Solutions for Homework 6 Foundations of Computational Science
1 Fall 2001

Problem 1

In this problem we derive an elementary orthogonal matrix and a transformation method used to solve
systems including least squares problems. (Stewart 1973)

(a) An elementary reflector or Householder transformation is given by

\[ H = I - 2uu^T \]

where \( u \in \mathbb{R}^n \) and \( u^Tu = 1 \). Show that \( H \) is a symmetric orthogonal matrix and that \( ||Hx||_2 = ||x||_2 \) for any \( x \in \mathbb{R}^n \).

**Solution:** The fact that \( H \) is symmetric is obvious from the form. For orthogonality we have

\[
H^TH = (I - 2uu^T)(I - 2uu^T) \\
= I - 4uu^T + 4(uu^T)(uu^T) \\
= I - 4uu^T + 4u(u^Tu)u^T \\
= I - 4uu^T + 4uu^T \\
= I
\]

To see that \( ||Hx||_2 = ||x||_2 \) for any \( x \in \mathbb{R}^n \) note

\[
Hx = (I - 2uu^T)x \\
= x - 2u(u^Tx) \\
= x - 2\mu u
\]

\[
||Hx||_2^2 = (x - 2\mu u)^T(x - 2\mu u) \\
= x^Tx - 4\mu u^Tx + 4\mu^2 u^Tu \\
= x^Tx - 4\mu(u^Tx) + 4\mu^2(u^Tu) \\
= x^Tx - 4\mu^2 + 4\mu^2 \\
= x^Tx
\]

(b) Let \( x \in \mathbb{R}^n \) and \( y \in \mathbb{R}^n \) with \( ||x||_2 = ||y||_2 \) where \( n > 1 \). Show that there exists an elementary reflector \( H \) such that \( Hx = y \). (Hint: Assume \( u \) is known and compare \( Hx \) to \( y \) to get an idea of the form of \( u \) – then fine tune the choice.)

**Solution:** All norms are taken to be 2-norms in this solution. Following the hint we see that \( u = \phi(x-y) \) is a good form to try. Since \( u^Tu = 1 \) we can take \( \phi = ||x-y||_2^{-1} \).

So we assume that \( x^Tx = y^Ty \) and take \( u = \phi(x - y) \) where \( \phi = ||x - y||_2^{-1} \).

We have a space, \( \mathcal{R}(u) = \mathcal{S} \), with dimension 1, that we can use to analyze the action of the reflector. We know we can decompose \( x \) and \( y \) based on \( \mathcal{S} \) as

\[
x = v_x + w_x \\
v_x \in \mathcal{S}^⊥ \\
w_x \in \mathcal{S} \\
y = v_y + w_y \\
v_y \in \mathcal{S}^⊥ \\
w_y \in \mathcal{S}
\]

We must deduce relationships between these vector using the reflector \( H \) and our assumptions. Note, since \( x - y \) is in \( \mathcal{S} \) and \( I - uu^T \) is a projector onto \( \mathcal{S}^⊥ \) we have
\[
0 = (I - uu^T)(x - y) \\
= (I - uu^T)x - (I - uu^T)y \\
\]
\[
(I - uu^T)x = (I - uu^T)y \\
v_x = v_y \\
= v \\
\]

So the component in \( S^\perp \) is the same in both vectors. That is,
\[
x = v + w_x \\
y = v + w_y \\
\]

Now consider \( w_x \) and \( w_y \).
\[
w_y = (uu^T)y \\
= (u^Ty)u \\
= \frac{(x^Ty - y^Ty)u}{\|x - y\|} \\
w_x = (uu^T)x \\
= (u^Tx)u \\
= \frac{(x^Tx - y^Tx)u}{\|x - y\|} \\
= \frac{(y^Ty - x^Ty)u}{\|x - y\|} \\
= -\frac{(x^Ty - y^Ty)u}{\|x - y\|} \\
= -w_y \\
\]

So the components in \( S \) differ only in sign.

We can know show the assertion easily,
\[
Hx = (I - 2uu^T)x \\
= (I - uu^T - uu^T)x \\
= (I - uu^T)x - (uu^T)x \\
= v - w_x \\
= v + w_y \\
= y \\
\]

To see intuitively what is happening note in the derivation how we write \( Hx = (I - uu^T)x - (uu^T)x \). The first matrix is a projector that removes the components in \( S \) from \( x \). The second term is a projector that determines the components in \( S \) that were removed from \( x \) by the first term and puts them back into the vector with the sign flipped. So essentially, the reflector \( H \) is built from two projectors relative to \( S \) such that the components in the vector to which \( H \) is applied that are in \( S \) a reflected back along \( x - y \). This generalizes a situation that is easy to see from a picture in \( \mathbb{R}^2 \).

(c) Suppose \( x \in \mathbb{R}^n \) and \( x \neq \sigma e_1 \). Use the results from (a) and (b) to derive the formulas in the notes for an elementary reflector \( H \) such that \( Hx = \gamma e_1 \) where \( \gamma \) is a real scalar.

Solution: The solution to (c) is contained in the notes.