fact, you can create a graph with any even degree you want if multiple edges are permitted. However, if you add more restrictions it may not be possible. Here are two typical questions the handshaking theorem may help you answer.

EXERCISE 1.7.1. *Is it possible to have a graph S with 5 vertices, each with degree 4, and 8 edges?*

EXERCISE 1.7.2. *A graph with 21 edges has 7 vertices of degree 1, three of degree 2, seven of degree 3, and the rest of degree 4. How many vertices does it have?*

1.8. Theorem 1.8.1.

THEOREM 1.8.1. *Every graph has an even number of vertices of odd degree.*

PROOF. Let V_o be the set of vertices of odd degree, and let V_e be the set of vertices of even degree. Since $V = V_o \cup V_e$ and $V_o \cap V_e = \emptyset$, the handshaking theorem gives us

$$2|E| = \sum_{v \in V} deg(v) = \sum_{v \in V_o} deg(v) + \sum_{v \in V_e} deg(v)$$

or

$$\sum_{v \in V_o} deg(v) = 2|E| - \sum_{v \in V_e} deg(v).$$

Since the sum of any number of even integers is again an even integer, the right-hand-side of this equations is an even integer. So the left-hand-side, which is the sum of a collection of odd integers, must also be even. The only way this can happen, however, is for there to be an even number of odd integers in the collection. That is, the number of vertices in V_o must be even. □

Discussion

Theorem 1.8.1 goes a bit further than our initial corollary of the handshaking theorem. If you have a problem with the last sentence of the proof, consider the following facts:

- odd + odd = even
- odd + even = odd
- even + even = even

If we add up an odd number of odd numbers the previous facts will imply we get an odd number. Thus to get an even number out of $\sum_{v \in V_o} deg(v)$ there must be an even number of vertices in V_o (the set of vertices of odd degree).

While there must be an even number of vertices of odd degree, there is no restrictions on the parity (even or odd) of the number of vertices of even degree.

This theorem makes it easy to see, for example, that it is not possible to have a graph with 3 vertices each of degree 1 and no other vertices of odd degree.

1.9. Handshaking Theorem for Directed Graphs.

THEOREM 1.9.1. *For any directed graph $G = (E, V)$,*

$$|E| = \sum_{v \in V} deg^-(v) = \sum_{v \in V} deg^+(v).$$

Discussion

When considering directed graphs we differentiate between the number of edges going into a vertex verses the number of edges coming out from the vertex. These numbers are given by the **in degree** and the **out degree**.

Notice that each edge contributes one to the in degree of some vertex and one to the out degree of some vertex. This is essentially the proof of Theorem 1.9.1.

1.10. Graph Invariants. The following are invariants under isomorphism of a graph G:

(1) G has r vertices.
(2) G has s edges.
(3) G has degree sequence $(d_1, d_2, ..., d_n)$.
(4) G is a bipartite graph.
(5) G contains r complete graphs K_n (as a subgraphs).
(6) G contains r complete bipartite graphs $K_{m,n}$.
(7) G contains r n-cycles.
(8) G contains r n-wheels.
(9) G contains r n-cubes.

Discussion

Recall that two simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are isomorphic if there is a bijection

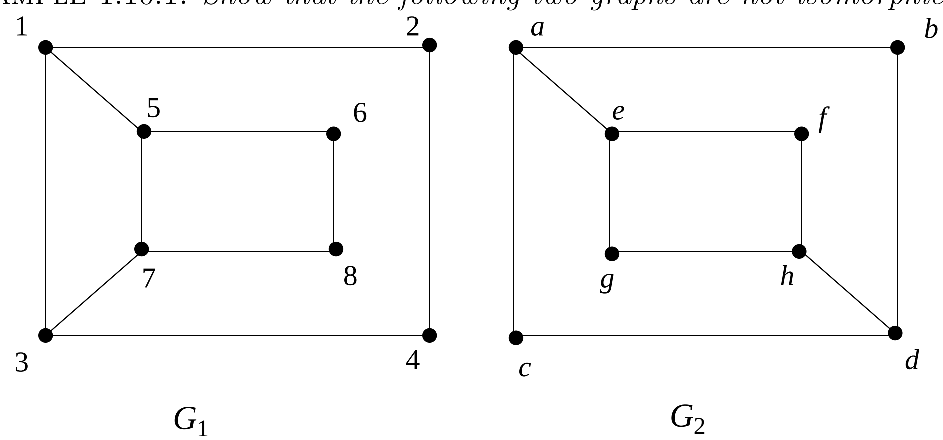
$$f : V_1 \to V_2$$

such that vertices u and v in V_1 are adjacent in G_1 if and only if $f(u)$ and $f(v)$ are adjacent in G_2. If there is such a function, we say f is an **isomorphism** and we write $G_1 \simeq G_2$.

It is often easier to determine when two graphs are *not* isomorphic. This is sometimes made possible by comparing *invariants* of the two graphs to see if they are different. We say a property of graphs is a **graph invariant** (or, just invariant) if, whenever a graph G has the property, any graph isomorphic to G also has the property. The **degree sequence** a graph G with n vertices is the sequence $(d_1, d_2, ..., d_n)$, where $d_1, d_2, ..., d_n$ are the degrees of the vertices of G and $d_1 \geq d_2 \geq \cdots \geq d_n$. Note that a graph could conceivably have infinitely many vertices. If the vertices are *countable* then the degree sequence would be an infinite sequence. If the vertices are not countable, then this degree sequence would not be defined.

The invariants in Section 1.10 may help us determine fairly quickly in some examples that two graphs are **not** isomorphic.

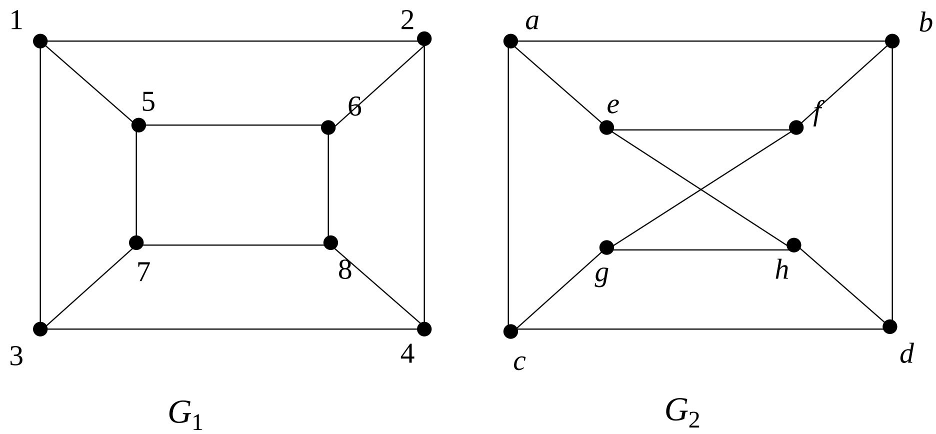
Example 1.10.1. *Show that the following two graphs are not isomorphic.*



The two graphs have the same number of vertices, the same number of edges, and same degree sequences $(3, 3, 3, 3, 2, 2, 2, 2)$. *Perhaps the easiest way to see that they are not isomorphic is to observe that* G_1 *has three 4-cycles, whereas* G_2 *has two 4-cycles. In fact, the four vertices of* G_1 *of degree 3 lie in a 4-cycle in* G_1*, but the four vertices of* G_2 *of degree 3 do not. Either of these two discrepancies is enough to show that the graphs are not isomorphic.*

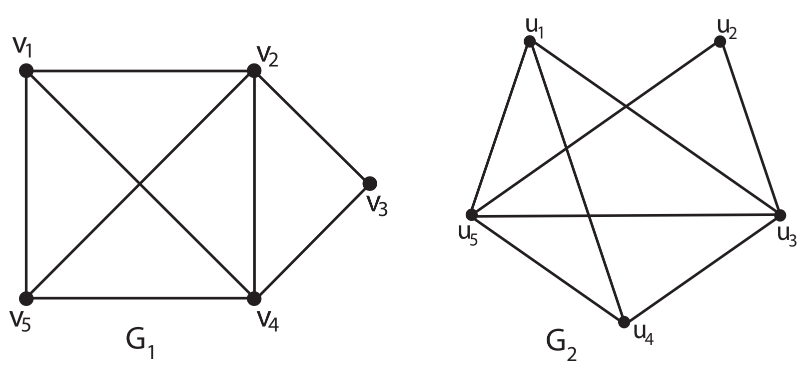
Another way we could recognize the graphs above are not isomorphic is to consider the adjacency relationships. Notice in G_1 *all the vertices of degree 3 are adjacent to 2 vertices of degree 3 and 1 of degree 2. However, in graph* G_2 *all of the vertices of degree 3 are adjacent to 1 vertex of degree 3 and 2 vertices of degree 2. This discrepancy indicates the two graphs cannot be isomorphic.*

Example 1.10.2. *The following two graphs are not isomorphic. Can you find an invariant that is different on the graphs.*



1.11. Example 1.11.1.

EXAMPLE 1.11.1. *Determine whether the graphs G_1 and G_2 are isomorphic.*



Solution

We go through the following checklist that might tell us immediately if the two are *not* isomorphic.

- They have the same number of vertices, 5.
- They have the same number of edges, 8.
- They have the same degree sequence $(4, 4, 3, 3, 2)$.

Since there is no obvious reason to think they are not isomorphic, we try to construct an isomorphism, f.
Note that the above does *not* tell us there *is* an isomorphism, only that there might be one.

The only vertex on each that have degree 2 are v_3 and u_2, so we must have $f(v_3) = u_2$.

Now, since $deg(v_1) = deg(v_5) = deg(u_1) = deg(u_4)$, we must have either

- $f(v_1) = u_1$ and $f(v_5) = u_4$, or
- $f(v_1) = u_4$ and $f(v_5) = u_1$.

It is possible only one choice would work or both choices may work (or neither choice may work, which would tell us there is no isomorphism).

We try $f(v_1) = u_1$ and $f(v_5) = u_4$.

Similarly we have two choices with the remaining vertices and try $f(v_2) = u_3$ and $f(v_4) = u_5$. This defines a bijection from the vertices of G_1 to the vertices of G_2. We still need to check that adjacent vertices in G_1 are mapped to adjacent vertices in G_2. To check this we will look at the adjacency matrices.

The adjacency matrix for G_1 (when we list the vetices of G_1 by v_1, v_2, v_3, v_4, v_5) is

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \end{bmatrix}$$

We create an adjacency matrix for G_2, using the bijection f as follows: since $f(v_1) = u_1$, $f(v_2) = u_3$, $f(v_3) = u_2$, $f(v_4) = u_5$, and $f(v_5) = u_4$, we rearrange the order of the vertices of G_2 to u_1, u_3, u_2, u_5, u_4. With this ordering, the adjacency matrix for G_2 is

$$B = \begin{bmatrix} 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \end{bmatrix}$$

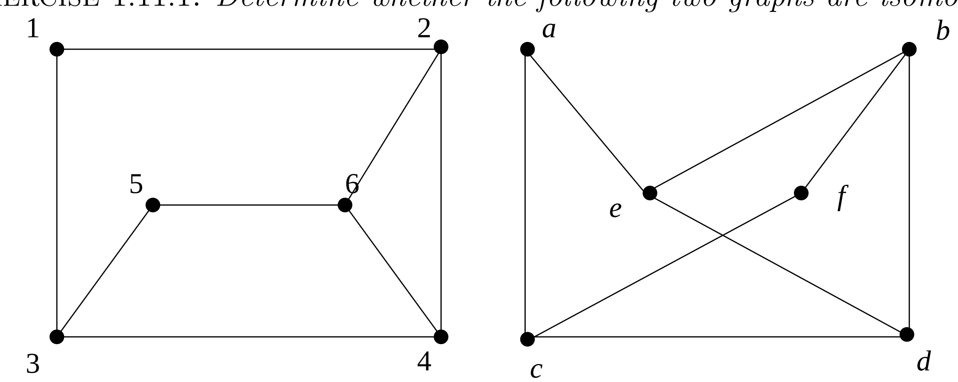
Since $A = B$, adjacency is preserved under this bijection. Hence the graphs are isomorphic.

Discussion

In this example we show that two graphs are isomorphic. Notice that it is *not* enough to show they have the same number of vertices, edges, and degree sequence. In fact, if we knew they were isomorphic and we were asked to prove it, we would proceed directly to try and find a bijection that preserves adjacency. That is, the check list is not necessary if you already know they are isomorphic. On the other hand, having found a bijection between two graphs that doesn't preserve adjacency doesn't tell us the graphs are not isomorphic, because some other bijection might work. If we go down this path, we would have to show that *every* bijection fails to preserve adjacency.

The advantage of the checklist is that it will give you a quick and easy way to show two graphs are *not* isomorphic if some invariant of the graphs turn out to be different. If you examine the logic, however, you will see that if two graphs have all of the same invariants we have listed so far, we still wouldn't have a proof that they are isomorphic. Indeed, there is no known list of invariants that can be efficiently checked to determine when two graphs are isomorphic. The best algorithms known to date for determining graph isomorphism have exponential complexity (in the number n of vertices).

Exercise 1.11.1. *Determine whether the following two graphs are isomorphic.*



G_1 G_2

Exercise 1.11.2. *How many different isomorphism (that is, bijections that preserve adjacencies) are possible from G_2 to itself in Example 1.10.1.*

Exercise 1.11.3. *There are 14 nonisomorphic pseudographs with 3 vertices and 3 edges. Draw all of them.*

Exercise 1.11.4. *Draw all nonisomorphic simple graphs with 6 vertices, 5 edges, and no cycles.*

Exercise 1.11.5. *Recall the equivalence relation on a set, $\mathcal{S}$, of graphs given by G_1 is related to G_2 if and only if $G_1 \simeq G_2$. How many equivalence classes are there if $\mathcal{S}$ is the set of all simple graphs with 6 vertices, 5 edges, and no cycles? Explain.*

1.12. Proof of Section 1.10 Part 3 for simple graphs.

Proof. Let G_1 and G_2 be isomorphic simple graphs having degree sequences. By part 1 of Section 1.10 the degree sequences of G_1 and G_2 have the same number of elements (finite or infinite). Let $f : V(G_1) \to V(G_2)$ be an isomorphism and let $v \in V(G_1)$. We claim $deg_{G_1}(v) = deg_{G_2}(f(v))$. If we show this, then f defines a bijection between the vertices of G_1 and G_2 that maps vertices to vertices of the same degree. This will imply the degree sequences are the same.

Proof of claim: Suppose $deg_{G_1}(v) = k$. Then there are k vertices adjacent to v, say $u_1, u_2, \dots, u_k$. The isomorphism maps each of the vertices to k distinct vertices adjacent to $f(u)$ in G_2 since the isomorphism is a bijection and preserves adjacency. Moreover, $f(u)$ will not be adjacent to any vertices other than the k vertices $f(u_1), f(u_2), \dots f(u_k)$. Otherwise, u would be adjacent to the preimage of such a vertex and this preimage would not be one of the vertices $u_1, u_2, \dots, u_k$ since f is an isomorphism. This would contradict that the degree of u is k. This shows the degree of $f(u)$ in G_2 must be k as well, proving our claim.

□

EXERCISE 1.12.1. *Prove the remaining properties listed in Section 1.10 for simple graphs using only the properties listed before each and the definition of isomorphism.*