

## 2. Closure of Relations

### 2.1. Definition of the Closure of Relations.

DEFINITION 2.1.1. Given a relation  $R$  on a set  $A$  and a property  $P$  of relations, the **closure** of  $R$  with respect to property  $P$ , denoted  $Cl_P(R)$ , is smallest relation on  $A$  that contains  $R$  and has property  $P$ . That is,  $Cl_P(R)$  is the relation obtained by adding the minimum number of ordered pairs to  $R$  necessary to obtain property  $P$ .

#### Discussion

To say that  $Cl_P(R)$  is the “smallest” relation on  $A$  containing  $R$  and having property  $P$  means that

- $R \subseteq Cl_P(R)$ ,
- $Cl_P(R)$  has property  $P$ , and
- if  $S$  is another relation with property  $P$  and  $R \subseteq S$ , then  $Cl_P(R) \subseteq S$ .

The following theorem gives an equivalent way to define the closure of a relation.

THEOREM 2.1.1. If  $R$  is a relation on a set  $A$ , then  $Cl_P(R) = \bigcap_{S \in \mathcal{P}} S$ , where

$\mathcal{P} = \{S \mid R \subseteq S \text{ and } S \text{ has property } P\}$ .

EXERCISE 2.1.1. Prove Theorem 2.1.1. [Recall that one way to show two sets,  $A$  and  $B$ , are equal is to show  $A \subseteq B$  and  $B \subseteq A$ .]

### 2.2. Reflexive Closure.

DEFINITION 2.2.1. Let  $A$  be a set and let  $\Delta = \{(x, x) \mid x \in A\}$ .  $\Delta$  is called the **diagonal relation** on  $A$ .

THEOREM 2.2.1. Let  $R$  be a relation on  $A$ . The **reflexive closure** of  $R$ , denoted  $r(R)$ , is the relation  $R \cup \Delta$ .

PROOF. Clearly,  $R \cup \Delta$  is reflexive, since  $(a, a) \in \Delta \subseteq R \cup \Delta$  for every  $a \in A$ .

On the other hand, if  $S$  is a reflexive relation containing  $R$ , then  $(a, a) \in S$  for every  $a \in A$ . Thus,  $\Delta \subseteq S$  and so  $R \cup \Delta \subseteq S$ .

Thus, by definition,  $R \cup \Delta \subseteq S$  is the reflexive closure of  $R$ .

□

## Discussion

Theorem 2.2.1 in Section 2.2 gives a simple method for finding the reflexive closure of a relation  $R$ .

## REMARKS 2.2.1.

- (1) Sometimes  $\Delta$  is called the **identity** or **equality** relation on  $A$ .
- (2)  $\Delta$  is the smallest reflexive relation on  $A$ .
- (3) Given the digraph of a relation, to find the digraph of its reflexive closure, add a loop at each vertex (for which no loop already exists).
- (4) Given the connection matrix  $M$  of a finite relation, the matrix of its reflexive closure is obtained by changing all zeroes to ones on the main diagonal of  $M$ . That is, form the Boolean sum  $M \vee I$ , where  $I$  is the identity matrix of the appropriate dimension.

## 2.3. Symmetric Closure.

**THEOREM 2.3.1.** *The **symmetric closure** of  $R$ , denoted  $s(R)$ , is the relation  $R \cup R^{-1}$ , where  $R^{-1}$  is the inverse of the relation  $R$ .*

## Discussion

## REMARKS 2.3.1.

- (1) To get the digraph of the inverse of a relation  $R$  from the digraph of  $R$ , reverse the direction of each of the arcs in the digraph of  $R$ .
- (2) To get the digraph of the symmetric closure of a relation  $R$ , add a new arc (if none already exists) for each (directed) arc in the digraph for  $R$ , but with the reverse direction.
- (3) To get the connection matrix of the inverse of a relation  $R$  from the connection matrix  $M$  of  $R$ , take the transpose,  $M^t$ .
- (4) To get the connection matrix of the symmetric closure of a relation  $R$  from the connection matrix  $M$  of  $R$ , take the Boolean sum  $M \vee M^t$ .
- (5) The composition of a relation and its inverse is not necessarily equal to the identity. A bijective function composed with its inverse, however, is equal to the identity.

## 2.4. Examples.

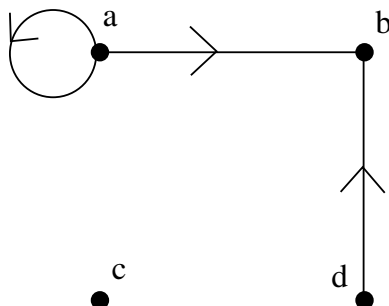
**EXAMPLE 2.4.1.** *If  $A = \mathbb{Z}$ , and  $R$  is the relation  $(x, y) \in R$  iff  $x \neq y$ , then*

- $r(R) = \mathbb{Z} \times \mathbb{Z}$ .
- $s(R) = R$ .

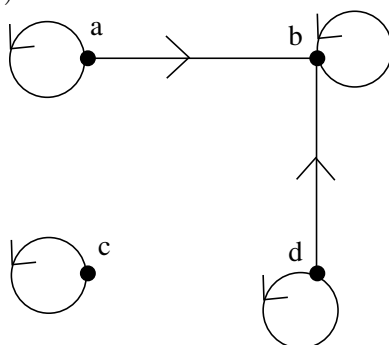
**EXAMPLE 2.4.2.** *If  $A = \mathbb{Z}^+$ , and  $R$  is the relation  $(x, y) \in R$  iff  $x < y$ , then*

- $r(R)$  is the relation  $(x, y) \in r(R)$  iff  $x \leq y$ .
- $s(R)$  is the relation  $(x, y) \in s(R)$  iff  $x \neq y$ .

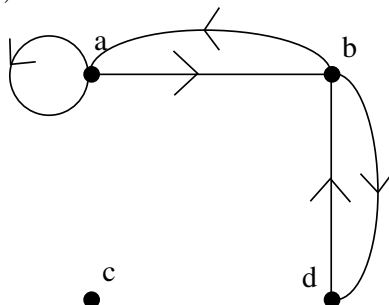
EXAMPLE 2.4.3. Let  $R$  be represented by the digraph



- The digraph of  $r(R)$  is



- The digraph of  $s(R)$  is



Discussion

In Section 2.4 we give several reflexive and symmetric closures of relations. Here is another example using the connection matrix of a relation.

EXAMPLE 2.4.4. Suppose  $R$  is a relation with connection matrix

$$M = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

- The connection matrix of the reflexive closure is

$$M_r = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

- The connection matrix for the symmetric closure is

$$M_s = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

## 2.5. Relation Identities.

**THEOREM 2.5.1.** *Suppose  $R$  and  $S$  are relations from  $A$  to  $B$ . Then*

- (1)  $(R^{-1})^{-1} = R$
- (2)  $(R \cup S)^{-1} = R^{-1} \cup S^{-1}$
- (3)  $(R \cap S)^{-1} = R^{-1} \cap S^{-1}$
- (4)  $(A \times B)^{-1} = B \times A$
- (5)  $\emptyset^{-1} = \emptyset$
- (6)  $(\overline{R})^{-1} = \overline{R^{-1}}$
- (7)  $(R - S)^{-1} = R^{-1} - S^{-1}$
- (8) If  $A = B$  then  $(R \circ S)^{-1} = S^{-1} \circ R^{-1}$
- (9) If  $R \subseteq S$  then  $R^{-1} \subseteq S^{-1}$

### Discussion

The properties given in Theorem 2.5.1 may be useful when trying to find the inverse of a relation.

**EXAMPLE 2.5.1.** *Suppose  $R$  is the relation on the real numbers defined by  $xRy$  iff  $x + y = 1$  or  $x - y = 1$ .  $R$  is the union of the two relations  $R_1$  and  $R_2$  defined by  $xR_1y$  iff  $x + y = 1$  and  $xR_2y$  iff  $x - y = 1$ . Since  $R_1^{-1}$  is defined by  $xR_1^{-1}y$  iff  $y + x = 1$  and  $R_2^{-1}$  is defined by  $xR_2^{-1}y$  iff  $y - x = 1$ , we apply the property  $(R_1 \cup R_2)^{-1} = R_1^{-1} \cup R_2^{-1}$  to see  $xR^{-1}y$  is defined by  $y + x = 1$  or  $y - x = 1$ .*

**EXERCISE 2.5.1.** *Describe the symmetric closure of the relation,  $R$ , in Example 2.5.1.*

**PROOF THAT**  $(R \cup S)^{-1} = R^{-1} \cup S^{-1}$ .  $(x, y) \in (R \cup S)^{-1}$

$\Leftrightarrow (y, x) \in R \cup S$ , by the definition of the inverse relation,

$\Leftrightarrow (y, x) \in R$  or  $(y, x) \in S$ , by the definition of union,

$\Leftrightarrow (x, y) \in R^{-1}$  or  $(x, y) \in S^{-1}$ , by the definition of inverse relations,

$\Leftrightarrow (x, y) \in R^{-1} \cup S^{-1}$ , by the definition of union.

□

EXERCISE 2.5.2. *Prove Property 1 in Theorem 2.5.1.*

EXERCISE 2.5.3. *Prove Property 3 in Theorem 2.5.1.*

EXERCISE 2.5.4. *Prove Property 4 in Theorem 2.5.1.*

EXERCISE 2.5.5. *Prove Property 5 in Theorem 2.5.1.*

PROOF OF PROPERTY 6 IN THEOREM 2.5.1. Let  $a \in A$  and  $b \in B$ . Then

$$\begin{aligned} (b, a) \in (\overline{R})^{-1} &\Leftrightarrow (a, b) \in (\overline{R}) && \text{by the definition of the inverse of a relation} \\ &\Leftrightarrow (a, b) \notin R && \text{by the definition of the complement} \\ &\Leftrightarrow (b, a) \notin R^{-1} && \text{by the definition of the inverse of a relation} \\ &\Leftrightarrow (b, a) \in \overline{R^{-1}} && \text{by the definition of the complement} \end{aligned}$$

□

EXERCISE 2.5.6. *Prove Property 7 in Theorem 2.5.1.*

EXERCISE 2.5.7. *Prove Property 8 in Theorem 2.5.1.*

EXERCISE 2.5.8. *Prove Property 9 in Theorem 2.5.1.*

## 2.6. Characterization of Symmetric Relations.

THEOREM 2.6.1. *Let  $R$  be a relation on a set  $A$ . Then  $R$  is symmetric iff  $R = R^{-1}$ .*

PROOF. First we show that if  $R$  is symmetric, then  $R = R^{-1}$ . Assume  $R$  is symmetric. Then

$$\begin{aligned} (x, y) \in R &\Leftrightarrow (y, x) \in R, && \text{since } R \text{ is symmetric} \\ &\Leftrightarrow (x, y) \in R^{-1}, && \text{by definition of } R^{-1} \end{aligned}$$

That is,  $R = R^{-1}$ .

Conversely, Assume  $R = R^{-1}$ , and show  $R$  is symmetric. To show  $R$  is symmetric we must show the implication “if  $(x, y) \in R$ , then  $(y, x) \in R$ ” holds. Assume  $(x, y) \in R$ . Then  $(y, x) \in R^{-1}$  by the definition of the inverse. But, since  $R = R^{-1}$ , we have  $(y, x) \in R$ .

□

## Discussion

Theorem 2.6.1 in Section 2.6 gives us an easy way to determine if a relation is symmetric.

**2.7. Paths.**

DEFINITION 2.7.1.

(1) A **path of length  $n$**  in a digraph  $G$  is a sequence of edges

$$(x_0, x_1)(x_1, x_2)(x_2, x_3) \cdots (x_{n-1}, x_n).$$

(2) If  $x_0 = x_n$  the path is called a **cycle** or a **circuit**.

(3) If  $R$  is a relation on a set  $A$  and  $a$  and  $b$  are in  $A$ , then a **path of length  $n$**  in  $R$  from  $a$  to  $b$  is a sequence of ordered pairs

$$(a, x_1)(x_1, x_2)(x_2, x_3) \cdots (x_{n-1}, b)$$

from  $R$ .

## Discussion

The reflexive and symmetric closures are generally not hard to find. Finding the transitive closure can be a bit more problematic. It is not enough to find  $R \circ R = R^2$ .  $R^2$  is certainly contained in the transitive closure, but they are not necessarily equal. Defining the transitive closure requires some additional concepts.

Notice that in order for a sequence of ordered pairs or edges to be a path, the terminal vertex of an arc in the list must be the same as the initial vertex of the next arc.

EXAMPLE 2.7.1. If  $R = \{(a, a), (a, b), (b, d), (d, b), (b, a)\}$  is a relation on the set  $A = \{a, b, c, d\}$ , then

$$(d, b)(b, a)(a, a)(a, a)$$

is a path in  $R$  of length 4.

**2.8. Paths v.s Composition.**

THEOREM 2.8.1. If  $R$  be a relation on a set  $A$ , then there is a path of length  $n$  from  $a$  to  $b$  in  $A$  if and only if  $(a, b) \in R^n$ .

PROOF. (by induction on  $n$ ):

*Basis Step:* An arc from  $a$  to  $b$  is a path of length 1, and it is a path of length 1 in  $R$  iff  $(a, b) \in R = R^1$ .

*Induction Hypothesis:* Assume that there is a path of length  $n$  from  $a$  to  $b$  in  $R$  iff  $(a, b) \in R^n$  for some integer  $n$ .

*Induction Step:* Prove that there is a path of length  $n + 1$  from  $a$  to  $b$  in  $R$  iff  $(a, b) \in R^{n+1}$ .

Assume

$$(a, x_1)(x_1, x_2) \cdots (x_{n-1}, x_n)(x_n, b)$$

is a path of length  $n + 1$  from  $a$  to  $b$  in  $R$ . Then

$$(a, x_1)(x_1, x_2) \cdots (x_{n-1}, x_n)$$

is a path of length  $n$  from  $a$  to  $x_n$  in  $R$ . By the induction hypothesis,  $(a, x_n) \in R^n$ , and we also have  $(x_n, b) \in R$ . Thus by the definition of  $R^{n+1}$ ,  $(a, b) \in R^{n+1}$ .

Conversely, assume  $(a, b) \in R^{n+1} = R^n \circ R$ . Then there is a  $c \in A$  such that  $(a, c) \in R$  and  $(c, b) \in R^n$ . By the induction hypothesis there is a path of length  $n$  in  $R$  from  $c$  to  $b$ . Moreover,  $(a, c)$  is a path of length 1 from  $a$  to  $c$ . By concatenating  $(a, c)$  onto the beginning of the path of length  $n$  from  $c$  to  $b$ , we get a path of length  $n + 1$  from  $a$  to  $b$ . This completes the induction step.

This shows by the principle of induction that There is a path of length  $n$  from  $a$  to  $b$  iff  $(a, b) \in R^n$  for any positive integer  $n$ .

□

### Discussion

Notice that since the statement is stated for all positive integers, induction is a natural choice for the proof.

## 2.9. Characterization of a Transitive Relation.

**THEOREM 2.9.1.** *Let  $R, S, T, U$  be relations on a set  $A$ .*

- (1) *If  $R \subseteq S$  and  $T \subseteq U$ , then  $R \circ T \subseteq S \circ U$ .*
- (2) *If  $R \subseteq S$ , then  $R^n \subseteq S^n$  for every positive integer  $n$ .*
- (3) *If  $R$  is transitive, then so is  $R^n$  for every positive integer  $n$ .*
- (4) *If  $R^k = R^j$  for some  $j > k$ , then  $R^{j+m} = R^{k+m}$  for every positive integer  $m$ .*
- (5)  *$R$  is transitive iff  $R^n$  is contained in  $R$  for every positive integer  $n$ .*

### Discussion

Theorem 2.9.1 in Section 2.9 give several properties that are useful in finding the transitive closure. The most useful are the last two. The second to the last property

stated in the theorem tells us that if we find a power of  $R$  that is the same as an earlier power, then we will not find anything new by taking higher powers. The very last tells us that we must include all the powers of  $R$  in the transitive closure.

**PROOF OF PROPERTY 1.** Assume  $T$  and  $U$  are relations from  $A$  to  $B$ ,  $R$  and  $S$  are relations from  $B$  to  $C$ , and  $R \subseteq S$  and  $T \subseteq U$ . Prove  $R \circ T \subseteq S \circ U$ .

Let  $(x, y) \in R \circ T$ . Then by the definition of composition, there exists a  $b \in B$  such that  $(x, b) \in T$  and  $(b, y) \in R$ . Since  $R \subseteq S$  and  $T \subseteq U$  we have  $(x, b) \in U$  and  $(b, y) \in S$ . This implies  $(x, y) \in S \circ U$ . Since  $(x, y)$  was an arbitrary element of  $R \circ T$  we have shown  $R \circ T \subseteq S \circ U$ .

□

**EXERCISE 2.9.1.** Prove property 2 in Theorem 2.9.1. Hint: use induction and property 1.

**EXERCISE 2.9.2.** Prove property 3 in Theorem 2.9.1. Hint: use induction on  $n$ .

**EXERCISE 2.9.3.** Prove property 4 in Theorem 2.9.1.

**EXERCISE 2.9.4.** Prove property 5 in Theorem 2.9.1.

## 2.10. Connectivity Relation.

**DEFINITION 2.10.1.** The **connectivity relation** of the relation  $R$ , denoted  $R^*$ , is the set of ordered pairs  $(a, b)$  such that there is a path (in  $R$ ) from  $a$  to  $b$ .

$$R^* = \bigcup_{n=1}^{\infty} R^n$$

## 2.11. Characterization of the Transitive Closure.

**THEOREM 2.11.1.** Given a relation  $R$  on a set  $A$ , the transitive closure of  $R$ ,  $t(R) = R^*$ .

**PROOF.** Let  $S$  be the transitive closure of  $R$ . We will prove  $S = R^*$  by showing that each contains the other.

- (1) Proof of  $R^* \subseteq S$ . By property 5 of Theorem 2.9.1,  $S^n \subseteq S$  for all  $n$ , since  $S$  is transitive. Since  $R \subseteq S$ , property 2 of Theorem 2.9.1 implies  $R^n \subseteq S^n$  for all  $n$ . Hence,  $R^n \subseteq S$  for all  $n$ , and so

$$R^* = \bigcup_{n=1}^{\infty} R^n \subseteq S.$$



(2) Proof of  $S \subseteq R^*$ . We will prove this by first showing that  $R^*$  is transitive.

Suppose  $(a, b)$  and  $(b, c)$  are in  $R^*$ . Then  $(a, b) \in R^m$  for some  $m \geq 1$  and  $(b, c) \in R^n$  for some  $n \geq 1$ . This means there is a path of length  $m$  in  $R$  from  $a$  to  $b$  and a path of length  $n$  in  $R$  from  $b$  to  $c$ . Concatenating these two paths gives a path of length  $m + n$  in  $R$  from  $a$  to  $c$ . That is,  $(a, c) \in R^{m+n} \subseteq R^*$ . Thus,  $R^*$  is transitive. Since  $S$  is the transitive closure of  $R$ , it must be contained in any transitive relation that contains  $R$ . Thus,  $S \subseteq R^*$ .

□

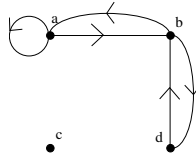
### Discussion

Theorem 2.11.1 in Section 2.11 identifies the transitive closure,  $t(R)$ , of a relation  $R$ . In the second part of the proof of Theorem 2.11.1 we have used the definition of the closure of a relation with respect to a property as given in Section 2.1. (See also the discussion immediately following Section 2.1.)

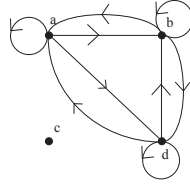
### 2.12. Example.

EXAMPLE 2.12.1. Let  $R$  be the relation on the set of integers given by  $\{(i, i+1) | i \in \mathbb{Z}\}$ . Then the transitive closure of  $R$  is  $R^* = \{(i, i+k) | i \in \mathbb{Z} \text{ and } k \in \mathbb{Z}^+\}$ .

EXAMPLE 2.12.2.  $S$  is the relation



The transitive closure is



### Discussion

Here is another example:

**EXAMPLE 2.12.3.** Let  $A$  be the set of all people and  $R$  the relation  $\{(x, y) \mid \text{person } x \text{ is a parent of person } y\}$ . Then  $R^*$  is the relation  $\{(x, y) \mid \text{person } x \text{ is an ancestor of person } y\}$ .

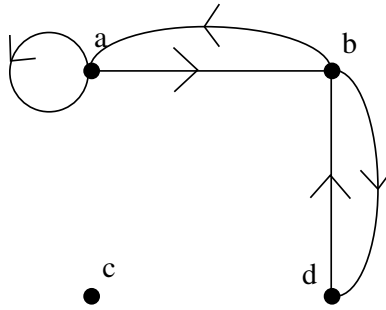
### 2.13. Cycles.

**THEOREM 2.13.1.** If  $R$  is a relation on a set  $A$  and  $|A| = n$ , then any path of length greater than or equal to  $n$  must contain a cycle.

**PROOF.** Suppose  $x_0, x_1, x_2, \dots, x_m$  is a sequence in  $A$  such that  $(x_{i-1}, x_i) \in R$  for  $i = 1, \dots, m$ , where  $m \geq n$ . That is, there is a path in  $R$  of length greater than or equal to  $n$ . Since  $|A| = n$ , the pigeon hole principle implies that  $x_i = x_j$  for some  $i \neq j$ . Thus, (assuming  $i < j$ ) the path contains a cycle from  $x_i$  to  $x_j (= x_i)$ .  $\square$

### Discussion

An easy way to understand what Theorem 2.13.1 in Section 2.13 is saying is to look at a digraph. Take the digraph



Take *any* path of length greater than 4. Say,  $(a, b)(b, d)(d, b)(b, a)(a, b)$ . By the theorem the path must have a cycle contained within it (a path that begins and ends at the same vertex). In this particular example there are several cycles; e.g.,  $(a, b)(b, d)(d, b)(b, a)$ . Notice if you eliminate this cycle you will have a shorter path with same beginning and ending vertex as the original path.

One of the points of Theorem 2.13.1 is that if  $|A| = n$  then  $R^k$  for  $k \geq n$  does not contain any arcs that did not appear in at least one of the first  $n$  powers of  $R$ .

**2.14. Corollaries to Theorem 2.13.1.**

COROLLARY 2.14.0.1. *If  $|A| = n$ , the transitive closure of a relation  $R$  on  $A$  is*

$$R^* = R \cup R^2 \cup R^3 \cup \dots \cup R^n.$$

COROLLARY 2.14.0.2. *We can find the connection matrix of  $R^*$  by computing the join of the first  $n$  Boolean powers of the connection matrix of  $R$ .*

Discussion

The corollaries of Theorem 2.13.1 in Section 2.14 give us powerful algorithms for computing the transitive closure of relations on finite sets. Now, it is *possible* for  $R^* = R \cup R^2 \cup \dots \cup R^k$  for some  $k < n$ , but the point is that you are guaranteed that  $R^* = R \cup R^2 \cup \dots \cup R^n$  if  $|A| = n$ .

**2.15. Example 10.**

EXAMPLE 2.15.1. *Let  $R$  be the relation  $\{(a, b), (c, b), (d, a), (c, c)\}$  in  $A = \{a, b, c, d\}$ . We find the transitive closure by using the corollary in Section 2.14.*

*The connectivity matrix for  $R$  is*

$$M = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad M^{[2]} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix},$$

$$M^{[3]} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \text{ and } M^{[4]} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Therefore the connectivity matrix for  $R^*$  is

$$M \vee M^{[2]} \vee M^{[3]} \vee M^{[4]} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

Discussion

In this example we use the basic algorithm to find the transitive closure of a relation. Recall  $M^{[n]} = \odot_{k=1}^n M$ . After taking the Boolean powers of the matrix, we take the join of the four matrices. While it did not happen on this example, it is possible for a Boolean power of the matrix to repeat before you get to the cardinality of  $A$ . If this happens you may stop and take the join of the powers you have, since nothing new will be introduced after that.

### 2.16. Properties v.s Closure.

EXERCISE 2.16.1. *Suppose  $R$  is a relation on a set  $A$  that is reflexive. Prove or disprove*

- (a)  $s(R)$  is reflexive
- (b)  $t(R)$  is reflexive

EXERCISE 2.16.2. *Suppose  $R$  is a relation on a set  $A$  that is symmetric. Prove or disprove*

- (a)  $r(R)$  is symmetric
- (b)  $t(R)$  is symmetric

EXERCISE 2.16.3. *Suppose  $R$  is a relation on a set  $A$  that is transitive. Prove or disprove*

- (a)  $r(R)$  is transitive
- (b)  $s(R)$  is transitive