COT5405: Fall 2006

Lecture 11

LP for Max-SAT

Max: $\Sigma_c w_c z_c$
Subject to:
$\Sigma_i y_i$ [i varies over non-negated variables of c] + $\Sigma_i (1-y_i)$ [i varies over negated variables of c] $\geq z_c$, $\forall c$
$y_i, z_c \in \{0,1\}$
Relaxation: $0 \leq y_i, z_c \leq 1$.

Algorithm 3:

1. Compute optimal solution y^*, z^* for relaxed LP.
2. Set $x_i = 1$ with probability y^*_i, and 0 otherwise.

Theorem: The value of the solution W from algorithm 3 satisfies: $E(W) \geq (1 - e^{-1})\ OPT$.

Proof:

Without loss of generality, let a clause c be $\{x_1, x_2, ..., x_k\}$.

$E(W_c) = w_c$ (1 – *probability that none of the* x_i*s are rounded to 1*)
$= w_c (1 - \Pi(1-y^*_k))$.

From the relationship of geometric to arithmetic mean,

$[\Pi(1-y^*_k)]^{1/k} \leq (1/k)\ \Sigma(1-y^*_k) \Rightarrow \Pi(1-y^*_k) \leq [(1/k)\ \Sigma(1-y^*_k)]^k = [1 - (1/k)\ \Sigma y^*_k]^k \leq [1 - z^*_c/k]^k$
$\Rightarrow E(W_c) \geq w_c (1 - [1 - z^*_c/k]^k)$.

Let $\beta_k = 1 - (1 - 1/k)^k$. We can show that $1 - [1 - z/k]^k \geq \beta_k z,\ z \in [0,1]$ (review question, Lec 10).
So, $E(W_c) \geq w_c \beta_k z^*_c \geq w_c z^*_c (1 - e^{-1})$, using the result of review question 2 of Lec 10.
So, $E(W) \geq (1 - e^{-1})\ \Sigma\ w_c z^*_c \geq (1 - e^{-1})\ OPT$.

Algorithm 4: Derandomize algorithm 3, using either $E(W_c) \geq w_c \beta_k z^*_c$ for each clause, or
$E(W) \geq \beta_k \Sigma w_c z^*_c$, where k is the largest number of variables in any clause.

Algorithm 5: Randomly select algorithm 1 or algorithm 3 with probability ½ each. It is expected factor ¾ algorithm for the following reason:
In algorithm 1, $E[W_c] = (1 - 2^{-k})w_c \geq (1 - 2^{-k})w_c z^*_c$.
In algorithm 2, $E[W_c] \geq [1 - (1 - 1/k)^k]w_c z^*_c$.
So, $E[W_c] \geq 0.5\ [1 - (1 - 1/k)^k]w_c z^*_c + 0.5\ (1 - 2^{-k})w_c z^*_c$. Note that $[1 - (1 - 1/k)^k] + (1 - 2^{-k}) \geq 3/2$.

Algorithm 6: Try algorithms 2 and 4, and choose the better one.