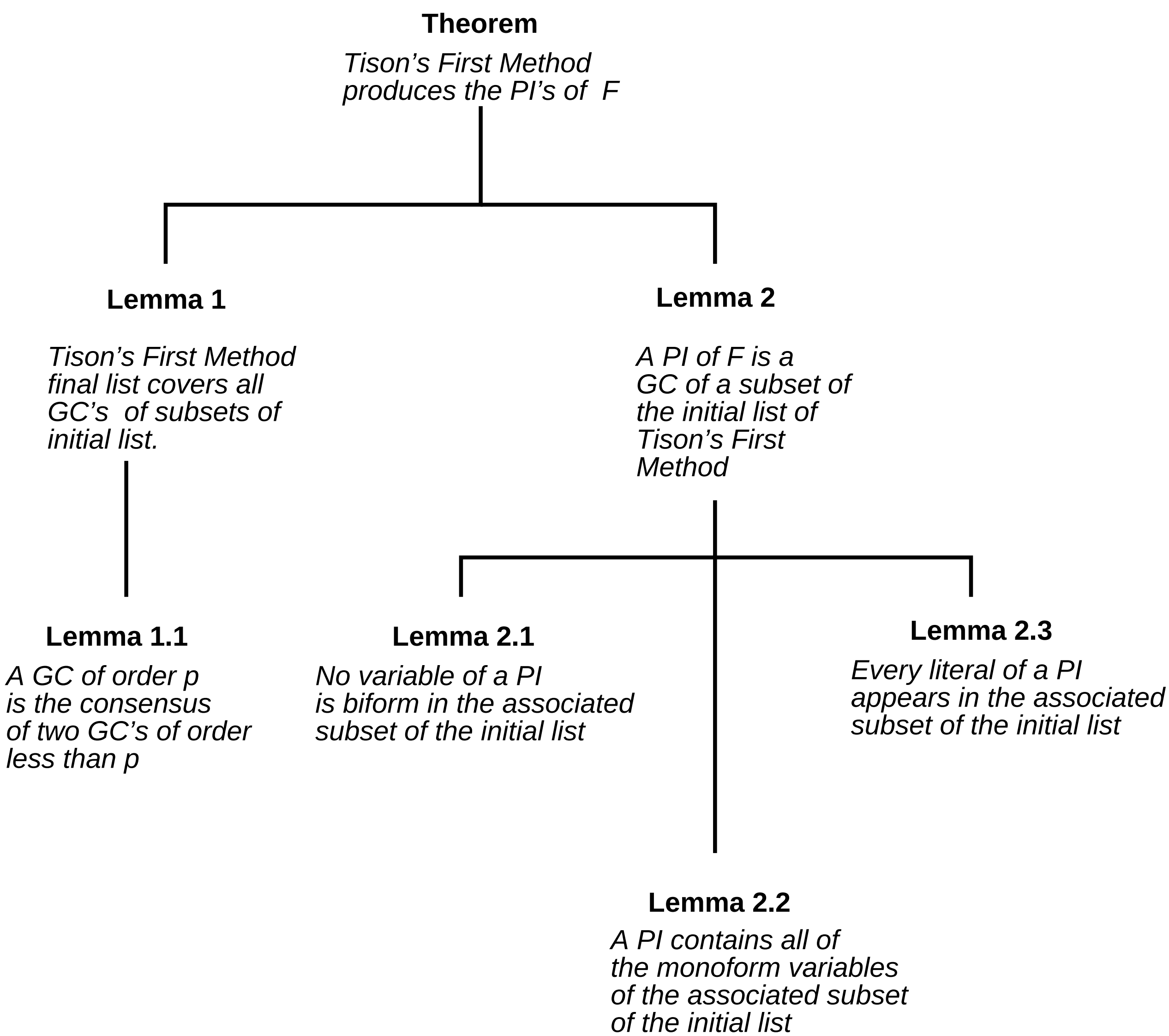
The next task is to prove the following theorem:

Theorem:(Correctness of Tison's First Method)

> Let $\mathcal{F} = T_1 + T_2 + \cdots + T_p$ be a sum-of-products expression for the switching function $\mathcal{F}$. If Tison's First Method is applied to the set of products $\mathcal{T} = (T_1, \ldots, T_p)$ it produces a final set of products $L = (L_1, \ldots, L_k)$ such that each L_i is a prime implicant of $\mathcal{F}$ and any prime implicant of $\mathcal{F}$ is equal to some L_j.

In order to do this we need a progression of 6 related lemmas. The relationships are shown in the following diagram.


Theorem
Tison’s First Method produces the PI’s of F
Lemma 1
Tison’s First Method final list covers all GC’s of subsets of initial list.
Lemma 2
A PI of F is a GC of a subset of the initial list of Tison’s First Method
Lemma 1.1
A GC of order p is the consensus of two GC’s of order less than p
Lemma 2.1
No variable of a PI is biform in the associated subset of the initial list
Lemma 2.3
Every literal of a PI appears in the associated subset of the initial list
Lemma 2.2
A PI contains all of the monoform variables of the associated subset of the initial list

Lemma 1.1:

Let Q_i $i = 1, \ldots, p$ be products of literals and $X = GC(Q_1, \ldots, Q_p)$ be the generalized consensus of order p of those products. There exists a variable x and two products R and S such that

$$\begin{array}{rcl} R & = & GC(Q_{i_1}, \ldots, Q_{i_r}) \\ S & = & GC(Q_{j_1}, \ldots, Q_{j_s}) \\ r \leq p & and & s \leq p \\ X & = & cons(R, S, x) \end{array}$$

In fact, the proof of this lemma shows that such a consensus can be formed for each biform variable in $(Q_1, \ldots, Q_p)$. Note that this is one part of the basic step of both of Tison's methods.

Proof of Lemma 1.1:

See Handout on Tison's Method

Lemma 1.1 allows us to prove the following result that forms a large part of the proof of the correctness theorem.

Lemma 1:

> Let $\mathcal{T} = (T_1, \ldots, T_n)$ be a set of products and let $X = GC(T_{i_1}, \ldots, T_{i_k})$, $k \leq n$ be a generalized consensus of a subset of $\mathcal{T}$. If Tison's First method is applied to $\mathcal{T}$ then there exists a product L_j such that
>
> $$\begin{array}{ccc} L_j & \in & L = (L_1, \ldots, L_m) \\ X & \rightarrow & L_j \end{array}$$
>
> where L is the final list produced by Tison's First method.

In other words all generalized consensi of subsets of $\mathcal{T}$ are covered by elements of L.

Proof of Lemma 1:

Without loss of generality, assume that $T_{i_j} = T_j$,$j = 1, \ldots, k$, i.e., given X and $i_1, \ldots, i_k$, renumber the products so that those involved in the generalized consensus appear first. Let $x_1, x_2, \ldots, x_b$ be the biform variables in order of use in generating consensi and let Stage i refer to computing all consensi w.r.t. x_i. Let t be the smallest integer such that all biform variables of $(T_1, \ldots, T_k)$ are contained in $x_1, \ldots, x_t$. (Note that this implies x_t is biform in $(T_1, \ldots, T_k)$.) We will use induction on t to show that at the end of stage t L has a term which covers X.

Assume $t = 1$. Then there is a single biform literal, x, in $(T_1, \ldots, T_k)$. Therefore, B_i, the biform variable part of T_i must be either x or x'. (We will assume the literal x is also an uncomplemented switching variable. If it is not the following reasoning is easily modified substituting x' for x.) Since for $X = GC(T_1, \ldots, T_k)$ to exist we must have $B_1 + \cdots + B_k = 1$ irredundantly we must have $k = 2$. If $k = 1$ the sum is not always 1 and if $k \geq 3$ it cannont be irredundant since there are only two possible literals. So $k = 2$ and $X = RS$ where $T_1 = xR$ and $T_2 = x'S$.

On the first step of Tison's method, the consensus of $T_1 = xR$ and $T_2 = x'S$ w.r.t. x will be evaluated. Since it is known that R and S contain no other biform variables the definition of consensus is satisfied and RS is added to the list L. Since $RS = GC(T_1, \ldots, T_k)$ this completes the case for $t = 1$.

Now suppose $t > 1$. Since x_t is biform in $(T_1, \ldots, T_k)$ by Lemma 1.1, we know that $X = cons(Y, Z, x_t)$, where Y is a generalized consensus of some subset, $\mathcal{Y}$, of terms in $(T_1, \ldots, T_k)$ and Z is a generalized consensus of another subset $\mathcal{Z}$.

Note, by definition, Y and Z are products of variables that are monoform in $\mathcal{Y}$ and $\mathcal{Z}$ respectively. x_t must be monoform in $\mathcal{Y}$ and $\mathcal{Z}$ and must appear as x_t in one (assume it is $\mathcal{Y}$) and x'_t in the other since X is formed by consensus w.r.t. x_t. So $X = Y_0 Z_0$ where $Y = x_t Y_0$ and $Z = x'_t Z_0$.

Since by definition of t, x_t is the last biform variable in $(T_1, \ldots, T_k)$, all biform variables in $\mathcal{Y}$ and $\mathcal{Z}$ must be among $x_1, \ldots, x_{t-1}$. By the induction hypothesis that all GC of sets whose biform variables are included in $x_1, \ldots, x_{t-1}$ have terms in L that cover them, two terms Y^* and Z^* exist in L such that

$$\begin{aligned} Y &\leq Y^* \\ Z &\leq Z^* \end{aligned}$$

There are only 3 possible cases:

If x_t not in Y^*, then Y^* contains only literals from Y_0 and therefore X is covered:

$$X = Y_0 Z_0 \leq Y_0 \leq Y^*$$

If x'_t not in Z^*, then Z^* contains only literals from Z_0 and therefore X is covered:

$$X = Y_0 Z_0 \leq Z_0 \leq Z^*$$

If x'_t in Z^* and x_t in Y^* then we have $Y = x_t Y_0$ and $Y^* = x_t Y_0^*$. Since $Y \leq Y^*$ it follows that $Y_0 \leq Y_0^*$. Similarly, $Z^* = x'_t Z_0^*$ and $Z_0 \leq Z_0^*$. Therefore,

$$X = Y_0 Z_0 \leq Y_0^* Z_0^* = cons(Y^*, Z^*, x_t) = X^*$$

But, X^* is produced by Tison's method when taking consensi w.r.t. x_t.

Once any product that covers X is placed in L it will not be removed except by another product that covers X. □

We need three fairly simple lemmas to prove our second significant contribution to the proof of the correctness theorem. Note that $\leq$ is used to denote the implication coverage relationship, i.e., $x \to y$ or y covers x is denoted $x \leq y$.

Lemma 2.1:

> Suppose P is a prime implicant of a switching function $\mathcal{F}$ and $(Q_1, \ldots, Q_k)$ is a set of products that are implicants of $\mathcal{F}$ such that
>
> $$P \leq Q_1 + Q_2 + \ldots + Q_k \leq \mathcal{F}$$
>
> irredundantly (no Q_i can be removed without destroying the coverage).
>
> If u is a literal in P then u' is not in $(Q_1, \ldots, Q_k)$, i.e., no variable in P is biform in $(Q_1, \ldots, Q_k)$.

Proof of Lemma 2.1:

Suppose that $P = uR$ and, without loss of generality, $Q_1 = u'S$.Since $P \leq Q_1 + \cdots + Q_k$, for any assignment such that $P = 1$ we have $P(Q_1 + \cdots + Q_k)' = 0$. If $P = 0$ the product must also be 0 so we have for all assignments to the switching variables

$$\begin{aligned} 0 &= P(Q_1 + \cdots + Q_k)' \\ &= PQ'_1 \cdots Q'_k \\ &= uR(u + S')Q'_2 \cdots Q'_k \\ &= (uR + uRS')Q'_2 \cdots Q'_k \\ &= uRQ'_2 \cdots Q'_k \\ &= PQ'_2 \cdots Q'_k \end{aligned}$$

This says that $P \leq Q_2 + \cdots + Q_k$ which contradicts the assuption of irredundancy. $\square$

Lemma 2.2:

Suppose P is a prime implicant of a switching function $\mathcal{F}$ and $(Q_1, \ldots, Q_k)$ is a set of products that are implicants of $\mathcal{F}$ such that

$$P \leq Q_1 + Q_2 + \ldots + Q_k \leq \mathcal{F}$$

irredundantly (no Q_i can be removed without destroying the coverage).

If u is a literal that is monoform in $(Q_1, \ldots, Q_k)$ then u appears in P.

Proof of Lemma 2.2:

Suppose a monoform u appears in $Q_1, \ldots, Q_m$ but not in $P, Q_{m+1}, \ldots, Q_k$. As above we have

$$P(Q_1 + \cdots Q_m + Q_{m+1} + \cdots + Q_k)' = 0$$

for all assignments to the variables. Now consider all assignments such that $u = 0$. $Q_1 = \cdots = Q_m = 0$ and $P, Q_{m+1}, \ldots, Q_k$ are unaffected, i.e., they take on the same set of values that they achieve for ALL ASSIGNMENTS. We therefore have for ALL ASSIGNMENTS

$$P(Q_{m+1} + \cdots + Q_k)' = 0$$

which says that $P \leq Q_{m+1} + \cdots + Q_k$. This contradicts the irredundancy assumption. □

Lemma 2.3:

Suppose P is a prime implicant of a switching function $\mathcal{F}$ and $(Q_1, \ldots, Q_k)$ is a set of products that are implicants of $\mathcal{F}$ such that

$$P \leq Q_1 + Q_2 + \ldots + Q_k \leq \mathcal{F}$$

irredundantly (no Q_i can be removed without destroying the coverage).

If u is a literal in P then u occurs in $(Q_1, \ldots, Q_k)$.

Proof of Lemma 2.3: Suppose $P = uR$ and u is not in $(Q_1, \ldots, Q_k)$. As above we have

$$P(Q_1 + \cdots + Q_k)' = 0.$$

Set $u = 1$. $Q_1, \cdots, Q_k$ are unaffected so

$$R(Q_1 + \cdots + Q_k)' = 0$$

for all assignments to the variables. It follows that

$$R \leq Q_1 + \cdots + Q_k \leq \mathcal{F}$$

which contradicts the primeness of P. □

Corollary 2.4:

Suppose P is a prime implicant of a switching function $\mathcal{F}$ and $(Q_1, \ldots, Q_k)$ is a set of products that are implicants of $\mathcal{F}$ such that

$$P \leq Q_1 + Q_2 + \ldots + Q_k \leq \mathcal{F}$$

irredundantly (no Q_i can be removed without destroying the coverage).

Let S_Q,S_m, and S_b be the sets of all literals, all monoform literals, and all biform literals in $(Q_1, \ldots, Q_k)$, respectively. If S_P is the set of literals in the product P, then $S_m = S_P$.

Proof: By Lemma 2.3, $S_P \subseteq S_Q = S_b \cup S_m$. By Lemma 2.1, $S_P \cap S_b = \emptyset$. Therefore, $S_P \subseteq S_m$. By Lemma 2.2, $S_m \subseteq S_P$. Therefore, $S_P = S_m$. □

We can now state the following which associates a prime implicant with a generalized consensus.

Lemma 2:

Suppose P is a prime implicant of a switching function $\mathcal{F}$ and $\mathcal{Q} = (Q_1, \ldots, Q_n)$ is a set of products that are implicants of $\mathcal{F}$ such that

$$\mathcal{F} = Q_1 + Q_2 + \ldots + Q_n.$$

Then there exists a subset of $\mathcal{Q}$ with k implicants and $k \leq n$, such that

$$P = GC(Q_{i_1}, \ldots, Q_{i_k})$$

Proof of Lemma 2: By the assumptions of the lemma,

$$P \leq Q_1 + Q_2 + \ldots + Q_n = \mathcal{F}.$$

Since $\mathcal{F} = 1$ does not necessarily mean that $P = 1$ there may be redundant implicants Q_j on the right side of the inequality. These can be removed while preserving the coverage of P by the resulting sum. Since P is prime none of its literals can be removed without violating the implication of $\mathcal{F}$. So after removing the Q_j which are redundant with respect to coverage of P we have

$$P \leq Q_{i_1} + \ldots + Q_{i_k} \leq \mathcal{F}.$$

By the summary of Lemmas 2.1, 2.2, and 2.3 expressed in Corollary 2.4, the literals of P are the monoform literals of $(Q_{i_1}, \ldots, Q_{i_k})$. Write $Q_{i_j} = B_{i_j} X_{i_j}$ by splitting the literals into biform and monoform respectively. Therefore, $P = X_{i_1} \cdots X_{i_k}$ and we must consider $B_{i_1} + \cdots + B_{i_k}$.

Choose an assignment of the variables so that all monoform literals evaluate to 1. Since $P \leq Q_{i_1} + \cdots + Q_{i_k}$ it follows that FOR THIS ASSIGNMENT

$$\begin{aligned} 1 &= Q_{i_1} + \cdots + Q_{i_k} \\ &= B_{i_1} X_{i_1} + \cdots + B_{i_k} X_{i_k} \\ &= B_{i_1} + \cdots + B_{i_k} \end{aligned}$$

Recall, however, that by the definitions of B_j and X_j there are no literals in common between P and any B_j so setting the assignment above does not affect the values of literals in any B_j. Therefore,

$$B_{i_1} + \cdots + B_{i_k} = 1$$

FOR ALL ASSIGNMENTS of variables in $(Q_{i_1}, \ldots, Q_{i_k})$.

The irredundancy of the sum follows from the irredundancy of $P \leq Q_{i_1} + \cdots + Q_{i_k}$. If, say, B_{i_1} was a redundant term then when $P = 1$

$$\begin{aligned} 1 &= B_{i_2} + \cdots + B_{i_k} \\ &= B_{i_2} X_{i_2} + \cdots + B_{i_k} X_{i_k} \\ &= Q_{i_2} + \cdots + Q_{i_k} \end{aligned}$$

which is a contradiction that $P \leq Q_{i_1} + \cdots + Q_{i_k}$ irredundantly (Q_{i_1} could have been removed).

Therefore,

$$P = GC(Q_{i_1}, \ldots, Q_{i_k})$$

□

We now have all of the theoretical groundwork to state and prove the correctness theorem.

Theorem:(Correctness of Tison's First Method)

> Let $\mathcal{F} = T_1 + T_2 + \cdots + T_p$ be a sum-of-products expression for the switching function $\mathcal{F}$. If Tison's First Method is applied to the set of products $\mathcal{T} = (T_1, \ldots, T_p)$ it produces a final set of products $L = (L_1, \ldots, L_k)$ such that each L_i is a prime implicant of $\mathcal{F}$ and any prime implicant of $\mathcal{F}$ is equal to some L_j.

Proof of Correctness Theorem: The proof consists of two parts:

1. Tison's First Method generates all prime implicants.

2. All implicants in the final list L are prime implicants.

The first part results directly from Lemma 2. For any prime implicant P we have

$$P \leq T_1 + \cdots + T_p = \mathcal{F}.$$

By Lemma 2, there must be a subset of terms of this sum such that

$$P \leq T_{i_1} + \cdots + T_{i_k} \leq \mathcal{F}$$

with $P = GC(T_{i_1}, \ldots, T_{i_k})$.

By Lemma 1, there must exist an $L_j \in L$ such that and $P \leq L_j$. Since L_j is built by a series of consensus operations from an initial set of implicants of $\mathcal{F}$ it must also be an implicant. Since P is prime, it follows that $P = L_j$, i.e., P cannot be covered by another implicant that is not prime.

The second part follows from the first part. Suppose L contains all of the prime implicants and one implicant Q which is not prime. Since Q is not prime some of its literals can be removed while preserving the implication property. Denote the product of the redundant literals by R. It follows that $Q = PR$ where

$$P \leq PR = Q \leq \mathcal{F}$$

Since no more literals can be removed P must be a prime implicant and by the first part of the proof $P = L_j$ for some j. Therefore, Q would be removed from L by the fact that it is covered by L_j. So only prime implicants appear in the final list L. $\square$