Formulas

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2} \tag{1}$$

Proof. Add the sum to itself (with the terms in reverse order): $2\sum_{k=1}^{n} k$
$= [1 + 2 + \ldots + (n-1) + n] + [n + (n-1) + \ldots + 2 + 1]$
$= [1 + n] + [2 + (n-1)] + \ldots + [(n-1) + 2] + [n + 1] = n(n+1)$.

$$\sum_{k=0}^{n} b^k = \frac{1 - b^{n+1}}{1 - b}, b \neq 1 \tag{2}$$

Proof. Expand the product $(1-b)\sum_{k=0}^{n} b^k = \sum_{k=0}^{n} b^k - \sum_{k=1}^{n+1} b^k = b^0 - b^{n+1}$.

$$\sum_{k=0}^{\infty} b^k = \frac{1}{1 - b}, |b| < 1 \tag{3}$$

Proof. Take the limit of $\sum_{k=0}^{n} b^k = \frac{1-b^{n+1}}{1-b}$ as $n \to \infty$.

$$\sum_{k=0}^{n} \ln k = \Theta(n \ln n) \tag{4}$$

Proof. First note that the sum is an approximation sum for an integral: a lower sum for $\int_1^n \ln x dx$ and an upper sum for $\int_2^n \ln x dx$. It follows that

$$\int_2^n \ln x dx < \sum_1^n \ln k < \int_1^n \ln x dx$$

Thus $\sum_1^n \ln k = \Theta(\int_1^n \ln x dx)$. Evaluation of the integral by parts yields $\int_1^n \ln x dx = n \ln n - (n-1) = \Theta(n \ln n)$.

$$\sum_{k=1}^{n} \frac{1}{k} = \Theta(\ln n) \tag{5}$$

Proof. As in the argument above, the sum is a lower approximation sum for $\int_1^n (1/x)dx$ and an upper approximation for $\int_2^n (1/x)dx$, hence the sum is $\Theta(\int_1^n dx/x)$. But this integral is the definition of natural logarithm: $\int_1^n dx/x = \ln n$

$$\ln n! = \Theta(n \ln n) \tag{6}$$

Proof. Applying Sterling's approximation, we have $\ln n! = \log(\sqrt{2\pi n}) + n \ln(n/e) + \ln(1 + \Theta(1/n)) = \Theta(n \ln n)$.

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \left(1 + \Theta\left(\frac{1}{n}\right)\right) \tag{7}$$

Proof. This is Sterling's approximation - proof beyond the scope of this page.