

Tight Timing Estimation With the Newton-Gregory Formulae

Robert van Engelen

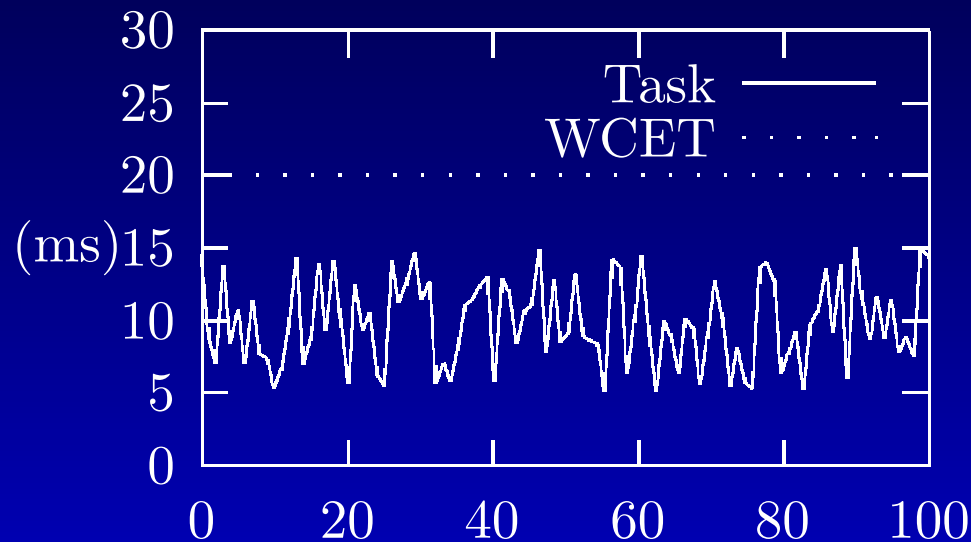
Kyle Gallivan

Burt Walsh

Florida State University

Introduction

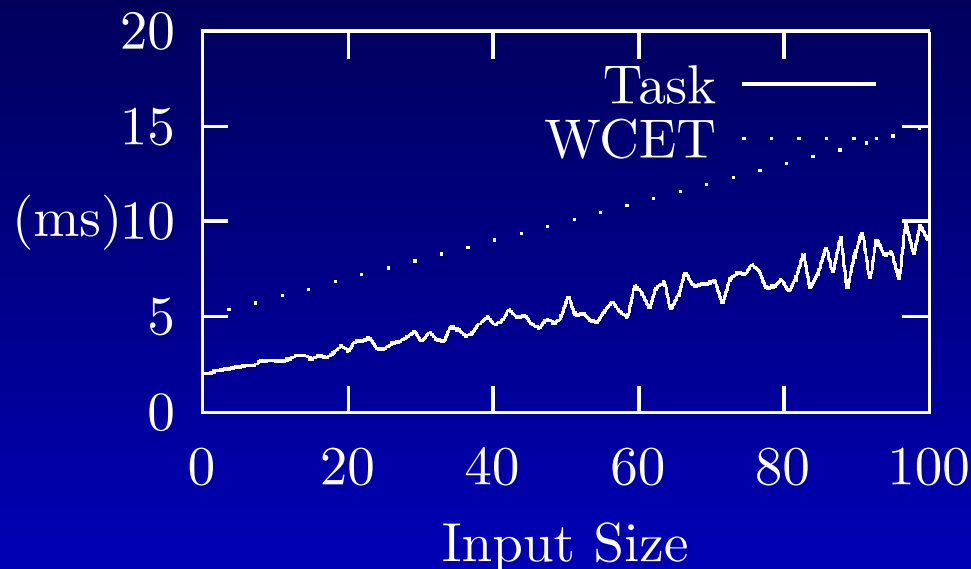
- *Worst-case execution time (WCET)* estimation has important applications in scheduling and real-time embedded systems.



The WCET bound is obtained by static (compiler) analysis of a task and is used to determine if the task will finish within a given time frame.

Parameterized WCET

- *Parametric timing analysis* produces formulae, where the unknown values affecting the execution are parameterized.



$$WCET = 5.0 + 0.1 N$$

where N is the task input size.

Related Work

- Presburger formulas [Pugh, 1994]
- Summations over τ functions [Haghighat, 1995]
- Ehrhart polynomials [Clauss, 1996]
- Guarded sums [Sakellariou, 1997]
- Parametric timing analysis [Vivancos et al., 2001]

Experiences

- We incorporated Sakellariou's guarded sums into a WCET estimation framework which required linking a small computer algebra system with the VPO compiler [Healy, 2000]
- The timing estimation can be very loose for loops with slightly more complicated control flow

```
for  $i = 0$  to 9 do
  if  $a[i] > 0$  then
    for  $j = 1$  to  $i$  do
       $b[j] = b[i] + a[i]$ 
  else
     $b[i] = a[i]$ 
```

Motivation

- WCET analysis focus on loops, because most of the execution time of a task is typically spent in loops.
- Loops may have symbolic bounds and exhibit complicated control flow.

```
for  $i = 0$  to  $N$  do
  if ... then
    for  $j = 0$  to  $i$  do
      ...
  else
    for  $j = 0$  to  $i$  do
      for  $k = 0$  to  $j$  do
        ...
```

Motivation (cont'd)

Existing methods...

- are computationally expensive and require extensive (symbolic) manipulation.
- can lead to an over estimation of the WCET of loops with complicated control flow due to the decoupling of the WCET of the loop body and the iteration count

$$\begin{aligned} WCET &= WCET(loopHeader) \\ &+ WCET(loopBody) * iterCount \end{aligned}$$

Loop Execution Time

- Loops are assumed to have been normalized

$$\boxed{\begin{array}{l} \text{for } i = a \text{ to } b \text{ step } s \text{ do} \\ S \end{array}} \Rightarrow \boxed{\begin{array}{l} \text{for } i = 0 \text{ to } \lfloor \frac{b-a}{s} \rfloor \text{ do} \\ S' \end{array}}$$

- The unknown loop execution time functions are

$\omega(H(i))$ = Loop header execution time

$\omega(S'(i))$ = Loop body execution time

- The (unknown) real loop execution time ω is

$$\omega = \omega(H(0)) + \sum_{i=0}^{\lfloor \frac{b-a}{s} \rfloor} \omega(S'(i)) + \omega(H(i+1))$$

Loop Execution Time Bound

- First we recursively determine (parameterized) WCET bounds on the unknowns

$$\omega(H(0)) \leq c_0$$

$$\omega(S'(i)) + \omega(H(i+1)) \leq p(i)$$

- Then, the WCET bound on ω is

$$\omega \leq WCET = c_0 + \sum_{i=0}^{n-1} p(i)$$

provided that the number of loop iterations

$$n = \lfloor \frac{b-a}{s} \rfloor + 1 > 0.$$

Newton-Gregory

- Let $p(i) = p_0 + p_1 i + \cdots + p_k i^k$ be a polynomial.
There exist *Newton series coefficients*

$$\Phi(i) = \langle \phi_0, \phi_1, \dots, \phi_k \rangle_i$$

such that

$$p(i) = \sum_{j=0}^k \phi_j \binom{i}{j}$$

Newton-Gregory WCET

Lemma.

- To determine the WCET of a loop, we can replace the WCET iteration summation with a constant summation

$$WCET = c_0 + \sum_{i=0}^{n-1} p(i) = \sum_{j=0}^k \phi_j \binom{n}{j+1}$$

for all (symbolic) $n > 0$.

Newton Series Conversions

- Given coefficients $[p_0, p_1, \dots, p_k]_i$ of polynomial $p(i) = p_0 + p_1 i + \dots + p_k i^k$, then the Newton series is

$$\Phi(i) = \mathbf{N}_k p(i)$$

where \mathbf{N}_k is the constant integer $(k + 1) \times (k + 1)$ “*Newton triangle matrix*”.

- Given Newton series coefficients $\Phi(i) = \langle \phi_0, \phi_1, \dots, \phi_k \rangle_i$, then

$$p(i) = \mathbf{N}_k^{-1} \Phi(i)$$

where \mathbf{N}_k^{-1} is the constant rational $(k + 1) \times (k + 1)$ “*inverse Newton triangle matrix*”.

WCET of a Loop

- Let $\Phi(i) = \langle \phi_0, \dots, \phi_k \rangle_i$ denote the Newton series of a polynomial in i over $i = 0, \dots, n - 1$, $n \geq 0$. We define

$$\sigma(\Phi(i), n) \stackrel{\text{def}}{=} \sum_{j=0}^k \phi_j \binom{n}{j+1}$$

- The WCET of a (possibly zero trip) loop $i = a, \dots, b$, stride s , and iteration WCET polynomial $p(i)$ with Newton series $\Phi(i)$ is

$$WCET = c_0 + \sigma(\Phi(i), \max(0, \frac{b-a}{s} + 1))$$

A Useful Property

Lemma.

- The coefficients s_0, \dots, s_{k+1} of the polynomial $s(n) = \sigma(\Phi(i), n)$ are given by the matrix-vector product

$$s(n) = [s_0, \dots, s_{k+1}]_n = \mathbf{N}_{k+1}^{-1} \langle 0, \phi_0, \dots, \phi_k \rangle_i$$

where $\Phi(i) = \langle \phi_0, \phi_1, \dots, \phi_k \rangle_i$.

Application

- Hence, the WCET of a loop is

$$\begin{aligned} WCET &= c_0 + \sigma(\Phi(i), \max(0, \frac{b-a}{s} + 1)) \\ &= c_0 + s(\max(0, \frac{b-a}{s} + 1)) \end{aligned}$$

where $s(n) = \mathbf{N}_{k+1}^{-1} \langle 0, \phi_0, \dots, \phi_k \rangle_i$.

An Example

```
for  $I = 1$  to  $N$  do /* header  $H_1$  */  
  for  $J = I$  to  $I * I - 2$  step 2 do /* header  $H_2$  */  
     $S$ 
```

The normalized iteration space is

$$i = 0, \dots, N - 1$$

$$j = 0, \dots, \lfloor \frac{i+i^2}{2} \rfloor - 1$$

The (recursively determined) execution time bounds

$$\omega(H_1(0)) \leq c_0$$

$$\omega(H'_2(i, 0)) \leq c_1$$

$$\omega(S'(i, j)) + \omega(H'_2(i, j + 1)) \leq c_2$$

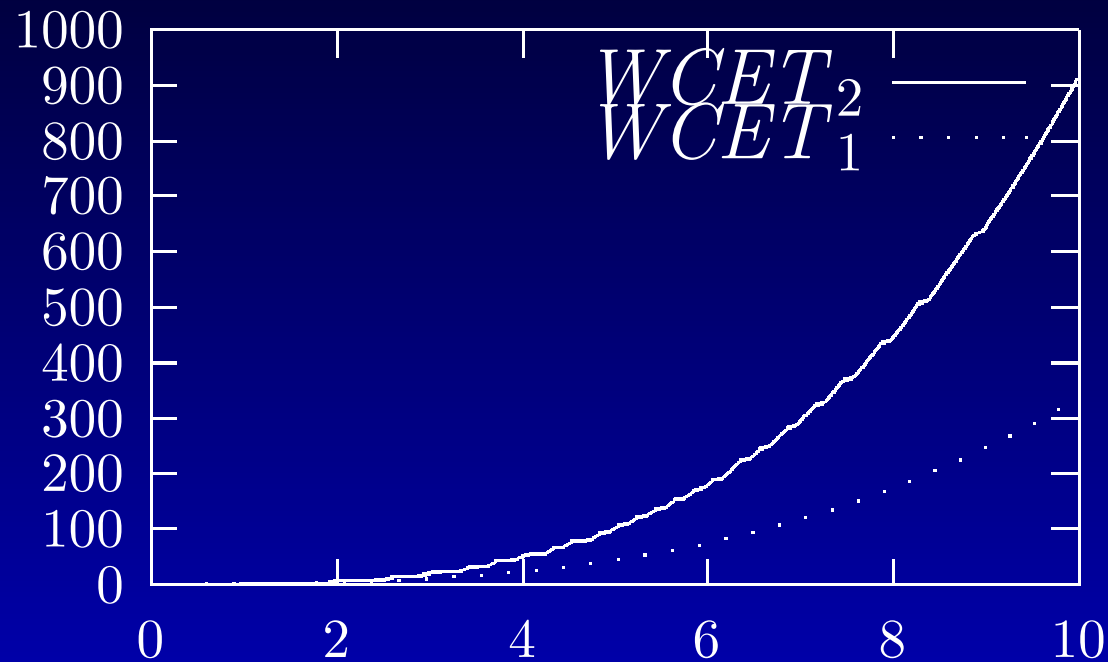
An Example (cont'd)

$WCET_1$

$$\begin{aligned} &= c_0 + \sigma(\mathbf{N}_1(c_1 + \sigma(\mathbf{N}_0 c_2, \max(0, \frac{i+i^2}{2}))), \max(0, N)) \\ &\quad (i + i^2 \geq 0, \text{ so replace } \max(0, \frac{i+i^2}{2}) \text{ by } \frac{i+i^2}{2}) \\ &= c_0 + \sigma(\mathbf{N}_1(c_1 + \sigma(\mathbf{N}_0 c_2, \frac{i+i^2}{2})), \max(0, N)) \\ &= c_0 + \sigma(\mathbf{N}_1(c_1 + \sigma(c_2, \frac{i+i^2}{2})), \max(0, N)) \\ &= c_0 + \sigma(\mathbf{N}_1(c_1 + \frac{1}{2}c_2(i + i^2)), \max(0, N)) \\ &= c_0 + \sigma(\langle c_1, c_2, c_2 \rangle_i, \max(0, N)) \\ &= c_0 + (c_1 - \frac{1}{6}c_2) \max(0, N) + \frac{1}{6}c_2 \max(0, N)^3 \end{aligned}$$

An Example (cont'd)

For $c_0 = 1$, $c_1 = 1$, and $c_2 = 2$ we have



where the conventional WCET estimation of the example fragment gives the over estimation

$$WCET_2 = c_0 + c_1 N + c_2 N \left\lfloor \frac{N^2 - N}{2} \right\rfloor$$

Loop Bounds Analysis

- If the loop iteration space size is non-negative ($n = \lfloor \frac{b-a}{s} \rfloor + 1 \geq 0$), the max operation can be eliminated from the WCET expression
$$WCET = c_0 + \sigma(\Phi(i), \max(0, \frac{b-a}{s} + 1)).$$
- Otherwise, splitting with guards is used.
- Use symbolic range propagation [Blume, 1995].
- Or we can use Newton series representations directly.

Monotonicity

- If the values of a (symbolic) expression are monotonic in a loop, its extreme values can be easily determined (symbolically).
- Most array index and loop bound expressions are monotonic (in one loop dimension).
- But how can we determine monotonicity without too much effort?

Monotonic Polynomials

- Let $\Phi(i) = \langle \phi_0, \phi_1, \dots, \phi_k \rangle_i$ denote the Newton series of a polynomial $p(i)$ in i .
- If $h(i) = \mathbf{N}_{k-1}^{-1} \langle \phi_1, \dots, \phi_k \rangle_i \geq 0$ for all $i = 0, \dots, n - 2$, $n \geq 0$, then $p(i)$ is monotonically increasing on the interval $[0, n - 1]$.
- Similarly, if $h(i) \leq 0$ for all $i = 0, \dots, n - 2$, then $p(i)$ is monotonically decreasing on the interval $[0, n - 1]$.

Monotonicity Example

- $p(i) = \frac{i^2-i}{2}$ with coefficients $[0, -\frac{1}{2}, \frac{1}{2}]_i$ has Newton series $\Phi(i) = \mathbf{N}_2[0, -\frac{1}{2}, \frac{1}{2}]_i = \langle 0, 0, 1 \rangle_i$.
- Because $h(i) = \mathbf{N}_1^{-1} \langle 0, 1 \rangle_i = i \geq 0$ for all $i \geq 0$, $p(i)$ is monotonically increasing with i .
- The value range of $p(i)$ on $i = 0, \dots, N$ is $[p(0), p(N)] = [0, \frac{N^2-N}{2}]$.
- While conventional value range analysis gives $\frac{1}{2}[0, N]^2 - \frac{1}{2}[0, N] = [-\frac{1}{2}N, \frac{1}{2}N^2]$.

Value Range Analysis

Value range of expression $E = [\mathcal{L}(E), \mathcal{U}(E)]$

Rewrite rules: $\mathcal{L}(E_1 + E_2) = \mathcal{L}(E_1) + \mathcal{L}(E_2)$

$$\mathcal{L}(\Phi(i)) = \begin{cases} \mathcal{L}(\phi_0) & \text{if } \mathcal{L}(\langle \phi_1, \dots, \phi_k \rangle_i) \geq 0 \\ \mathcal{L}(\mathbf{N}_k^{-1} \Phi(n-1)) & \text{if } \mathcal{U}(\langle \phi_1, \dots, \phi_k \rangle_i) \leq 0 \\ \mathcal{L}(\mathbf{N}_k^{-1} \Phi(i)) & \text{otherwise} \end{cases}$$

$$\mathcal{U}(\Phi(i)) = \begin{cases} \mathcal{U}(\phi_0) & \text{if } \mathcal{U}(\langle \phi_1, \dots, \phi_k \rangle_i) \leq 0 \\ \mathcal{U}(\mathbf{N}_k^{-1} \Phi(n-1)) & \text{if } \mathcal{L}(\langle \phi_1, \dots, \phi_k \rangle_i) \geq 0 \\ \mathcal{U}(\mathbf{N}_k^{-1} \Phi(i)) & \text{otherwise} \end{cases}$$

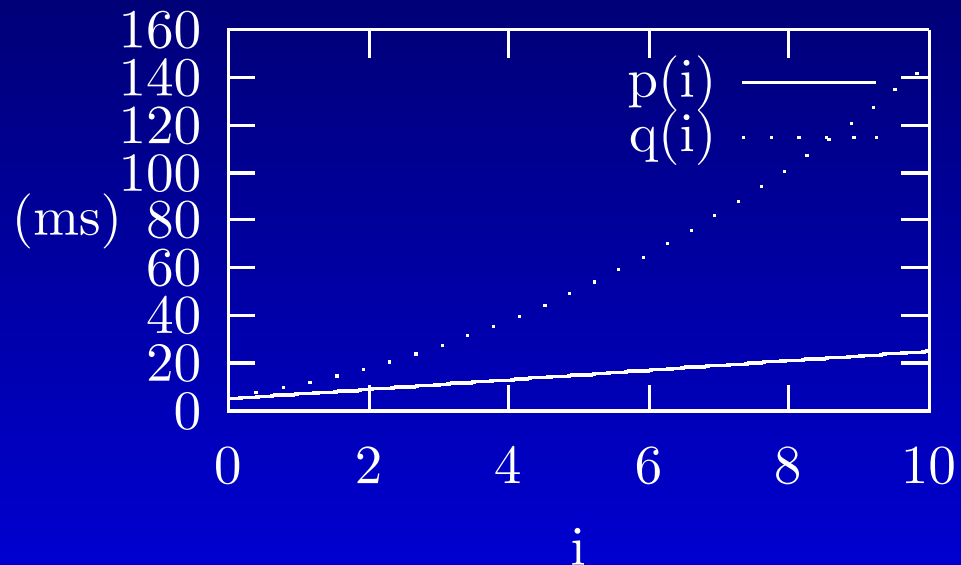
Finding Critical Paths

The objective is to find the critical path

```
for  $i = 0$  to  $n$  do
  if  $C$  then
     $S_1$  /* path with parametric WCET cost  $p(i)$  */
  else
     $S_2$  /* path with parametric WCET cost  $q(i)$  */
```


Comparing WCET Polynomials

- Let $\Phi(i)$ and $\Psi(i)$ be Newton series of polynomials $p(i)$ and $q(i)$.
- If $\mathcal{L}(\Phi(i) - \Psi(i)) \geq 0$ then $p(i) \geq q(i)$.
- If $\mathcal{U}(\Phi(i) - \Psi(i)) \leq 0$ then $p(i) \leq q(i)$.



Example

```
for  $i = 0$  to  $N$  do
  if  $a[i] > 0$  then
    for  $j = 0$  to  $i$  do
       $S_4$ 
    else
      for  $j = 0$  to  $i$  do
        for  $k = 0$  to  $j$  do
           $S_8$ 
```

$$\omega(H_1(0)) \leq 1$$

$$\omega(C_2(i)) \leq 1$$

$$\omega(H_3(i, 0)) \leq 1$$

$$\omega(S_4(i, j)) + \omega(H_3(i, j + 1)) \leq 2$$

$$\omega(H_6(i, 0)) \leq 1$$

$$\omega(H_7(i, 0)) \leq 1$$

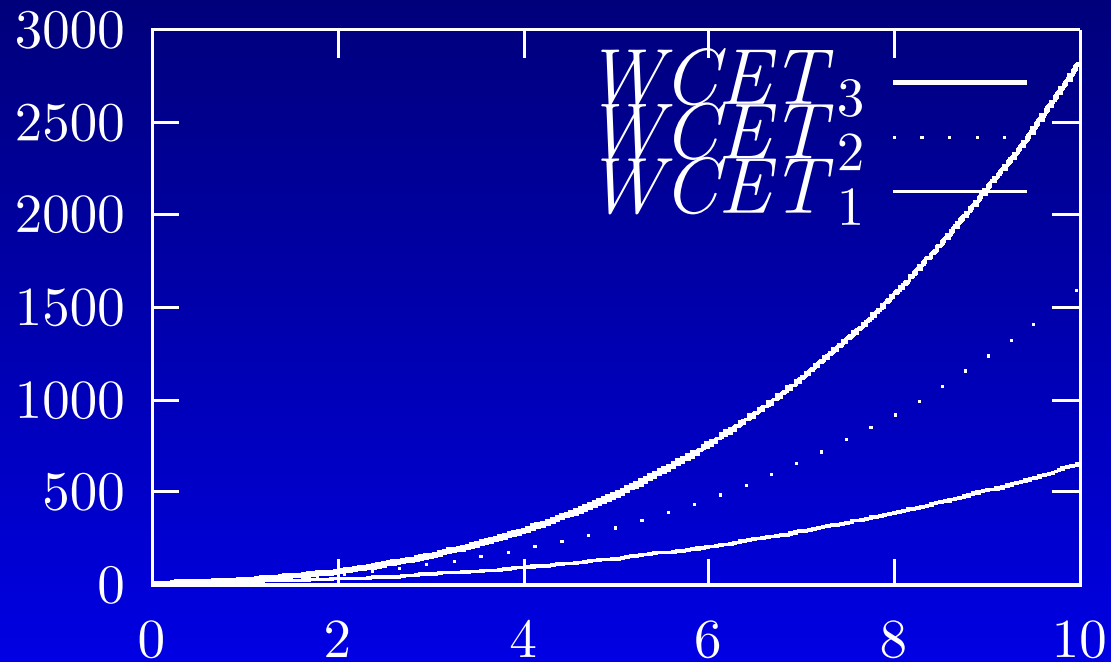
$$\omega(S_8(i, j)) + \omega(H_7(i, j + 1)) \leq 2$$

Example (cont'd)

$$WCET_1 = 7 + 6\frac{1}{6}N + 2\frac{1}{2}N^2 + \frac{1}{3}N^3$$

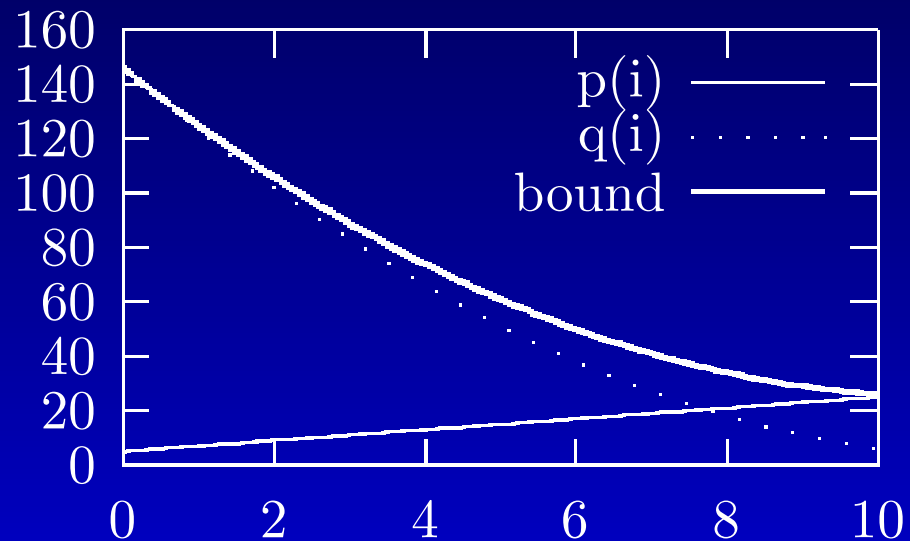
$$WCET_2 = 1 + (N + 1) \max(5 + 2N, 6 + 4N + N^2)$$

$$WCET_3 = 1 + (N + 1) \max(5 + 2N, 6 + 5N + 2N^2)$$



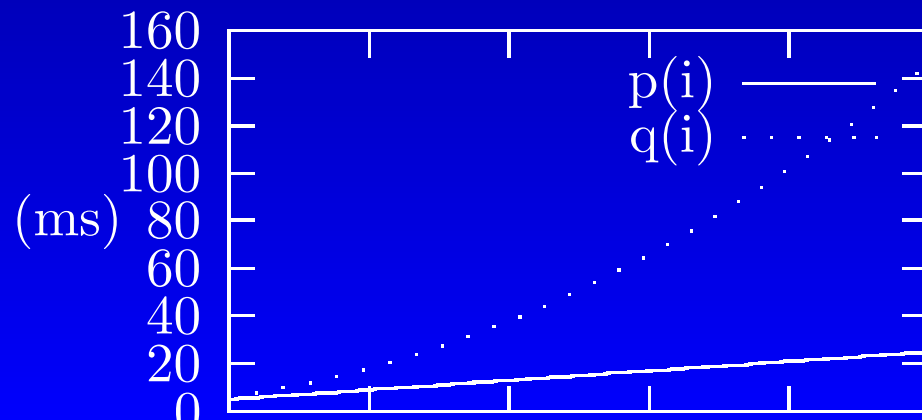
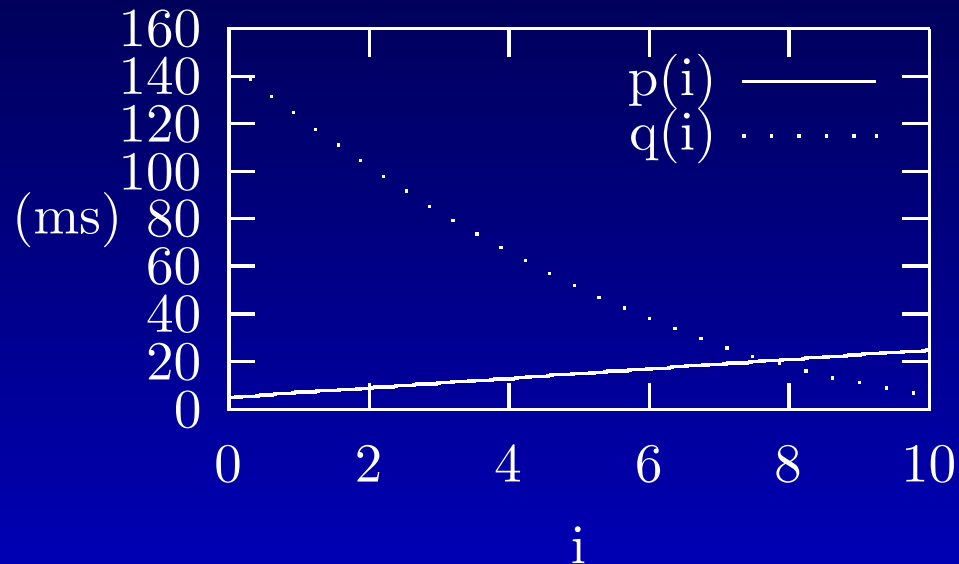
Bounding WCET Polynomials

- When one of the WCET polynomials does not dominate, we need to split the iteration space (root finding!) or derive an overall bounding polynomial.



Iteration Reversal

When one of the WCET polynomials does not dominate and the paths are loop invariant, the iteration direction can be reversed.



Conclusions

- Parametric WCET has important applications.
- Timing of loops with complicated control flow is difficult.
- Relatively simple numeric/symbolic method.
- Approach also contributes to value range analysis.
- Implementation is work in progress.