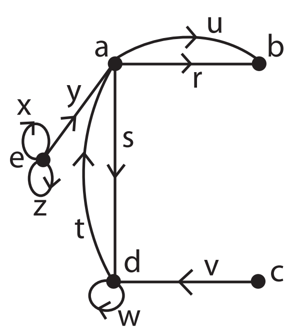
MAD 3105 PRACTICE TEST 2 SOLUTIONS

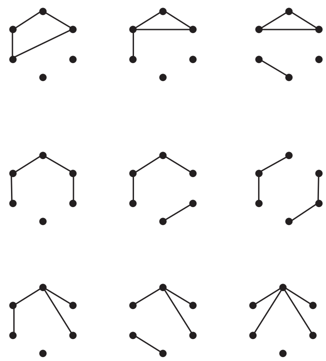
1. Define a graph G with $V(G) = \{a, b, c, d, e\}$, $E(G) = \{r, s, t, u, v, w, x, y, z\}$ and γ, the function defining the edges, is given by the table

ϵ	r	s	t	u	v	w	x	y	z
$\gamma(\epsilon)$	(a, b)	(a, d)	(d, a)	(a, b)	(c, d)	(d, d)	(e, e)	(e, a)	(e, e)

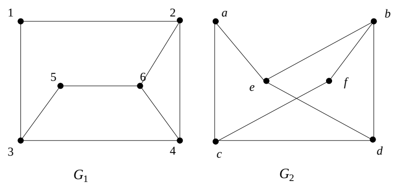
   (a) Draw a picture of G.



   (b) directed
   (c) $deg(b) = 2$ and $deg(e) = 5$
2. List 5 properties that are invariant under isomorphism.
   See section 10 in the lecture notes for Introduction to Graphs.
3. Sketch a graph of each of the following when $n = 5$. For what positive value(s) of $n > 2$ is the graph bipartite?
   Look in the text in section 8.2 for graphs.
   (a) K_n Bipartite if $n = 2$ only
   (b) C_n Bipartite if n is even and at least 3.
   (c) W_n Never Bipartite
   (d) Q_n Bipartite for all $n > 2$.
4. Draw all the nonisomorphic simple graphs with 6 vertices and 4 edges.



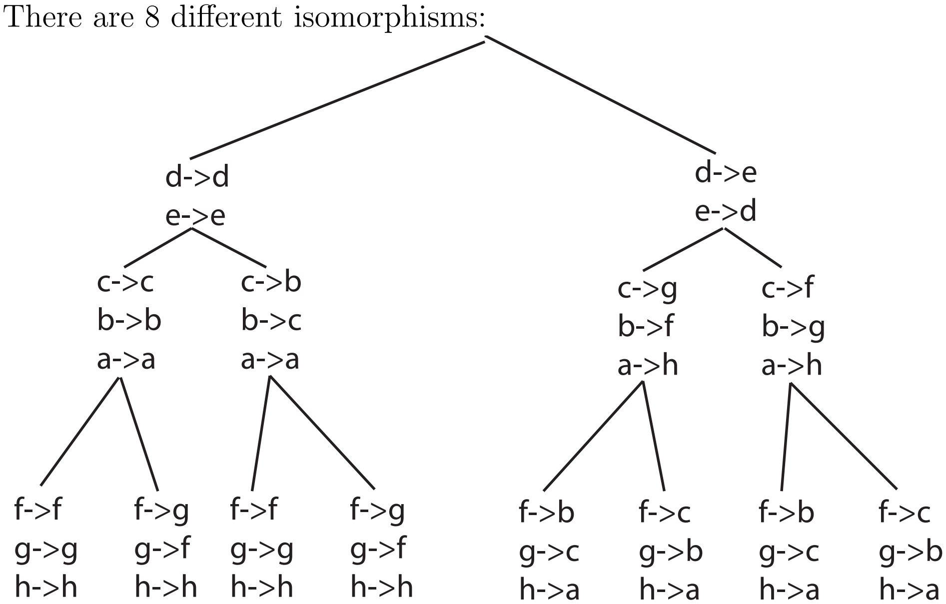
5. Determine which of the following graphs are isomorphic.



are isomorphic. None of the others are isomorphic to another.

6. Given the graph G below, how many different isomorphisms are there from G to G. Briefly explain.

There are 8 different isomorphisms:



7. Can an undirected graph have 5 vertices, each with degree 6?

Yes, Take for example the complete graph with 5 vertices and add a loop at each vertex.

8. Can a simple graph have 5 vertices, each with degree 6?

No, the complete graph with 5 vertices has 10 edges and the complete graph has the largest number of edges possible in a simple graph.

9. A graph has 21 edges has 7 vertices of degree 1, three of degree 2, seven of degree 3, and the rest of degree 4. How many vertices does it have?

   The graph has 19 vertices.

10. How many edges does a graph with 5 vertices have if 2 of the vertices have degree 3, 1 vertex has degree 2, and the rest of the vertices have degree 1?

    By the Hand Shaking Theorem, $2|E| = 2 \cdot 3 + 1 \cdot 2 + 2 \cdot 1 = 10$, so $|E| = 5$.

11. Let S be a set of simple graphs and define the relation R on S as follows:

    Let $G, H \in S$. Then $(G, H) \in R$ if and only if $G \simeq H$.

    This is equivalent to

    Let $G, H \in S$. Then $(G, H) \in R$ if and only if there exists an isomorphism $f : V(G) \to V(H)$.

    Prove this relation is an equivalence relation.

    This is the proof of Theorem 4 in the lecture notes on Equivalence Relations.

12. Suppose G and H are isomorphic simple graphs. Show that their complimentary graphs $\overline{G}$ and $\overline{H}$ are also isomorphic.

    Proof. Since G and H are isomorphic there is a function $f : V(G) \to V(H)$ that is a bijection and preserves adjacency. Note that a complimentary graph has the exact same vertex set, so f may also be thought of as a function from $V(\overline{G})$ to $V(\overline{H})$. Moreover, f is still a bijection. The question that remains is "does f preserve adjacency in the complimentary graphs?"

    Let $u, v \in V(\overline{G}) = V(G)$.

    $\{u, v\}$ is an edge in $\overline{G}$ iff $\{u, v\}$ is *not* an edge in G by the definition of complimentary graph.

    $\{u, v\}$ is *not* an edge in G iff $\{f(u), f(v)\}$ is *not* an edge in H since f preserves adjacency.

    $\{f(u), f(v)\}$ is *not* an edge in H iff $\{f(u), f(v)\}$ is an edge in $\overline{H}$ by the definition of complimentary graphs.

    Thus $\{u, v\}$ is an edge in $\overline{G}$ iff $\{f(u), f(v)\}$ is an edge in $\overline{H}$. This shows f preserves adjacency.

    □

13. Give the number of cut vertices and cut edges of the following graphs.
    (a) K_n, $n \geq 2$

       For $n = 2$ there is one cut edge. For $n \geq 3$ there are no cut edges. For $n \geq 2$ there are no cut vertices.

    (b) W_n, $n \geq 3$

       There are no cut edges nor vertices.

    (c) $K_{m,n}$, $m, n \geq 1$

       If $m = n = 1$ there is one cut edge and no cut vertices. If $m = 1$ and $n > 1$ there are n cut edges and one cut vertex. If $n = 1$ and $m > 1$ there are m cut edges and one cut vertex. If $m, n \geq 2$ there are no cut edges nor vertices.

14. For what values of n does each graph have (i) and Euler circuit? (ii) a Hamilton circuit?
    (a) K_n

(i) n odd and at least 3
(ii) all n

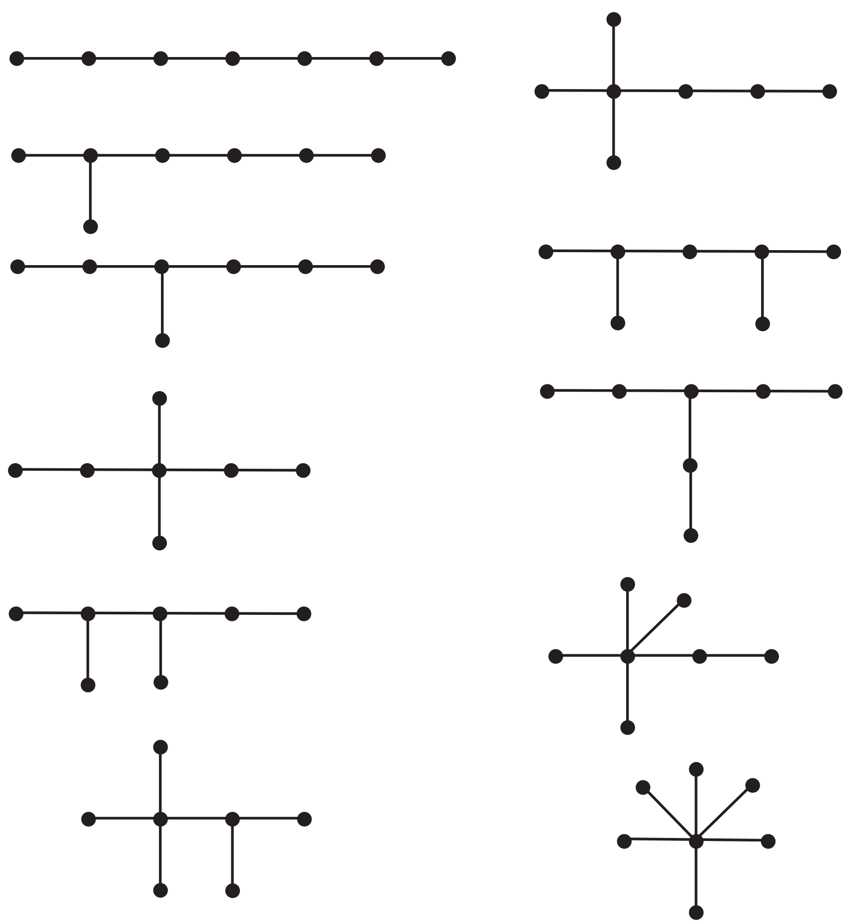
(b) C_n
(i) and (ii) All $n \geq 3$

(c) W_n
(i) no n
(ii) All $n \geq 3$

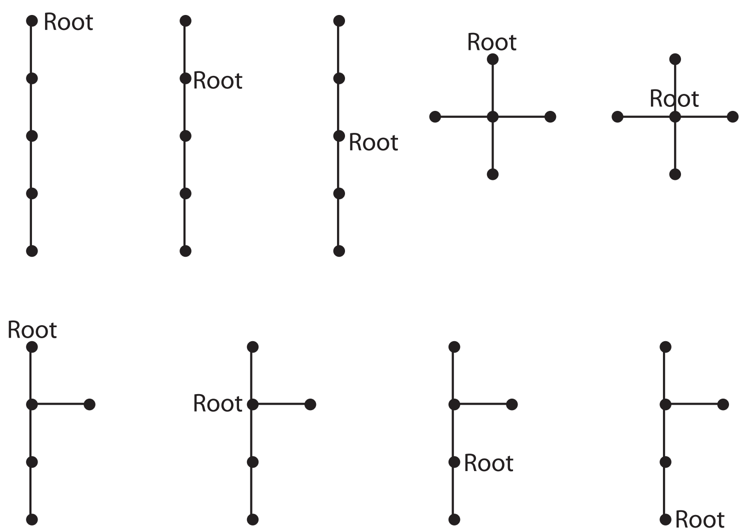
(d) Q_n
(i) n even and at least 2
(ii) all n.

15. Does the Theorem given imply the graph below has a Hamilton circuit?
The given theorem does not imply anything about the graph. The condition of the theorem is not satisfied.

16. Draw all the nonisomorphic (unrooted) trees with 6 edges.



17. Draw all the nonisomorphic rooted trees with 4 edges.



18. Given G is a finite simple graph. Give *six different* completions to the sentence. G is a tree if and only if ...

   Look in the lecture notes *Introduction to Trees* pages 1 (definition, Theorem 1) and 7 (Theorem 5).

19. Answer the following questions. Explain.
   (a) How many leaves does a full 3-ary tree with all leaves at height 4 have?
   $L = 3^{height} = 3^4 = 81$
   (b) How many leaves does a 3-ary tree have if it has 15 parents and every parent has exactly 3 children?
   There are $15 \cdot 3 + 1 = 46$ vertices since every vertex except the root is the child of one of these 15 parents. Thus the number of leaves is $46 - 15 = 31$.

20. Prove
   (a) Prove that every connected graph with at least 2 vertices has at least two non-cut vertices.
   Text section 8.4, problem 29.
   (b) Prove a connected graph with n vertices has at least $n - 1$ edges.
   One version uses the first principal of induction and problem 20a. The following is another possible version.

Proof. **Basis:** A graph with a single vertex clearly has at least 0 edges.

Induction Step: Let n be a positive integer and suppose any connected graph with k vertices has at least $k - 1$ edges for any integer k with $1 \le k \le n$.

Let G be a connected graph with $n + 1$ vertices and let v be a vertex of G. Remove v and all incident edges to v from G. The resulting graph, which we will call G', may or may not be connected, but we do know it has n vertices. Suppose G' has s components where $1 \le s \le n$. Each component is a connected graph with n or fewer vertices, so we may apply the induction hypothesis to each component. For each component, if the component has k vertices then it has at least $k-1$ edges. Thus G' must have at least $n-s$ edges. When v was removed from G we only removed edges incident to v to obtain G'. For each component in G' there must be an edge from v to a vertex in that component since G was connected. Thus there must be at least s more edges in G than there are in G', i.e. the number of edges in G is at least $(n-s)+s=n$. □

(c) *Proof.* Let G be a finite graph with all vertices of degree at least 2. We prove this theorem by recursively creating a path in G that must eventually contain a cycle.

Basis: Choose an arbitrary vertex, v_0, in G. Since $deg(v_0) \geq 2$ there is an edge, $\{v_0, v_1\}$, incident to v_0. Take this edge as the first edge in the path.

Recursive Step: Suppose we have a path $P_n : \{v_0, v_1\}, \{v_1, v_2\}, \dots,$ $\{v_{n-1}, v_n\}$ in G.
If v_n is not equal to any of the vertices $v_0, v_1, \dots, v_{n-1}$, then the only edge in P_n incident to v_n is $\{v_{n-1}, v_n\}$. Since $deg(v_n) \geq 2$, there must be an edge $\{v_n, v_{n+1}\}$ in G different from any of the edges in the path P_n. Add this edge onto the end of the path P_n to obtain a path P_{n+1}.
If v_n is equal to one of the vertices $v_0, v_1, \dots, v_{n-1}$, then we stop with P_n.

Notice the recursive step must be possible until the ending condition is met and the ending condition must eventually be met since there are only finitely many vertices in G. The resulting path must contain a cycle since the path repeats a vertex. The cycle would go from the first occurrence of the vertex to the second occurrence. □

(d) Prove a graph with n vertices and at least n edges contains a cycle for all positive integers n. You may use that a graph with all vertices of degree at least 2 contains a cycle.

Proof. **Basis:** Prove a graph with 1 vertex and at least 1 edge contains a cycle. Notice that is there is only one vertex, then any edge must begin and end at that vertex. Thus, in this case the graph contains a loop, which is a cycle.

Induction Step: Let $k \in \mathbb{Z}^+$. Assume that any graph with k vertices and at least k edges contains a cycle. Prove any graph with $k+1$ vertices and at least $k+1$ edges contains a cycle.
Let G be a graph with $k+1$ vertices and at least $k+1$ edges. We break the rest of the proof into 2 cases.

Case 1: Assume all the vertices have degree at least 2. Problem 20c tells us in this case, that there must be a cycle contained in G.

Think about why you cannot use something similar to what we do below in this case.

Case 2: Assume there is at least one vertex of degree 0 or 1. *It is not necessarily true that there is a vertex with degree 0 AND one with degree 1.*

Case A: Suppose there exists some vertex, say v, in G with degree 0. This vertex must be isolated. In other words, there are no edges attached to v. Consider The graph $G' = G - \{v\}$. G' has k vertices and at least k edges. *It actually has at least* $k+1$ *edges, but that's ok.* We can apply the induction hypothesis to G' to get that G' must contain a cycle. However, G' is a subgraph of G, so a cycle in G' is also a cycle in G.

Case B: Suppose there exists some vertex, say v, in G with degree 1. This vertex must have exactly one edge, say e, attached to it. Let G' be the subgraph of G obtained by removing v and e. G' has k vertices and at least k edges. We can apply the induction hypothesis to G' to get that G' must contain a cycle. However, G' is a subgraph of G, so a cycle in G' is also a cycle in G.

We have shown in all possible cases G must contain a cycle, therefore, by the principle of mathematical induction, any graph with at least as many edges as vertices must contain a cycle. □

(e) Prove the complimentary graph of a disconnected graph is connected.

Proof. Let G be a graph that is disconnected and let u and v be vertices in the complimentary graph, which we will denote by G'. Recall that the vertex set of G and G' are the same so u and v are also vertices in G. We break the proof into two cases. First the case where u and v are not in the same component of G and second the case where they are in the same component.
Case 1: Assume u and v are not in the same component in G. Then $\{u, v\}$ is not and edge in G. By the definition of the complimentary graph, $\{u, v\}$ must be an edge in G'. This edge provides a path in G' from u to v.
Case 2: Assume u and v are in the same component in G. There there is a vertex w in G that is a different component than the one u and v are in. Since it is in a different component, $\{u, w\}, \{w, v\}$ are not edges in G, so $\{u, w\}, \{w, v\}$ are edges in G'. These edges give us a path from u to v in G'. This shows G' is connected. □

(f) A full m-ary tree with i internal vertices has $n = mi + 1$ vertices
Lecture notes *Introduction to Trees*, Theorem 3 part 1.

(g) An m-ary tree with height h has at most m^h leaves.
Lecture notes *Introduction to Trees*, Theorem 4.

(h) Let G be a simple graph. G is a tree if and only if G is acyclic but the addition of any edge between any two vertices in G will create a cycle in G.
Lecture notes *Introduction to Trees*, Theorem 5.

(i) Let G be a simple graph. G is a tree if and only if G is connected but the removal of any edge in G produces a disconnected graph.

Proof. Suppose G is a tree. Then by defintion, G is connected. Let $\{u, v\}$ be an arbitrary edge in G. Then $\{u, v\}$ is clearly a path from u to v. Since G is a tree this is the *only* path in G between u and v, so if the edge $\{u, v\}$ is removed from G there will be no path from u to v in the resulting graph. Thus the resulting graph is disconnected.

Conversely, suppose G is connected but the removal of any edge in G produces a disconnected graph. We need to prove G is a tree. We already have G is connected, so what we still need to show is that G is acyclic. We use contradiction. Suppose G contains a cycle, C, and $\{u, v\}$ is an edge in C. If we remove this edge we claim the resulting graph, say G', is still connected. Let w and x be vertices in G'. Since G is connected there is a path from w to x in G.

If this path uses the edge $\{u, v\}$, then we can replace this edge in the path with the part of the cycle left from C after removing $\{u, v\}$. The new path is a path from w to x that does not use $\{u, v\}$, so it is a path in G'.

If the original path does not use $\{u, v\}$ then it is a path in G'.

Thus there is a path in G' from w to x. since these vertices were arbitrary we have shown G' is connected. This contradicts our assumption that the removal of any edge in G produces a disconnected graph. Thus G does not contain a cycle and must, therefore, be a tree.

□

(j) Let G be a simple graph with n vertices. G is a tree is and only if G is connected and has $n - 1$ edges.

Text p. 642, exercise 15.

(k) Let G be a simple graph with n vertices. G is a tree if and only if G is acyclic and has $n - 1$ edges.

Proof. Let G be a simple graph with n vertices. Assume G is a tree. Then by definition G is acyclic. Select a root v and direct the edges away from v. Then there are as many edges as there are terminal vertices of the edges and every vertex except v is a terminal vertex of some edge.

Conversely, assume G is acyclic and has $n - 1$ edges and we will prove G is a tree. To show G is a tree we only need to show it is connected. Suppose G has k connected components. Then the vertices and edges of G must be distributed among the components. Each component must be a tree since the graph is acyclic. Since each component is a tree each component must have exactly one less edge than vertices in that component. But then the total number of edges in G would be $n - k$. Hence $k = 1$ and so G must be connected.

□

21. Find the prefix, postfix, and infix forms for the algebraic expressions $ab + c/d \uparrow 3$ and $a((b+c)/d) \uparrow 3$ (in infix form).
    prefix: $+ \ * \ a \ b \ / \ c \ \uparrow \ d \ 3; \uparrow \ * \ a \ / \ + \ b \ c \ d \ 3$
    postfix: $a \ b \ * \ c \ d \ 3 \ \uparrow \ / \ +; a \ b \ c \ + \ d \ / \ * \ 3 \ \uparrow$
22. What is the value of the prefix expression: $+ - \uparrow 3 \ 2 \uparrow 2 \ 3 \ / \ 6 - 4 \ 2 = 4$
23. What is the value of the postfix expression: $9 \ 3 \ / \ 5 \ + \ 7 \ 2 \ - \ * = 40$