\[ x \mathcal{H}_{1-V} - = (x - x) \]
\[ 0 = x \mathcal{H} + (x - x) \mathcal{V} \]
\[ q = x \mathcal{V} - \]
\[ q = x (\mathcal{H} + \mathcal{V}) \]

A norm-based bound is easily derived:

\[ q = x \mathcal{V} \]

Suppose we want to bound the effect of a perturbation to all or Perturbation Theorems for Linear Systems (Stewart 1998)
The condition number with respect to inversion is \( (\mathcal{A})^y \)

\[
\begin{align*}
\| I - \mathcal{V} \| \| \mathcal{V} \| &= (\mathcal{V})^y \\
\frac{\| \mathcal{V} \|}{\| \mathcal{A} \|} (\mathcal{V})^y &= \\
\| \mathcal{A} \| \| I - \mathcal{V} \| &> \\
\| \mathcal{A}_{I - \mathcal{V}} \| &> \frac{\| x \|}{\| x - \bar{x} \|}
\end{align*}
\]

So we have...
\[
\frac{\|V\|(V)y - I}{\|\mathcal{E}\| (V)y} \geq \frac{\|x\|}{\|x - \hat{x}\|}
\]

and if I then \( \geq \|\mathcal{E}\|_{\mathcal{I}-V} \)

\[
\frac{\|\mathcal{E}_{\mathcal{I}-V}\| - I}{\|\mathcal{E}_{\mathcal{I}-V}\|} \geq \frac{\|x\|}{\|x - \hat{x}\|}
\]

and exists and \( \mathcal{E}_{\mathcal{I}-V} \) then \( \geq \|\mathcal{E}_{\mathcal{I}-V}\| \) if

additional assumptions

We can find the error relative to the true solution given some
components may have larger relative errors. Only reasonable or predictive for the larger components and smaller elements of $\varepsilon$ are roughly the same magnitude. Otherwise they are all

These norm-based bounds are reasonable at a component level if all
It concerns only the accuracy in the data.

Note that this has nothing to do with the computation of the solution.

You lose $k$ digits of accuracy in the solution relative to the accuracy in the data $(A)$. If

$$\gamma + 10^{-\gamma} \approx \frac{\|x\|}{\|x - x\|}$$

then the perturbed solution satisfies

$$\gamma = \frac{\|A\|}{\|A\|}$$

For $\gamma = 10^{\gamma}$ and $A$ is known to $\gamma$ digits, i.e.

**Rule of Thumb**
\[
\begin{pmatrix}
1.96170e-05 \\
-5.0895e-03 \\
-3.7159e-01
\end{pmatrix}
= q
\]

\[
\begin{pmatrix}
1 \\
1 \\
1
\end{pmatrix}
= x
\]

\[
\begin{pmatrix}
1.15870e-04 & -7.8646e-05 \\
1.15870e-04 & -9.8288e-04 \\
8.2041e-03 & 1.1535e-02 \\
-2.4828e-02 & -1.0263e-02 \\
3.9856e-02 & -1.0263e-02
\end{pmatrix}
= A
\]

Since the bound admits perturbation matrix it is permissible. Consider the example from Stewart 1998.
The bound is incredibly pessimistic.

\[
3 \cdot 10^{-3} = \frac{\|V\|}{\|\hat{A}\|}(V)^{y}
\]

\[
3 \cdot 10^{-3} = \frac{\|x\|}{\|x - \hat{x}\|}
\]

Solving yields

\[
q = x(\hat{A} + A)\]

\[
\begin{pmatrix}
7.7933e-11 & 1.1048e-11 & 6.4209e-11 \\
6.9456e-09 & 6.8435e-09 & -3.1642e-09 \\
3.9372e-07 & 1.262e-08 & 2.434e-08
\end{pmatrix} = \hat{A}
\]
The bound is accurate.

\[
0.5 = \frac{\| \mathcal{V} \|}{\| \mathcal{H} \|} (\mathcal{V})_y
\]

\[
0.2 = \frac{\| x \|}{\| x - \bar{x} \|}
\]

Solving \( q = zx(\mathcal{H} + \mathcal{V}) \) yields

\[
\begin{pmatrix}
6.4209e-07 & 1.1048e-06 & 2.7933e-07 \\
1.9456e-07 & 8.4355e-07 & 1.1642e-06 \\
3.9372e-07 & 1.1262e-06 & 2.4346e-08
\end{pmatrix}
\]
is available.

assuming that a factorization of \( A \) that have \( \mathcal{O}(m^2) \) complexity can be estimated using condition estimation techniques.

From norm-based worst case bound predicts.

We may have solved a system much more accurately than the perturbation matrix.

So the actual error depends on the structure/values of the
\[ |x| \| \mathcal{H} \|_{\text{I-V}} \geq |x - x| \]

and if then

\[ |x| \| \mathcal{H} \|_{\text{I-V}} \geq |x - x| \]

\textbf{THEOREM (Bauer-Skeel)}

i.e. replace each element by its absolute value.

\[ | \mathcal{A} \mathcal{V} | \leq | \mathcal{A} | \mathcal{V} | \leq | \mathcal{A} | \mathcal{V} | \]

\textbf{DEFINITION:} The absolute value of a matrix \( \mathcal{A} \) is denoted Component-wise Bounds
somewhat unattractive.

estimate reasonably well the elements of $A^{-1}$ makes this bound

In practice, the requirement that we must compute (or at least

which is quite good.

$$
\begin{pmatrix}
9.7e-05 \\
6.4e-05 \\
1.0e-06 \\
6.0e-05
\end{pmatrix}
= ||A||_{1}^{-1} ||A||_{1} = 1.0e-06 \Rightarrow \text{we have a Bauer-Skek bound of}

Estimating $||A||_{1} \approx 1.0e-06 \Rightarrow A \approx I$ we have a Bauer-Skek bound of

$$
\begin{pmatrix}
-2.0e-05 \\
-2.0e-05 \\
-2.0e-05
\end{pmatrix}
= \epsilon I
$$

Using $x^1$ from the example above where we had a bad norm-based
the value of $\|A^{-1}E\|\|\hat{x}\|$ can be estimated via condition estimation techniques. A mixed bound is possible

\[
\frac{\|\hat{x} - x\|}{\|\hat{x}\|} \leq \frac{\|A^{-1}E\|\|\hat{x}\|}{\|\hat{x}\|}
\]
Clearly, given \( A \) is nonsingular \( \iff 0 = \|r\| \) and only if \( \|r\| = 0 \).

\[
x = \frac{x - q}{A}
\]

The residual vector given a guess at the solution of a system is

**Definition:** The residual quality

Given an approximate solution \( x \) to a system \( Ax = q \), can we assess its

A posteriori bounds
\[ \frac{\|V\|}{\|\mathcal{H}\|} \geq \frac{\|x\|^2 \|V\|}{\|t\|} \]

then \( q = x(\mathcal{H} + V) \) \( \forall \) \( \mathcal{I} \) (2)

\[ \frac{\|x\|^2 \|V\|}{\|t\|} = \frac{\|V\|}{\|\mathcal{H}\|} \]

\[ q = x(\mathcal{H} + V) \]

\[ \frac{\|x\|}{\|x\|} = \mathcal{H} \]

Theorem (Wilkinson): If there exists a matrix \( \mathcal{H} \) such that the system \( \mathcal{A}x = q \) is solvable, then the system \( \mathcal{A}x = q \) can be solved as follows:

solve the system \( \mathcal{A}x = q \) with a guess \( x \), then the system attempts to solve.
system is solved well relatively. is small, i.e., the residual is small relative to the size of $a$ and so the

$$\frac{\|x\|}{\|A\|}$$

Suppose we have (3). Then by (4), $\|A\|/\|A\|$ is small. We say that $x$ was computed stably if $x$ is small.

Consider a well-conditioned system. What does this say?
and $q = x(\mathcal{H} + V)$ with $\mathcal{H}$ exists with $\mathcal{H}$ is small.

Then by (1) we have that exists with $\mathcal{H}$ is small.

Now suppose

\[
\frac{\|x\| \|V\|}{\|\mathcal{I}\|}.
\]
relatively small if and only if $x$ was stably produced.

Assume the system is well-conditioned.

Putting the two together yields:
Stewart 1973 gives the following example hold.

III-conditioning changes the conclusion (although the bounds still
\[
a \bar{x}_2 = \bar{x}_2 = b_2 = 0 \quad \begin{pmatrix} 1 \\ -1 \end{pmatrix}
\]
i.e., a small residual, even when the solution has no digits accuracy.

\[
\begin{pmatrix}
0 \\
10^{-3}
\end{pmatrix} = A x_1 - q
\]

but we have

\( x_1 \) and \( x_1 \) are not close (no digits agree)

Consider \( A x_1 = q_1 \).
has been approximated well.

i.e., the residual is of the same size as $q$, even when the solution

$$
\begin{pmatrix}
1 & \\
0 & \\
\end{pmatrix}
= \alpha x^2 - \tau q
$$

but we have

$x^2$ and $\tau q$ are close (3 digits agree)

Consider $\alpha x^2 = \tau q$.  

The theorem still holds but when \( r \) is not relatively small we must have \( \| \tilde{r} \| \) large, i.e., \( \kappa(A) \| b \| \) is large.
\[ h = x \cap T \]

3. Solve \( h = x \cap T \)

\[ q = h \cap T \]

2. Solve \( q = h \cap T \)

\[ \cap T = V \]

1. Factor \( V \)

We have a three-stage process:

Recall is the unit roundoff and due to technical reasons in solving systems (Steewart 1998).
elements in the lower right of the matrix. Where the $u$ is pessimistic and tends not to show up except in the

$$|\Omega| |\mathbf{T}|_{\text{wff}} \geq |\mathcal{F}|$$

The factorization yields where componentwise $\mathcal{F} + \mathcal{V}$ where conditioning solves (3) does not magnify error. III-conditioning can show up in but it tends to be artificial so accurately.

Elements are bounded by $I$ in magnitude so the (2) is solved. With partial or complete pivoting $\mathcal{T}$ is well-conditioned and
\[ \frac{\|V\|}{\|\Omega\|\|T\|} = (V)_{\cdot} \]

\[ \mathcal{W}(V)_{\cdot}u(\mathcal{W}u + \varepsilon) \geq \frac{\|V\|}{\|H\|} \]

\[ |\Omega| |T| \mathcal{W}u(\mathcal{W}u + \varepsilon) \geq |H| \]

\[ q = x(H + V) \]

The computed solution \( x \) using the three stage algorithm satisfies
factorization relative to the maximum element in $A$. i.e., the maximum element in the active parts encountered in the

$$\max_{j} \frac{\left| \sum_{i} \alpha_i \right|}{\max_{j} \left| \sum_{i} \alpha_i \right|} = u_{j}$$

We estimate $\left\{ \frac{\alpha}{\left\| \sum_{i} \alpha_i \right\|} \right\}_j$ (via the growth factor)
growth factor is less than $2^n$. For $n = 1000$ we have $\frac{1}{1000} \approx 0.001$. But typically the

$$\frac{1}{10} \frac{u}{u} \times \cdots \times \frac{1}{2} \frac{2}{2} \frac{u}{u} \geq u$$

- Complete pivoting
- Partial pivoting

achievable but not common in practice.

$$\frac{1}{u} \geq u$$
We have $\gamma_n = 1$ for

- A when totally positive - every square submatrix has positive determinant
- A when diagonally dominant - $|a_{ii}| \geq \sum_{j=1}^{n} |a_{ij}|$ for $i = 1, \ldots, n.$
- Symmetric $A$ when positive definite - $x^TAx > 0$ for $x \neq 0.$
reduce arithmetic ill-conditioning

• give a good pivot sequence

This helps

\[ \mathcal{D} \big| \mathcal{W} \big| \mathcal{T}^{\frac{n}{2}} \mathcal{W}^{hn} \geq |\mathcal{E}| \]

the factorization backward error inherits the scaling, i.e.,

\[ \mathcal{D}^{\frac{n}{2}} \mathcal{A} \mathcal{D}^{\frac{n}{2}} \]

diagonal matrices

scale the rows and columns of \( \mathcal{A} \) by multiplying by positive

Scaling
equilibrium.

The pivot sequence so not needed. This is called row $D_A$ does not have some norm (typically 1-norm). A known to unit roundoff and partial pivoting used, scale so has elements almost equal in magnitude.

$A^\mathsf{r}D$

That suppose you know A contains observation errors, scale A so