MAD 3105 PRACTICE FOR FINAL EXAM

Information about the final:
- The final exam is "cumulative", covering Boolean Algebra, Graphs, Trees, and Relations. There is a bit more emphasis on Boolean Algebra, because the other material has been sampled for the previous two exams.
- There will be one proof that requires using the Principle of Mathematical Induction (PMI). This question will count 20be assessed with particular attention to the logical structure, on top of the math particular to the subject of the proof.
- More generally, the exam looks like this:
  (1) Thirteen True/False questions covering a variety of facts and small problems. These count a total of 26 percent.
  (2) Several questions on boolean topics, covering canonical normal forms (disjunctice or conjunctive), using boolean algebra axioms and factoids to prove other factoids, simplification of boolean functions using Karnaugh maps or Quine-McClusky. These are worth about 34 percent.
  (3) Miscellaneous questions about trees, graphs, and relations, perhaps including a proof or two, worth a total of 20 percent.
  (4) One PMI proof, from one of the areas covered by the exam, worth 20 percent.
- Here are some of the "rules and constraints" that will be in effect:
  (1) When a proof specifies a method, you must use that method to receive full credit.
  (2) On proofs that do not specify a method you may use any method.
  (3) On single, short answer problems that do not specifically say to explain no explanation is required.
  (4) On problems that say to explain or show work, you must do so to receive full credit. However, you do not have to give a formal proof. Show your work in the spaces provided on the exam or *clearly* indicate where work is shown.
  (5) The problems on the exam are about the same level of difficulty as the problems on this review.
  (6) The topics covered on the exam are a subset of the topics covered on this review.
  (7) No calculators or material other than writing utensils will be permitted. Scap paper will be provided if needed.

1. Graphs

(1) Define a graph G with $V(G) = \{a, b, c, d, e\}$, $E(G) = \{r, s, t, u, v, w, x, y, z\}$ and γ, the function defining the edges, is given by the table

ϵ	r	s	t	u	v	w	x	y	z
$\gamma(\epsilon)$	(a,b)	(a,d)	(d,a)	(a,b)	(c,d)	(d,d)	(e,e)	(e,a)	(e,e)

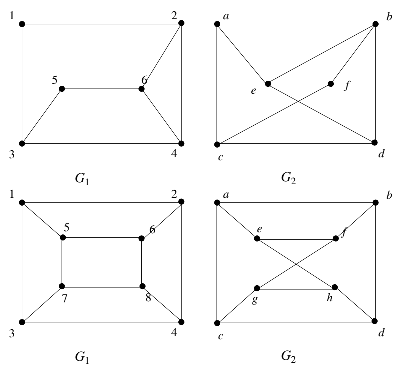
(a) Draw a picture of G. Label all vertices and edges.
(b) Is G directed or undirected?
(c) If G is undirected find the degrees of b and e. If G is directed find the degrees of b and e in the underlying undirected graph.

(2) List 5 properties that are invariant under isomorphism.

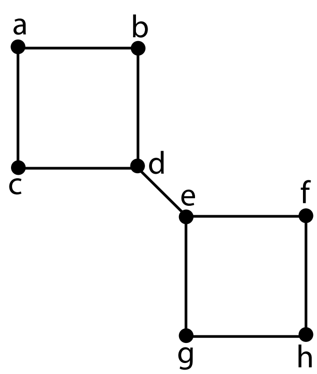
(3) Sketch a graph of each of the following when $n = 5$. For what positive value(s) of $n > 2$ is the graph bipartite?
(a) K_n
(b) C_n
(c) W_n
(d) Q_n

(4) Draw all the nonisomorphic simple graphs with 6 vertices and 4 edges.

(5) Determine which of the following graphs are isomorphic.



(6) Given the graph G below, how many different isomorphisms are there from G to G.



(7) Can an undirected graph have 5 vertices, each with degree 6?

(8) Can a simple graph have 5 vertices, each with degree 6?

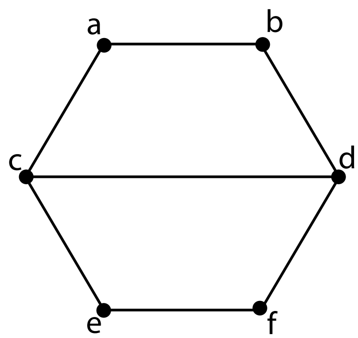
(9) A graph has 21 edges has 7 vertices of degree 1, three of degree 2, seven of degree 3, and the rest of degree 4. How many vertices does it have?

(10) How many edges does a graph with 5 vertices have if 2 of the vertices have degree 3, 1 vertex has degree 2, and the rest of the vertices have degree 1?

(11) Give the number of cut vertices and cut edges of the following graphs.
  (a) K_n, $n \geq 2$
  (b) W_n, $n \geq 3$
  (c) $K_{m,n}$, $m, n \geq 1$

(12) For what values of n does each graph have (i) and Euler circuit? (ii) a Hamilton circuit?
  (a) K_n
  (b) C_n
  (c) W_n
  (d) Q_n

(13) Does the Theorem given imply the graph below has a Hamilton circuit?



Theorem (Ore's Theroem). *If G is a simple graph with n vertices with $n \geq 3$ such that $deg(u) + deg(v) \geq n$ for every pair of nonadjacent vertices u andv in G, then G has a Hamilton circuit.*

(14) Draw all the nonisomorphic (unrooted) trees with 6 edges.

(15) Draw all the nonisomorphic rooted trees with 4 edges.

(16) Given G is a finite simple graph. Give *six different* completions to the sentence. G is a tree if and only if ...

(17) Answer the following questions. Explain.
  (a) How many leaves does a full 5-ary tree with all leaves at height 40 have?
  (b) How many leaves does a 3-ary tree have if it has 15 parents and every parent has exactly 3 children?

(18) Prove
  (a) There is a simple path between every pair of distinct vertices in a connected graph.
  (b) Prove a connected graph with n vertices has at least $n - 1$ edges.
  (c) Prove a finite graph with all vertices of degree at least 2 contains a cycle.

(d) Prove a graph with n vertices and at least n edges contains a cycle for all positive integers n. You may use that a graph with all vertices of degree at least 2 contains a cycle.
(e) Prove the complimentary graph of a disconnected graph is connected.
(f) An m-ary tree with height h has at most m^h leaves.
(g) Let G be a graph with at least two vertices. G is a tree if and only if for each pair of distinct vertices in G, there is a unique simple path between the vertices.
(h) Let G be a simple graph. G is a tree if and only if G is acyclic but the addition of any edge between any two vertices in G will create a cycle in G.
(i) Let G be a simple graph with at least 2 vertices. G is a tree if and only if G is connected but the removal of any edge in G produces a disconnected graph.
(j) Let G be a simple graph with n vertices. G is a tree if and only if G is connected and has $n-1$ edges.
(k) Let G be a simple graph with n vertices. G is a tree if and only if G is acyclic and has $n-1$ edges.

2. Relations

(19) Let R be the relation defined below. Determine which properties, reflexive, irreflexive, symmetric, antisymmetric, transitive, the relation satisfies. Prove each answer.
(a) R is the relation on a set of all people enrolled in courses at FSU given by two people a and b are such that $(a, b) \in R$ if and only if a and b are enrolled in the same course at FSU.
(b) R is the relation on $\{a, b, c\}$, $R = \{(a, b), (b, a), (b, b), (c, c)\}$
(c) R is the relation on the set of positive integers given by mRn if and only if $gcd(m, n) > 1$.
(d) R is the relation on the set of positive real numbers given by xRy if and only if x/y is a rational number.
(e) Let $f : \mathbb{R} \to \mathbb{R}$ be the function $f(x) = \lfloor x \rfloor$. Define the relation R on the set of real numbers by $R = graph(f)$.
(f) Let $f : \mathbb{R} \to \mathbb{R}$ be the function $f(x) = \lfloor x \rfloor$. Define the relation R on the set of real numbers by aRb iff $f(a) = f(b)$.

(20) Let R be the relation $R = \{(a, c), (b, b), (b, c), (c, a)\}$ and S the relation $S = \{(a, a), (a, b), (b, c), (c, a)\}$ is a relation on $A = \{a, b, c\}$.
(a) Find R^2.
(b) Find $S \circ R$.
(c) Find R^{-1}

(21) The matrix below is the matrix for a relation.

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

(a) Which of the following properties does the relation satisfy: reflexive, irreflexive, symmetric, antisymmetric, asymmetric, transitive?
(b) Find the matrix that represents R^{-1}.
(c) Find the matrix that represents R^2.
(d) Find the matrix that represents $r(R)$.
(e) Find the matrix that represents $s(R)$.
(f) Find the matrix that represents $t(R)$.

(22) Suppose R is a relation on A. Using the property that composition is associative and mathematical induction, prove that $R^n \circ R = R \circ R^n$ for any positive integer n.

(23) Prove that a relation R on a set A is transitive if and only if $R^n \subseteq R$ for all positive integers n.

(24) Let A be a set and let R and S be relations on A. If R and S satisfy the property given, does the relation given have to satisfy the same property? Prove or disprove each answer.
(a) Reflexive, $R \cup S$
(b) Reflexive, R^{-1}
(c) Symmetric, $t(R)$
(d) Symmetric, $R \circ S$
(e) Antisymmetric, $R \oplus S$
(f) Antisymmetric, R^n for any positive integer n
(g) Transitive, $r(R)$
(h) Transitive, R^{-1}
(i) Equivalence Relation, $R - S$
(j) Equivalence Relation, R^n for any positive integer n
(k) Partial Order, $R \circ S$
(l) Partial Order, $R \oplus S$

(25) Let R_1 be a relation from S to T and let R_2 be a relation from T to U. Prove $(R_2 \circ R_1)^{-1} = R_1^{-1} \circ R_2^{-1}$.

(26) True or false. If R and S are relations on A, then
(a) $r(R \cap S) = r(R) \cap r(S)$.
(b) $r(R - S) = r(R) - r(S)$.
(c) $s(R \cup S) = s(R) \cup s(S)$.
(d) $s(R^n) = s(R)^n$ for any positive integer n.
(e) $t(R^{-1}) = (t(R))^{-1}$.
(f) $t(R \circ S) = t(R) \circ t(S)$.

(27) Which of the following relations are equivalence relations, which are partial orders, and which are neither? For the relations that are equivalence relations

find the equivalence classes. For the ones that are neither equivalence relations nor partial orders name the property(ies) that fails.

(a) The relation R on the set of Computer Science majors at FSU where aRb iff a and b are currently enrolled in the same course.
(b) The relation R on the set of integers where $(m, n) \in R$ if and only if $mn \equiv 2 (\text{mod } 2)$.
(c) The relation R on the set of ordered pairs of integers where $(a, b)R(c, d)$ iff $a = c$ or $b = d$
(d) Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by $f(x) = \lceil x \rceil$. Define the relation R on $\mathbb{R}$ by $(x, y) \in R$ if and only if $f(x) = f(y)$.
(e) The relation R on the set of all subsets of $\{1, 2, 3, 4\}$ where SRT means $S \subseteq T$
(f) The relation R on the set of positive integers where $(m, n) \in R$ if and only if $gcd(m, n) = max\{m, n\}$.
(g) Let $G = (V, E)$ be a simple graph. Let R be the relation on V consisting of pairs of vertices (u, v) such that there is a path from u to v or such that $u = v$.
(h) The relation R on the set of ordered pairs $\mathbb{Z}^+ \times \mathbb{Z}^+$ of positive integers defined by

$$(a, b)R(c, d) \Leftrightarrow a + d = b + c.$$

(28) Let R be the relation on the set of ordered pairs of positive integers such that $(a, b)R(c, d)$ if and only if $ad = bc$.
(a) Prove R is an equivalence relation.
(b) Find the equivalence class of (a, b) where $(a, b) \in \mathbb{Z}^+ \times \mathbb{Z}^+$:

(29) Let R be the relation on the set $\mathbb{R}$ real numbers defined by xRy iff $x - y$ is an integer. Prove that R is an equivalence relation on $\mathbb{R}$.

(30) Suppose $A = \{2, 4, 5, 6, 7, 10, 18, 20, 24, 25\}$ and R is the partial order relation $(x, y) \in R$ iff $x|y$.
(a) Draw the Hasse diagram for the relation.
(b) Find all minimal elements.
(c) Find all maximal elements.
(d) Find all upper bounds for $\{6\}$.
(e) Find all lower bounds for $\{6\}$.
(f) Find the least upper bound for $\{6\}$.
(g) Find the greatest lower bound for $\{6\}$.
(h) Find the least element.
(i) Find the greatest element.
(j) Is this a lattice?

(31) Suppose $A = \{2, 3, 4, 5\}$ has the usual "less than or equal" order on integers. Find each of the following for the case where R is the lexicographic order relation on $A \times A$ and where R is the product order relation on $A \times A$.
(a) Draw the Hasse diagram for the relation.
(b) Find all minimal elements.

(c) Find all maximal elements.
(d) Find all upper bounds for $\{(2,3),(3,2)\}$.
(e) Find all lower bounds for $\{(2,3),(3,2)\}$.
(f) Find the least upper bound for $\{(2,3),(3,2)\}$.
(g) Find the greatest lower bound for $\{(2,3),(3,2)\}$.
(h) Find the least element.
(i) Find the greatest element.
(j) Is this a lattice?

(32) Carefully prove the following relations are partial orders.
(a) Recall the product order: Let $(A_1, \preceq_1)$ and $(A_2, \preceq_2)$ be posets. Define the relation $\preceq$ on $A_1 \times A_2$ by $(a_1, a_2) \preceq (b_1, b_2)$ if and only if $a_1 \preceq_1 b_1$ and $a_2 \preceq_2 b_2$. Prove the product order is a partial order.
(b) Let $(B, \preceq_B)$ be a poset and let A be a set. Define the set $FUN(A,B)$ to be the set of all functions with domain A and codomain B. Prove the relation $\preceq_F$ on $FUN(A,B)$ defined by $f \preceq_F g$ iff $f(t) \preceq_B g(t)\ \forall t \in A$ is a partial order.
(c) Let $(A, \preceq_A)$ and $(B, \preceq_B)$ be posets and recall the lexicographic order, $\preceq_L$ on $A \times B$ is defined by

$$(a,b) \preceq_L (c,d) \Leftrightarrow [(a \preceq_A c) \wedge ((a = c) \rightarrow (b \preceq_B d))].$$

Prove the lexicographic order on $A \times B$ is a partial order.

(33) Suppose $(A, \preceq)$ is a poset such that every nonempty subset of A has a least element. Prove that $\preceq$ is a total ordering on A.

(34) Prove: Suppose $(A, \preceq)$ is a finite nonempty poset. Then A has a minimal element.

3. Boolean Algebra

(35) How many Boolean functions of degree n are there?

(36) Define the Boolean function, F, in the three variables, x, y, and z, by $F(1,1,0) = F(1,0,1) = F(0,0,0) = 1$ and $F(x,y,z) = 0$ for all other (x,y,z) in $\{0,1\}^3$.
(a) Find the sum-of-products form for F.
(b) Find the product-of-sums form for F.
(c) Find the dual of the expression in 36a.
(d) Sketch the logical network that has the same output as F, and uses the order of operations given in the expression in part (a).
(e) Sketch the logical network that has the same output as F, and uses the order of operations given in the expression in part (b).

(37) Determine if each set is functionally complete.
(a) $\{+\}$
(b) $\{\cdot\}$
(c) {-}
(d) $\{+, \cdot\}$
(e) {+, -}
(f) $\{\cdot$, -}

   (g) $\{\downarrow\}$
   (h) $\{|\}$

(38) True or false.
   (a) $\overline{x \downarrow y} = \overline{x}|\overline{y}$.
   (b) $(xx+1) = (x+1)(x+1)$
   (c) $\overline{yz+\overline{x}} = \overline{yz}x$

(39) Prove the law holds for an abstract Boolean algebra. Use only the properties of a Boolean algebra and the previously stated laws.
   (a) Idempotent Laws: $x + x = x$ and $x \cdot x = x$
   (b) Dominance Laws: $x + 1 = 1$ and $x \cdot 0 = 0$
   (c) Absorption Laws: $(x \cdot y) + x = x$ and $(x + y) \cdot x = x$
   (d) $w + z = 1$ and $w \cdot z = 0$ if and only if $z = \overline{w}$
   (e) Double Compliments Law: $\overline{\overline{x}} = x$
   (f) DeMorgan's Laws: $\overline{x \cdot y} = \overline{x} + \overline{y}$ and $\overline{x+y} = \overline{x} \cdot \overline{y}$

(40) Let B be a Boolean algebra and let $\preceq$ be the relation on B given by $x \preceq y$ if and only if $x + y = y$. Prove $\preceq$ is a partial order. You may use the properties of the Boolean operations complementation, $+$, and $\cdot$, but only the definition of the relation $\preceq$.

(41) Let B_1 and B_2 be Boolean algebras and let $\phi : B_1 \to B_2$ be a Boolean algebra isomorphism. Prove $a \preceq b$ in B_1 if and only if $\phi(a) \preceq \phi(b)$ in B_2 where $\preceq$ is the partial order on a Boolean algebra defined in 40.

(42) Provide the rest of the information in the table.

Boolean Algebra	0 element	1 element	an element that is neither 0 nor 1 nor an atom
$B = \{0, 1\}$			
B^5			
$BOOL(2)$			
$P(\{a, b, c, d\})$			
$FUN(\{a, b, c\}, B)$			
D_6			

(43) List the atoms of each of the Boolean algebras in the table in exercise 42.

(44) Express each element given in the 4th column in table 42 as a sum of atoms. Note that your answer depends on your answer to 42.

(45) How many elements does a finite Boolean algebra have if the number of atoms is 5?

(46) Define an isomorphism between $FUN(\{a, b, c\}, B)$ and B^3.

(47) Draw a Hasse diagram for each of the Boolean algebras in the table in exercise 42 and the partial order $\preceq$ defined in exercise 40.

(48) Use (a) Karnaugh maps and (b) the Quine McClusky method to find a minimal expansion for $wxyz + wxy\overline{z} + wx\overline{y}z + w\overline{x}\,\overline{y}z + w\overline{x}\,\overline{y}\,\overline{z} + \overline{w}x\overline{y}z + \overline{w}xyz + \overline{w}\,\overline{x}\,\overline{y}z$