

Derangements

Breno de Medeiros

Let $S = \{p_1, p_2, \dots, p_n\}$ be a finite set with n elements. A permutation of S is a function defined on S , with image in S , and one-to-one. The set of permutations of S is $\Pi = \{\pi(\cdot) : S \rightarrow S; p \neq q \Rightarrow \pi(p) \neq \pi(q)\}$. The number of distinct permutations of S is $|\Pi(n)| = n!$

Numerical example: $n = 5, |\Pi(5)| = 5! = 120$.

A fixed point p of a permutation $\pi(\cdot)$ is any element of S that is not moved by $\pi(\cdot)$, i.e., $\pi(p) = p$. An interesting type of permutation is a *derangement*. Derangements are permutations without *fixed points*. Call the set of derangements D . Clearly the number of derangements is at most $n!$, but clearly less.

The goal of this exercise is to count the number of derangements on n elements. Our first approximation is $|D(n)| \leq n!$. First we introduce some notation. The set of permutations that fix p_1 is denoted Π_{p_1} . The set of permutations that fix both p_1 and p_2 is denoted Π_{p_1, p_2} . Note that

$$\Pi_{p_1, p_2} \subset \Pi_{p_1}; \text{ and } \Pi_{p_1} \cap \Pi_{p_2} = \Pi_{p_1, p_2}.$$

We can approximate the number of derangements by:

$$\begin{aligned} |D| &\geq |\Pi| - |\Pi_{p_1}| - |\Pi_{p_2}| - \dots - |\Pi_{p_n}| \\ &= n! - n(n-1)! = 0. \end{aligned}$$

This equation says that the number of derangements is approximately the number of all permutations minus the number of permutations that fixes p_1 , minus the number of permutations that fixes p_2 , etc. However, the fact that the result equals 0 shows that the equation cannot be correct. We only conclude that $D(n) \geq 0$, which we already knew.

The problem with the above equation is that it subtracts some permutations twice. For instance, all permutations within Π_{p_1, p_2} were counted both as elements of Π_{p_1} and subtracted, and also as elements of Π_{p_2} and subtracted. Even worse, the permutations in Π_{p_1, p_2, p_3} have been subtracted three times, and so on. Therefore, we can improve the above approximation as:

$$\begin{aligned} |D| &\leq |\Pi| - |\Pi_{p_1}| - \dots - |\Pi_{p_n}| + |\Pi_{p_1, p_2}| + |\Pi_{p_1, p_3}| + \dots + |\Pi_{p_1, p_n}| + |\Pi_{p_2, p_3}| + \dots + |\Pi_{p_{n-1}, p_n}| \\ &= n! - n(n-1)! + \binom{n}{2}(n-2)! = \frac{n!}{2} \end{aligned}$$

We then notice that we have added back too much. Namely, the permutations that fix p_1 and p_2 and p_3 have been added back thrice: In Π_{p_1, p_2} , and Π_{p_1, p_3} , and Π_{p_2, p_3} . Since they had been subtracted three times in the first round, they have neither been added or subtracted from D at this point. But they should have been subtracted once, because they are not derangements. We

can improve this by subtracting back the triple-fixed point permutations such as the elements of Π_{p_1,p_2,p_3} :

$$\begin{aligned}
|D| &\simeq |\Pi| - |\Pi_{p_1}| - \cdots - |\Pi_{p_n}| + |\Pi_{p_1,p_2}| + |\Pi_{p_1,p_3}| + \cdots + |\Pi_{p_1,p_n}| + |\Pi_{p_2,p_3}| + \cdots + |\Pi_{p_{n-1},p_n}| \\
&\quad - |\Pi_{p_1,p_2,p_3}| - \cdots - |\Pi_{p_1,p_{n-1},p_n}| - |\Pi_{p_2,p_3,p_4}| - \cdots - |\Pi_{p_{n-2},p_{n-1},p_n}| \\
&= n! - \binom{n}{1}(n-1)! + \binom{n}{2}(n-2)! - \binom{n}{3}(n-3)!
\end{aligned}$$

It is possible to show by induction that the number of derangements is given by the extension of the above formula, alternating the signal (add, subtract) as one increases the number of fixed points. This general technique for counting the number of elements of a set is called the inclusion-exclusion principle.

$$\begin{aligned}
|D| &\simeq |\Pi| - |\Pi_{p_1}| - \cdots - |\Pi_{p_n}| + |\Pi_{p_1,p_2}| + |\Pi_{p_1,p_3}| + \cdots + |\Pi_{p_1,p_n}| + |\Pi_{p_2,p_3}| + \cdots + |\Pi_{p_{n-1},p_n}| \\
&\quad - |\Pi_{p_1,p_2,p_3}| - \cdots - |\Pi_{p_1,p_{n-1},p_n}| - |\Pi_{p_2,p_3,p_4}| - \cdots - |\Pi_{p_{n-2},p_{n-1},p_n}| \\
&\quad + \cdots - \cdots + \cdots + (-1)^n \Pi_{p_1,\dots,p_n} \\
&= n! - \binom{n}{1}(n-1)! + \binom{n}{2}(n-2)! - \binom{n}{3}(n-3)! + \cdots \\
&= \sum_{k=0}^n (-1)^k \binom{n}{k} (n-k)!
\end{aligned}$$

For instance, for $n = 5$ this equals 44. The number of derangements is also denoted $!n$, so we have: $!1 = 0, !2 = 1, !3 = 2, !4 = 9, !5 = 44$.

The fraction of all permutations that are derangements is:

$$|D|/n! = \sum_{k=0}^n (-1)^k \binom{n}{k} \frac{(n-k)!}{n!} \tag{1}$$

$$= \sum_{k=0}^n (-1)^k \frac{1}{k!} \tag{2}$$

And that value converges to $1/e$ as n becomes vary large. So for large n , only approximately 36.79% of all permutations are derangements.