Homework 1 Foundations of Computational Science 1 Spring 2001

Due date: 5pm Wednesday, 1/31/01

Problem 1

Suppose you have two vectors $x, y \in \mathbb{R}^n$ that are linearly independent but not orthogonal. Is it possible to create two vectors $u$ and $v$, via linear combinations of $x$ and $y$, so that the $\text{span}[u, v] = \text{span}[x, y]$, $u$ and $v$ are orthogonal, and $\|u\|_2 = \|v\|_2 = 1$?

Problem 2

Show that the following hold for all vectors $x \in \mathbb{C}^n$ or give a counterexample.

\[ \|x\|_\infty \leq \|x\|_1 \leq n \|x\|_\infty \]
\[ \|x\|_2 \leq \|x\|_1 \leq \sqrt{n} \|x\|_2 \]

Problem 3

Show that the definition of a matrix norm is not enough to guarantee that the matrix norm is consistent.

Problem 4

Consider a matrix $A \in \mathbb{R}^{m \times n}$ where $m < n$ and the underdetermined homogeneous equation

\[ Ax = 0 \]

Show that there must exist a vector $x \in \mathbb{R}^n$ such that $Ax = 0$ and $x \neq 0$. Show how to construct such a vector using the properties of $A$ and its rank relative to $m$.

Problem 5

Recall the definition of a nonsingular lower triangular matrix from set 3 of the notes:

$L = [\lambda_{ij}] \in \mathbb{R}^{m \times n}, \lambda_{ij} = 0$ if $i < j, \lambda_{ii} \neq 0$ if $i = 1, \ldots, n$.

Now consider a lower trapezoidal matrix where $L$ is $m \times n$ with $m > n$ and the same 0 and nonzero constraints as the triangular matrix.

Are the nonzero conditions enough to guarantee that $Lx = f$, has a unique solution? If not give examples of $L$ and $f$ that have a unique solution and that do not.

Problem 6

(a)

A first order linear recurrence is defined as follows:

\[ \alpha_0 = \gamma_0 \]
\[ \alpha_i = \beta_i \alpha_{i-1} + \gamma_i \]
\[ i = 1, \ldots, n \]

where $\alpha_i, \gamma_i, \beta_i$ are all scalars.

Show how this can be written as a system of equations. Comment on any structural properties of the matrix and how they might be exploited to solve the recurrence. How many operations are required?
(b)
Consider a polynomial of degree $n$,

$$P(\epsilon) = \prod_{i=0}^{n} \delta_i \epsilon^i$$

Show how the idea of a linear recurrence can be used to evaluate $P(\epsilon)$ while avoiding explicitly evaluating the powers $\epsilon^i$. Specifically show the system of linear equations that result from the recurrence and comment on an structure that exists in addition to that found in (a).