Homework 1
50 points

Context

Define a function f to be *admissable* iff it is defined on almost all of the non-negative integers and has positive real values on almost all of the domain. That is, there is an integer n_0 such that $f(n)$ is defined and $f(n) > 0$ for all integers $n > n_0$. Note that for polynomials, this condition requires that the leading coefficient is a positive number.

Instructions: Prove propositions 1-4 and work the exercise.

Proposition 1. If f and g are admissable and $f \leq \mathcal{O}(g)$ then $\Theta(f+g) = \Theta(g)$.

Hint. Apply the definitions directly.

Exercise. Prove or disprove: $\Theta(1+g) = \Theta(g)$ for any admissable g.

Proposition 2. Suppose that f and g are admissable and

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = C$$

where C is a non-negative constant. Then $f = \mathcal{O}(g)$ and $g = \Omega(f)$. Moreover, if $C > 0$ then $f = \Theta(g)$.

Hint. Start by showing that $f(n)/g(n) \leq C+1$ for almost all n.

Proposition 3. If a and b are positive numbers (not necessarily integers) and $a \leq b$ then $n^a \leq \mathcal{O}(n^b)$ and $n^b \geq \Omega(n^a)$.

Hint. Consider the quotient $n^a/n^b = n^{a-b}$ and apply Proposition 2 above.

Proposition 4. Suppose f is a polynomial in n of degree d, with the form

$$f(n) = \sum_{i=0}^{d} a_i n^i$$

with leading coefficient $a_d > 0$. Then $f(n) = \Theta(n^d)$.

Hint. Write $p(n)$ as the sum of $a_d n^d$ plus terms of lower degree, and apply various results above along with mathematical induction.